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A GENERAL ITERATIVE METHOD FOR APPROXIMATION OF FIXED POINTS AND THEIR APPLICATIONS

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Abstract. We propose a new iterative algorithm and prove strong and weak convergence theorems for computing fixed points of nonexpansive mappings in a Banach space. We showed that our iteration process is faster than Picard, Mann and S iteration processes. Our results are applied for finding solutions of variational inequality problem.

Keywords: Fixed Point; Nonexpansive mapping; Strong and weak convergence; Variational Inequality Problem.

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1. INTRODUCTION

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|.\|$, respectively, C is a nonempty closed convex subset of H, P_C is the metric projection of H onto C, and I is the identity mapping on C. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

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We denote the set of fixed points of T by $F(T)$. Recall that a mapping $A : C \to H$ is said to be

(1) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \forall x, y \in C,$$

(2) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta \left\| x - y \right\|^2, \ \forall x, y \in C,$$

(3) v-inverse strongly monotone if there exists a constant $v > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq v \left\| Ax - Ay \right\|^2, \ \forall x, y \in C,$$

(4) L-Lipschitzian if there exists constant a $L > 0$ such that

$$\left\| Ax - Ay \right\| \leq L \left\| x - y \right\|, \ \forall x, y \in C.$$

It is obvious that any v-inverse strongly monotone mapping A is monotone and $(1/v)$-Lipschitz continuous. A self-mapping $f : C \to C$ is a contraction if there exists a constant $k \in (0, 1)$ such that

$$\left\| f(x) - f(y) \right\| \leq k \left\| x - y \right\|, \ \forall x, y \in C.$$

Let $A : C \to H$ be a mapping. The classical variational inequality problem $VI(C\ , A)$ is to find a point $x \in C$ such that $\langle Ax, y - x\rangle \geq 0$ for all $y \in C$. The set of solutions of variational inequality problem is denoted by $\Omega(C\ , A)$. The variational inequalities were initially studied by Stampachhia [5, 6], and ever since have been widely studied.(see, [11, 12, 15, 16, 17, 18, 19, 21, 22, 24]). Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in H$ satisfying $0 = Au$ and so on. The existence and approximation of solutions are important aspects for the study of variational inequalities. It is well known that the variational inequality problem is equivalent to finding the set of fixed points of the operator $P_C(I - \mu A)$, i.e., $F(P_C(I - \mu A)) = \Omega(C\ , A)$, where $\mu > 0$ is a constant and P_C is a metric projection from H onto C. We know that if A is κ-Lipschitzian and η-strongly monotone, then the operator $P_C(I - \mu A)$ is also a contraction on C with $0 < \mu < 2\eta/\kappa^2$.

So, it follows from the Banach contraction principle that $VI\left(C,A\right)$ has a unique solution x^* and the sequence of the Picard iteration process, given by x^*,

$$x_{n+1} = P_C\left(I - \mu A\right) x_n,\ n \geq 1, \tag{1.1}$$

converges strongly to x^*. This method is called the projected gradient method [13]. It has been used widely in many practical problems, due partially to its fast convergence.

On the other hand, two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [7] and is defined as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$.

The second iteration process is referred to as Ishikawa's iteration process [9] which is defined recursively by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = \left(1-\beta_n\right) x_n + \beta_n T x_n,\ n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ is a real sequences in $(0,1)$ (see for example [7, 9, 10]). The Ishikawa process can be seen as a "double Mann iterative process" or a "hybrid of Mann process with itself".

In 2007, Agarwal-O'Regan-Sahu [1] introduced the S iteration process:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n = \left(1-\beta_n\right) x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ is a real sequences in $(0,1)$. The S iteration process is independent of all Picard, Mann and Ishikawa iterative processes. Sahu [4] also showed analytically that the process (1.4) converges faster than both Picard and Mann. See Theorem 3.6 [4].

In this paper, motivated and inspired by Agarwal-O'Regan-Sahu [1], we will introduce a new iterative scheme. Our process reads as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)\, T x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)\, T x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)\, x_n + \gamma_n T x_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ is a real sequences in $(0,1)$. (1.5) is called S* iteration process. S* iteration process is independent from the (1.1), (1.2), (1.3), and (1.4) iteration processes.

The purpose of this paper is to prove that our process (1.5) converges faster than all of Picard, Mann and S iterative processes for contractions in the sense of Berinde [3]. We support our analytical proof by a numerical example. We also prove a strong convergence theorem with the help of our process for the class of nonexpansive mappings in general Banach spaces and apply it to get a result in uniformly convex Banach spaces. We also prove some weak convergence results when the underlying space satisfies Opial's condition.

2. PRELIMINARIES

Let X be a Banach space and $S_X = \{x \in X : \|x\| = 1\}$ unit sphere on X. For all $\lambda \in (0,1)$, and $x, y \in S_X$ with $x \neq y$, if $\|(1-\lambda)\, x + \lambda y\| < 1$, then X is called strictly convex. If X is a *strictly convex* Banach space and $\|x\| = \|y\| = \|\alpha x + (1-\alpha)\, y\|$ for $x, y \in X$ and $\alpha \in (0,1)$, then $x = y$.

The space X is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x and y in S_X. In this case, the norm of X is called *Gateaux differentiable.* For all $y \in S_X$, if the limit (2.1) is attained uniformly for $x \in S_X$, then the norm is said to be *uniformly Gateaux differentiable or Frechet differentiable.*

We call the space X satisfies the opial condition [8] if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$.

A mapping $T : C \to X$ is demiclosed at $y \in X$ if for each sequence $\{x_n\}$ in C and each $x \in X$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

The following lemmas will be needed in the sequel for the proof of our main results:

Lemma 1. [23]*Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and T a nonexpansive mapping on C. Then, $I - T$ is demiclosed at zero.*

Lemma 2. [14]*Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 3. [2]*Let X be a reflexive Banach space satisfying the Opial condition, C a nonempty convex subset of X and $T : C \to X$ an operator such that $I - T$ demiclosed at zero and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

3. COMPARISON OF S* AND S ITERATION PROCESSES

Now, we give S* operator and some of its properties and then compare the rate of convergence of the S* iteration process with the S iteration process for contraction operators.

Definition 1. *Let C be a nonempty convex subset of a vector space X and $T : C \to C$ an operator. Then, $G_{\alpha,\beta,\gamma,T} : C \to C$ is called S^* operator generated by α, β, $\gamma \in (0, 1)$*

and T if

$$G_{\alpha,\beta,\gamma,T} = (1-\alpha)\,T + \alpha T\left((1-\beta)\,T + \beta T\left((1-\gamma)\,I + \gamma T\right)\right).$$

If T is a contraction operator with contractivity factor k, then it is easy to see that $G_{\alpha,\beta,\gamma,T}$ is a contraction with contractivity factor $k\left(1-\alpha\left(1-k\left(1-\beta\gamma\left(1-k\right)\right)\right)\right)$ and if T is a nonexpansive operator then $G_{\alpha,\beta,\gamma,T}$ is also nonexpansive.

Proposition 1. *Let X be a Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a contraction operator. Suppose that α, β, $\gamma \in (0,1)$. If $G_{\alpha,\beta,\gamma,T}$ is an S^* operator generated by α, β, γ and T, then $F\left(G_{\alpha,\beta,\gamma,T}\right) = F\left(T\right).$*

Proposition 2. *Let X be a strictly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive operator with $F\left(T\right) \neq \emptyset$. Suppose that α, β, $\gamma \in (0,1)$. If $G_{\alpha,\beta,\gamma,T}$ is an S^* operator generated by α, β, γ and T, then $F\left(G_{\alpha,\beta,\gamma,T}\right) = F\left(T\right).$*

Proof. It is clear from the definition of $G_{\alpha,\beta,\gamma,T}$ that $F\left(T\right) \subseteq F\left(G_{\alpha,\beta,\gamma,T}\right)$. Now we show that $F\left(G_{\alpha,\beta,\gamma,T}\right) \subseteq F\left(T\right)$. Let $z \in F\left(G_{\alpha,\beta,\gamma,T}\right)$ and $v \in F\left(T\right)$. We have

$$\begin{aligned}
\|z-v\| &= \|(1-\alpha)\,Tz + \alpha T\left((1-\beta)\,T + \beta T\left((1-\gamma)\,I + \gamma T\right)\right)z - v\| \\
&\leq (1-\alpha)\,\|Tz-v\| + \alpha\,\|(1-\beta)\,Tz + \beta T\left((1-\gamma)\,I + \gamma T\right)z - v\| \\
&\leq (1-\alpha)\,\|Tz-v\| + \alpha\left[(1-\beta)\,\|Tz-v\| + \beta\,\|T\left((1-\gamma)\,I + \gamma T\right)z - v\|\right] \\
&\leq (1-\alpha\beta)\,\|Tz-v\| + \alpha\beta\,\|T\left((1-\gamma)\,I + \gamma T\right)z - v\| \\
&\leq (1-\alpha\beta)\,\|Tz-v\| + \alpha\beta\,\|(1-\gamma)\,z + \gamma Tz - v\| \\
&\leq (1-\alpha\beta)\,\|Tz-v\| + \alpha\beta\left[(1-\gamma)\,\|z-v\| + \gamma\,\|Tz-v\|\right] \\
&\leq (1-\alpha\beta\gamma)\,\|z-v\| + \alpha\beta\gamma\,\|Tz-v\| \\
&\leq \|z-v\|.
\end{aligned}$$

This implies that

$$\begin{aligned} \|z-v\| &= \|Tz-v\| \\ &= \|T((1-\gamma)I+\gamma T)z-v\| \\ &= \|T((1-\beta)T+\beta T((1-\gamma)I+\gamma T))z-v\| \\ &= \|(1-\alpha)Tz+\alpha T((1-\beta)T+\beta T((1-\gamma)I+\gamma T))z-v\|. \end{aligned}$$

Since X is strinctly convex, we obtain $Tz = z$. Hence we have $F(G_{\alpha,\beta,\gamma,T}) \subseteq F(T)$, and so proof is completed. □

The following definitions about the rate of convergence are due to Berinde [3].

Definition 2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0$, then $\{a_n\}$ converges faster than $\{b_n\}$.*

Definition 3. *Suppose that for two fixed-point iteration processes $\{x_n\}$ and $\{u_n\}$, both converging to the same fixed point p, the error estimates*

$$\|x_n - p\| \leq a_n \text{ for all } n \geq 1,$$

$$\|u_n - p\| \leq b_n \text{ for all } n \geq 1,$$

are available where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{x_n\}$ converges faster than $\{u_n\}$ to p.

Theorem 1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator with contractivity factor $k \in [0,1)$ and fixed point p. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0,1)$ such that $\alpha \leq \alpha_n < 1$, $\beta \leq \beta_n < 1$ and $\gamma \leq \gamma_n < 1$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta, \gamma > 0$. For given $u_1 = x_1 \in C$, define sequences $\{x_n\}$ and $\{u_n\}$ in C as follows:*

S^* iteration process:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n &= (1-\beta_n)Tx_n + \beta_n Tz_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n \end{aligned}$$

S iteration process:

$$\begin{aligned} u_{n+1} &= (1-\alpha_n) Tu_n + \alpha_n Tv_n \\ v_n &= (1-\beta_n) u_n + \beta_n Tu_n. \end{aligned}$$

Then, we have the following:

(a) $\|x_{n+1}-p\| \leq k^n \left[1-\alpha\left(1-k\left(1-\beta\gamma\left(1-k\right)\right)\right)\right]^n \|x_1-p\|$

(b) $\|u_{n+1}-p\| \leq k^n \left[1-\alpha\beta\left(1-k\right)\right]^n \|u_1-p\|$.

Furthermore, the S^* iteration process is faster than the S iteration process.

Proof. (a) From the definition of the S* iteration process, we obtain

$$\begin{aligned} & \|x_{n+1}-p\| \\ \leq\; & (1-\alpha_n) k \|x_n-p\| + \alpha_n k \|y_n-p\| \\ \leq\; & (1-\alpha_n) k \|x_n-p\| + \alpha_n k \left[(1-\beta_n) k \|x_n-p\| + \beta_n k \|z_n-p\|\right] \\ \leq\; & (1-\alpha_n) k \|x_n-p\| \\ & +\alpha_n k \left[(1-\beta_n) k \|x_n-p\| + \beta_n k \left[(1-\gamma_n) \|x_n-p\| + \gamma_n k \|x_n-p\|\right]\right] \\ =\; & \|x_n-p\| \left[(1-\alpha_n) k + \alpha_n k^2 (1-\beta_n) + \alpha_n \beta_n k^2 (1-\gamma_n) + \alpha_n\beta_n\gamma_n k^3\right] \\ =\; & \|x_n-p\| k \left[1-\alpha_n\left(1-k\left(1-\beta_n\gamma_n\left(1-k\right)\right)\right)\right] \\ \leq\; & a_n \end{aligned}$$

where $a_n = \|x_1-p\| k^n \left[1-\alpha\left(1-k\left(1-\beta\gamma\left(1-k\right)\right)\right)\right]^n$.

(b) From the definition of the S iteration process, we have

$$\begin{aligned} \|u_{n+1}-p\| &\leq (1-\alpha_n) k \|u_n-p\| + \alpha_n k \|v_n-p\| \\ &\leq (1-\alpha_n) k \|u_n-p\| + \alpha_n k \left[(1-\beta_n) \|u_n-p\| + \beta_n k \|u_n-p\|\right] \\ &= \|u_n-p\| k \left[1-(1-k)\alpha_n\beta_n\right] \\ &\leq b_n \end{aligned}$$

where $b_n = \|u_1-p\| k^n \left[1-(1-k)\alpha\beta\right]^n$.

Moreover, it is clear that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Indeed, since

$$\begin{aligned}
\beta\left(1-k\gamma\right)<1 &\Rightarrow \beta\left(1-k\right)\left(1-k\gamma\right)<1-k \\
&\Rightarrow \beta\left(1-k\right)-k\beta\gamma\left(1-k\right)<1-k \\
&\Rightarrow \beta\left(1-k\right)<1-k+k\beta\gamma\left(1-k\right) \\
&\Rightarrow \left(1-k\right)\beta<1-k\left[1-\beta\gamma\left(1-k\right)\right] \\
&\Rightarrow \left(1-k\right)\alpha\beta<\alpha\left[1-k\left[1-\beta\gamma\left(1-k\right)\right]\right] \\
&\Rightarrow 1-\alpha\left[1-k\left[1-\beta\gamma\left(1-k\right)\right]\right]<1-\left(1-k\right)\alpha\beta,
\end{aligned}$$

we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\|x_1-p\|\,k^n\left[1-\alpha\left(1-k\left(1-\beta\gamma\left(1-k\right)\right)\right)\right]^n}{\|u_1-p\|\,k^n\left[1-\left(1-k\right)\alpha\beta\right]^n} \\
&= \lim_{n\to\infty} \frac{\left[1-\alpha\left(1-k\left(1-\beta\gamma\left(1-k\right)\right)\right)\right]^n}{\left[1-\left(1-k\right)\alpha\beta\right]^n} \\
&= 0.
\end{aligned}$$

So, the S^* iteration process is faster than the S iteration process. □

We support our above analytical proof by a numerical example.

Example 1. *Let* $X = \mathbb{R}$ *and* $C = [1, \infty)$. *Let* $T : C \to C$ *be an operator defined by* $Tx = \sqrt{x^2 - 9x + 54}$ *for all* $x \in C$. *It is not difficult to show that* T *is a contraction. Choose* $\alpha_n = \beta_n = \gamma_n = 3/4$ *for all* n *with initial value* $x_1 = 30$. *The comparison given in the following table shows that our iterative process (1.5) converges faster than all Picard, Mann and S iterative processes up to the accuracy of seven decimal places. All the processes converge to the same fixed point* $p = 6$. *First twelve iterations for all the processes are given below for the sake of comparison.*

A comparison of our process with other processes				
steps	S* iteration	S iteration	Picard iteration	Mann iteration
x_1	30.000000	30.000000	30.000000	30.000000
x_2	21.834770	24.050330	26.153390	27.115050
x_3	14.505540	18.437270	22.419180	24.290740
x_4	8.990788	13.393820	18.837380	21.542040
x_5	6.530200	9.372555	15.469660	18.889280
x_6	6.055566	6.993935	12.413040	16.360650
x_7	6.005067	6.186207	9.816627	13.995420
x_8	6.000455	6.028369	7.875057	11.847570
x_9	6.000041	6.004134	6.718706	9.986986
x_{10}	6.000004	6.000598	6.218734	8.490041
x_{11}	6.000000	6.000086	6.058386	7.408304
x_{12}	6.000000	6.000012	6.014862	6.724666

Remark. The above calculations have been repeated by taking different values of parameters α_n, β_n and γ_n. It has been verified every time that our iterative process S* converges faster than all Picard, Mann and S iterative processes. Moreover, it has been observed that as the values of α_n, β_n and γ_n go far below 0.5 and near 0 (above 0.5 and near 1), the convergence gets slower (faster), and it happens with every scheme except Picard as it has nothing to do with these parameters. For example, when $\alpha_n = \beta_n = \gamma_n = 1/4$ for all n, the values for the above four processes at the 12th iteration become 6.000679, 6.006863, 6.014862, 19.854840, respectively. The accuracy of seven decimal places is obtained by our process at the 16th iteration. The values at this iteration for the above processes are 6.000000, 6.000005, 6.000015 15.718300 . By the way, for $\alpha_n = \beta_n = \gamma_n = 1/2$ for all n, at the 14th iteration, we get 6.000000, 6.000029 6.000934 and 9.130115, respectively. For the initial value $x_1 = 2$ and $\alpha_n = \beta_n = \gamma_n = 3/4$ for all n, our iteration process is faster than the others. At the 7th iteration, S*, S, Picard and Mann iteration processes become 6.000000, 5.999999, 6.000362 and 5.989854, respectively.

In the light of Theorem 1, we give the following theorem.

Theorem 2. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator with contractivity factor $k \in [0,1)$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0,1)$ such that $\alpha \leq \alpha_n < 1$, $\beta \leq \beta_n < 1$ and $\gamma \leq \gamma_n < 1$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta, \gamma > 0$. Let $\{x_n\}$ be the S^* iteration process defined by (1.5) for the arbitrary initial value $x_1 \in C$ and $F(T) \neq \emptyset$. Then the iterative sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. For $p \in F(T)$, it follows from the proof of Theorem 1 and the definition of $\{x_n\}$ that

$$\lim_{n\to\infty} \|x_{n+1} - p\| \leq \lim_{n\to\infty} k^n \left[1 - \alpha\left(1 - k\left(1 - \beta\gamma\left(1-k\right)\right)\right)\right]^n \|x_1 - p\| = 0.$$

So, proof is completed. □

4. NONEXPANSIVE OPERATORS AND S* ITERATION PROCESS

Lemma 4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (1.5) and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F(T)$. Set $a_n := x_n - p$ for all $n \in \mathbb{N}$. From (1.5), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n \|Tx_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n \|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.1}$$

From (4.1) and (1.5), we have

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n) T x_n + \beta_n T z_n - p\| \\ &\leq (1-\beta_n)\|T x_n - p\| + \beta_n \|T z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|z_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.2}$$

By (4.2) and (1.5), we obtain

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n) T x_n + \alpha_n T y_n - p\| \\ &\leq (1-\alpha_n)\|T x_n - p\| + \alpha_n \|T y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|y_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

So, $\{\|a_n\|\}$ is nonincreasing and hence $\lim_{n\to\infty}\|a_n\|$ exists for all $p \in F(T)$. ☐

Lemma 5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (1.5) and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof. By Lemma 4, $\lim_{n\to\infty}\|x_n - p\|$ exists. Assume that $\lim_{n\to\infty}\|x_n - p\| = c$. From (4.1) and (4.2) we have

$$\limsup_{n\to\infty}\|y_n - p\| \leq c \text{ and } \limsup_{n\to\infty}\|z_n - p\| \leq c. \tag{4.3}$$

Since T is nonexpansive mappings, we have

$$\|Tx_n - p\| \leq \|x_n - p\| \text{ and } \|Ty_n - p\| \leq \|y_n - p\|$$

Taking $\limsup$ on both sides, we obtain

$$\limsup\|Tx_n - p\| \leq c \text{ and } \limsup\|Ty_n - p\| \leq c. \tag{4.4}$$

Since

$$c = \lim_{n\to\infty}\|x_{n+1} - p\| = \lim_{n\to\infty}\|(1-\alpha_n)(Tx_n - p) + \alpha_n (Ty_n - p)\|,$$

by using Lemma 2, we have

$$\lim_{n\to\infty} \|Tx_n - Ty_n\| = 0.$$

Now

$$\|x_{n+1} - p\| = \|(1-\alpha_n) Tx_n + \alpha_n Ty_n - p\| \leq \|Tx_n - p\| + \alpha_n \|Tx_n - Ty_n\|$$

yields that

$$c \leq \lim_{n\to\infty} \inf \|Tx_n - p\| \tag{4.5}$$

so that (4.4) and (4.5) gives

$$\lim_{n\to\infty} \|Tx_n - p\| = c.$$

On the other hand, we have

$$\|Tx_n - p\| \leq \|Tx_n - Ty_n\| + \|Ty_n - p\| \leq \|Tx_n - Ty_n\| + \|y_n - p\|,$$

so we write

$$c \leq \lim_{n\to\infty} \inf \|y_n - p\|. \tag{4.6}$$

From (4.3) and (4.6) we get

$$\lim \|y_n - p\| = c.$$

Since T is nonexpansive, by using (4.1) we have

$$\lim_{n\to\infty} \sup \|Tz_n - p\| \leq c. \tag{4.7}$$

From the inequality (4.4) and (4.7), by using Lemma 2 we obtain

$$\lim_{n\to\infty} \|Tx_n - Tz_n\| = 0.$$

Since

$$\|Tx_n - p\| \leq \|Tx_n - Tz_n\| + \|Tz_n - p\| \leq \|Tx_n - Tz_n\| + \|z_n - p\|$$

we write

$$c \leq \lim_{n\to\infty} \inf \|z_n - p\|. \tag{4.8}$$

From (4.3) and (4.8), it follows that

$$\lim_{n\to\infty} \|z_n - p\| = c.$$

On the other hand, since

$$\begin{aligned} c &= \lim_{n\to\infty} \|z_n - p\| \\ &= \lim_{n\to\infty} \|(1-\gamma_n)\, x_n + \gamma_n T x_n - p\| \\ &= \lim_{n\to\infty} \|(1-\gamma_n)(x_n - p) + \gamma_n (T x_n - p)\|, \end{aligned}$$

by Lemma 2, we have

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

So proof is completed. $\square$

By using Lemma 1, Lemma 3, Lemma 4 and Lemma 5, we can write the following theorem.

Theorem 3. *Let X be a real uniformly convex Banach space which satisfies the Opial condition, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by S^* iteration process. Then $\{x_n\}$ converges weakly to a fixed point of T.*

5. APPLICATIONS

Let H be a Hilbert space, C a closed convex subset of H, $v > 0$ a constant, $P_C : H \to C$ a metric projection and $A : C \to H$ a v- inverse strongly monoton mapping. It is well known that $P_C(I - \gamma A)$ is nonexpansive mapping provided that $\gamma \in (0, 2v)$. So, we derive the following theorems from Theorem 2, and Theorem 3.

Theorem 4. *Let H be a Hilbert space, C a nonempty closed convex subset of H and $A : C \to H$ a κ-Lipschitzian and η-strongly monoton operator. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha < \alpha_n$, $\beta < \beta_n$, and $\gamma < \gamma_n$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta, \gamma > 0$. Then for $\mu \in (0, 2\eta/\kappa^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$,* *and defined by*

$$\begin{cases} x_{n+1} & = (1-\alpha_n) P_C (I-\mu A) x_n + \alpha_n P_C (I-\mu A) y_n \\ y_n & = (1-\beta_n) P_C (I-\mu A) x_n + \beta_n P_C (I-\mu A) z_n \\ z_n & = (1-\gamma_n) x_n + \gamma_n P_C (I-\mu A) x_n, \end{cases}$$

converges strongly to $p \in \Omega(C, A)$.

Theorem 5. *Let* H *be a Hilbert space,* C *a nonempty closed convex subset of* H *and* $A : C \to H$ *a* κ*-Lipschitzian and* η*-strongly monoton operator. Let* $\{\alpha_n\}, \{\beta_n\}$, *and* $\{\gamma_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha \leq \alpha_n$, $\beta \leq \beta_n$, *and* $\gamma \leq \gamma_n$ *for all* $n \in \mathbb{N}$ *and for some* $\alpha, \beta, \gamma > 0$. *Then for* $\mu \in (0, 2\eta/\kappa^2)$, *the iterative sequence* $\{x_n\}$ *generated from* $x_1 \in C$ *for all* $n \in \mathbb{N}$, *and defined by*

$$x_{n+1} = (1-\alpha_n) P_C (I-\mu A) x_n + \alpha_n P_C (I-\mu A) [(1-\beta_n) P_C (I-\mu A) x_n \\ + \beta_n P_C (I-\mu A) [(1-\gamma_n) x_n + \gamma_n P_C (I-\mu A) x_n]]$$

converges strongly to $p \in \Omega(C, A)$.

Theorem 6. *Let* H *be a Hilbert space,* C *a closed convex subset of* H, $v > 0$ *a constant,* $P_C : H \to C$ *a metric projection and* $A : C \to H$ *a* v*- inverse strongly monoton mapping. Suppose that* $\Omega(C, A) \neq \emptyset$ *and* $\gamma \in (0, 2v)$. *Let* $\{x_n\}$ *be a sequence in* C, *with a arbitrarily initial value* x_1, *generated by*

$$\begin{cases} x_{n+1} & = (1-\alpha_n) P_C (I-\gamma A) x_n + \alpha_n P_C (I-\gamma A) y_n \\ y_n & = (1-\beta_n) P_C (I-\gamma A) x_n + \beta_n P_C (I-\gamma A) z_n \\ z_n & = (1-\gamma_n) x_n + \gamma_n P_C (I-\gamma A) x_n \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha \leq \alpha_n$, $\beta \leq \beta_n$ *and* $\gamma \leq \gamma_n$ *for all* $n \in \mathbb{N}$ *and for some* α, β, $\gamma > 0$. *Then* $\{x_n\}$ *converges weakly to a solution of the variational inequality* $VI(C, A)$.

Algorithms for signal and image processing are often iterative constrained optimization procedures designed to minimize a convex differentiable function $f(x)$ over a closed convex set C in H. It is well known that every L-Lipschitzian operator is $2/L$-ism. Therefore, the following scheme converges to minimizer of f.

Corollary 1. *Let H be a Hilbert space and C a nonempty closed convex subset of H, and f a convex and differentiable function on an open set D containing the set C. Suppose that ∇f is a Lipschitz continuous operator on D, $\gamma \in (0, 2/L)$ and minimizers of f relative to the set C exists. For arbitrarily initial value $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} & = (1-\alpha_n) P_C (I - \gamma \nabla f) x_n + \alpha_n P_C (I - \gamma \nabla f) y_n \\ y_n & = (1-\beta_n) P_C (I - \gamma \nabla f) x_n + \beta_n P_C (I - \gamma \nabla f) z_n \\ z_n & = (1-\gamma_n) x_n + \gamma_n P_C (I - \gamma \nabla f) x_n \end{cases}$$

for all $n \in \mathbb{N}$ where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha \leq \alpha_n$, $\beta \leq \beta_n$ and $\gamma \leq \gamma_n$ for all $n \in \mathbb{N}$ and for some α, β, $\gamma > 0$. Then $\{x_n\}$ converges weakly to a minimizer of f.

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