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COUPLED FIXED POINT THEOREMS FOR MAPPINGS SATISFYING A CONTRACTIVE CONDITION OF RATIONAL TYPE ON A PARTIALLY ORDERED METRIC SPACE

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Abstract. Recently, in the paper [J. Harjani, B. Lopez and K. Sadarangani; A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Abstract and Applied Analysis, Volume (2010), Article ID 190701, 8pages], some fixed point theorems were established for mappings satisfying a rational type contractive condition in partially ordered metric space. In this paper, we obtain some corresponding coupled fixed point theorems in partially ordered metric spaces by employing a rational type contractive condition. Our results generalize and extend some recently announced results in the literature.

Keywords: Coupled fixed point theorems; partially ordered metric spaces; rational type contractive condition.

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1. Introduction

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The notion of coupled fixed points was introduced by Chang and Ma [3]. Since then, the concept has been of interest to many researchers in metrical fixed point theory.

Bhaskar and Lakshmikantham [2] established coupled fixed point theorems in a metric space endowed with partial order by employing the following contractivity condition: For a mapping $T\colon X \times X \to X$, there exists $k \in (0,1)$ such that

$$d(T(x,y),T(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)], \forall\ x,\ y,\ u,\ v \in X,\ x \geq u,\ y \leq v. \tag{1}$$

Harjani et al [5] estabished some fixed point theorem in partially ordered metric space setting by using a contractive condition of rational type. That is, for a mapping $T\colon X \to X$, there exist some $\alpha,\ \beta \in [0,1]$, with $\alpha+\beta<1$, such that

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \beta d(x,y), \tag{2}$$

$\forall\ x,\ y \in X,\ x \neq y$.

The results of Harjani et al [5] are extensions of those of Jaggi [6]. Motivated by the works of Jaggi [6] and Harjani et al [5], in the present paper, we shall prove corresponding coupled fixed point theorems in partially ordered metric space by employing some notions of Bhaskar and Lakshmikantham [2] as well as a rational type contractive condition. The result of [2] has also been generalized and extended by Lakshmikantham and Ciric [7]. In the recent times, several papers have been devoted to the study of the concepts of coupled fixed points in partially ordered metric space. We refer to the reference section for detail.

2. Preliminaries

We consider the following definitions:

Definition 1.1: Let (X,d) be a metric space. An element $(x,y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $T\colon X \times X \to X$ if $T(x,y)=x$ and $T(y,x)=y$.

For the definitions above, we refer to [2, 4, 7].

Definition 1.2 [2]**:** Let $(X, \preceq$ be a partially ordered set and $T\colon X \times X \to X$. We say that T has the *mixed monotone property* if $T(x,y)$ is monotone nondecreasing in x and monotone nonincreasing in y, that is, $\forall\ x,\ y \in X$,

$$\forall\ x_1,\ x_2 \in X,\ x_1 \preceq x_2\ \Rightarrow\ T(x_1,y) \preceq T(x_2,y)$$

and

$$\forall\ y_1,\ y_2 \in X,\ y_1 \preceq y_2\ \Rightarrow\ T(x,y_1) \succeq T(x,y_2).$$

3. Main results

Let $(X, \preceq$ be a partially ordered set and d be a metric on X such that (X,d) is a complete metric space. We also endow the product space $X \times X$ with the following partial order:

for $(x,y),\ (u,v) \in X \times X,\ (u,v) \preceq (x,y)\ \Longleftrightarrow\ x \succeq u,\ y \preceq v.$

Theorem 2.1: Let $(X, \preceq)$ be a partially ordered metric set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T\colon X \times X \to X$ be a continuous mapping which has the mixed monotone property such that, for some $\alpha,\ \beta \in [0,1),\ \forall\ x,\ y,\ u,\ v \in X,\ x \neq u$, we have

$$d(T(x,y),T(u,v)) \preceq \alpha\frac{d(x,T(x,y)).d(u,T(u,v))}{d(x,u)} + \beta d(x,u),\ \alpha+\beta<1. \tag{3}$$

Then, T has a coupled fixed point.

Proof. Choose $(x_0,y_0) \in X \times X$ and set $x_1 = T(x_0,y_0),\ y_1 = T(y_0,x_0)$, and in general, $x_{n+1} = T(x_n,y_n),\ y_{n+1} = T(y_n,x_n)$.

With $x_0 \preceq T(x_0,y_0) = x_1$ (say) and $y_0 \succeq T(y_0,x_0) = y_1$ (say). By the iterative process above, $x_2 = T(x_1,y_1)$ and $y_2 = T(y_1,x_1)$. Therefore,

$$T^2(x_0,y_0) = T(T(x_0,y_0),T(y_0,x_0)) = T(x_1,y_1) = x_2,$$

and

$$T^2(y_0,x_0) = T(T(y_0,x_0),T(x_0,y_0)) = T(y_1,x_1) = y_2.$$

Due to the mixed monotone property of T, we obtain

$$x_2 = T^2(x_0, y_0) = T(x_1, y_1) \succeq T(x_0, y_0) = x_1,\ y_2 = T^2(y_0, x_0) = T(y_1, x_1) \preceq T(y_0, x_0) = y_1.$$

In general, we have that for $n \in \mathbb{N}$,

$$x_{n+1} = T^{n+1}(x_0, y_0) = T(T^n(x_0, y_0), T^n(y_0, x_0)),\ y_{n+1} = T^{n+1}(y_0, x_0) = T(T^n(y_0, x_0), T^n(x_0, y_0)).$$

It is obvious (as in [2]) that

$$x_0 \preceq T(x_0, y_0) = x_1 \preceq T^2(x_0, y_0) = x_2 \preceq \cdots \preceq T^n(x_0, y_0) = x_n \preceq \cdots,$$

and

$$y_0 \succeq T(y_0, x_0) = y_1 \succeq T^2(y_0, x_0) = y_2 \succeq \cdots \succeq T^n(y_0, x_0) = y_n \succeq \cdots.$$

Therefore, we have by condition (3) that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n, y_n), T(x_{n-1}, y_{n-1})) \\ &\preceq \alpha \tfrac{d(x_n, T(x_n, y_n)).d(x_{n-1}, T(x_{n-1}, y_{n-1}))}{d(x_n, x_{n-1})} + \beta d(x_n, x_{n-1}) \\ &= \alpha \tfrac{d(x_n, x_{n+1}).d(x_{n-1}, x_n)}{d(x_n, x_{n-1})} + \beta d(x_n, x_{n-1}) \\ &= \alpha d(x_n, x_{n+1}) + \beta d(x_n, x_{n-1}), \end{aligned}$$

from which it follows that

$$d(x_n, x_{n+1}) \preceq \left(\frac{\beta}{1-\alpha}\right) d(x_n, x_{n-1}). \tag{4}$$

Similarly, we have by (3) again that

$$\begin{aligned} d(y_{n+1}, y_n) &= d(T(y_n, x_n), T(y_{n-1}, x_{n-1})) \\ &\preceq \alpha \tfrac{d(y_n, T(y_n, x_n)).d(y_{n-1}, T(y_{n-1}, x_{n-1}))}{d(y_n, y_{n-1})} + \beta d(y_n, y_{n-1}) \\ &= \alpha d(y_n, y_{n+1}) + \beta d(y_n, y_{n-1}), \end{aligned}$$

which yields

$$d(y_n, y_{n+1}) \preceq \left(\frac{\beta}{1-\alpha}\right) d(y_n, y_{n-1}). \tag{5}$$

We have from (4) and (5) that

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \preceq \left(\frac{\beta}{1-\alpha}\right) \left[d(x_n, x_{n-1}) + d(y_n, y_{n-1})\right]. \tag{6}$$

Let $\delta_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$ and $\lambda = \frac{\beta}{1-\alpha}$. Then, we have from (6) that

$$\delta_n \preceq \lambda\delta_{n-1} \preceq \lambda^2\delta_{n-2} \preceq \cdots \preceq \lambda^n\delta_0. \tag{7}$$

If $\delta_0 = 0$, then (x_0, y_0) is a coupled fixed point of T.

Suppose that $\delta_0 > 0$. Then, for each $r \in \mathbb{N}$, we obtain by (7) and the repeated application of triangle inequality that

$$\begin{aligned} d(x_n, x_{n+r}) + d(y_n, y_{n+r}) &\preceq [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+r-1}, x_{n+r})] \\ &\quad +[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+r-1}, y_{n+r})] \\ &= [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] \\ &\quad + \cdots + [d(x_{n+r-1}, x_{n+r}) + d(y_{n+r-1}, y_{n+r})] \\ &\preceq \delta_n + \delta_{n+1} + \cdots + \delta_{n+r-1} \\ &\preceq \tfrac{\lambda^n(1-\lambda^r)\delta_0}{1-\lambda} \rightarrow \ 0 \text{ as n} \rightarrow \infty. \ (8) \end{aligned}$$

Therefore, $\{x_n\},\ \{y_n\}$ are Cauchy sequences in (X, d).

Since (X, d) is a complete metric space, there exist $x^*,\ y^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$ and $\lim\limits_{n\to\infty} y_n = y^*$. We now show that (x^*, y^*) is a coupled fixed point of T :

Let $\epsilon > 0$. Continuity of T at (x^*, y^*) implies that, for a given $\frac{\epsilon}{2} > 0$, there exists a $\delta > 0$, such that $d(x^*, u) + d(y^*, v) < \delta$ implies $d(T(x^*, y^*), T(u, v)) < \frac{\epsilon}{2}$.

Since $\{x_n\} \rightarrow\ x$ and $\{y_n\} \rightarrow\ y$, for $\zeta = \min(\frac{\epsilon}{2}, \frac{\delta}{2}) > 0$, there exist $n_0,\ m_0$, such that, for $n \geq n_0,\ m \geq m_0$, we have $d(x_n, x^*) \prec \zeta$, and $d(x_m, x^*) \prec \zeta$.

Therefore, for $n \in \mathbb{N},\ n \geq \max\{\mathrm{n_0}, \mathrm{m_0}\}$,

$$\begin{aligned} d(T(x^*, y^*), x^*) &\preceq d(T(x^*, y^*), x_{n+1}) + d(x_{n+1}, x^*) \\ &= d(T(x^*, y^*), T(x_n, y_n)) + d(x_{n+1}, x^*) \prec \tfrac{\epsilon}{2} + \zeta \preceq \epsilon, \end{aligned}$$

from which it follows that $T(x^*, y^*) = x^*$. In a similar manner, we can show that $T(y^*, x^*) = y^*$.

Hence, (x^*, y^*) is a coupled fixed point of T.

This complete the proof.

We state the next result without proof.

Theorem 2.2: Let the hypotheses of Theorem 2.1 hold. In addition, suppose that there exists $z \in X$ which is comparable to x and y, $\forall\ x,\ y \in X$. Then, T has a unique coupled fixed point.

Suppose that there exist $(x^*, y^*), (x', y') \in X \times X$ are coupled fixed points of T.

Case(I): If x^*, x' are comparable and y^*, y' are also comparable, and $x^* \neq x', y^* \neq y'$, then by the contractive condition, we have

$$\begin{aligned} d(x^*, x') &= d(T(x^*, y^*), T(x', y')) \\ &\preceq \alpha \tfrac{d(x^*, T(x^*, y^*) \cdot d(x', T(x', y')))}{d(x^*, x')} + \beta d(x^*, x') \\ &= \alpha \tfrac{d(x^*, x^*).d(x', x')}{d(x^*, x')} + \beta d(x^*, x') \\ &= \beta d(x^*, x'), \end{aligned}$$

which gives $d(x^*, x') \preceq 0,\ \beta < 1$ (a contradiction). Thus, $x^* = x'$.

Also, $d(y^*, y') = d(T(y^*, x^*), T(y', x')) \preceq \frac{\alpha \cdot d(y^*, T(y^*, x^*)) \cdot d(y', T(y', x'))}{d(y^*, y')} + \beta d(y^*, y')$,

from which it follows(as above) that $d(y^*, y') \preceq 0$ (a contradiction).

Hence, $y^* = y'$. Therefore, (x^*, y^*) is a unique coupled fixed point of T.

Case II: If x^* is not comparable to x' and y^* is not comparable to y', then by the contractive condition, there exists w comparable to x^* and x', and there exists v comparable to y^* and y'.

Monotonicity implies that w_n is comparable to $x_n^* = T(x_{n-1}^*, y_{n-1}^*) = x^*$, and w_n is comparable to w_1. Also, monotonicity implies that y_n^* is comparable to v and y_n^* is also comparable to w_2.

On the other hand, if $x_n^* \neq w_1, x_n' \neq w_1$, then by the contractive condition, we get

$$d(w_1, x_n^*) = d(T(w_1, w_2), T(x_{n-1}^*, y_{n-1}^*))$$

Case III: If (x^*, y^*) is not comparable to (x', y'), then there exists (w, v) comparable to (x^*, y^*) and (x', y'). Monotonicity implies that $(T^n(w, v), T^n(v, w))$

$$d\left(\left(\begin{array}{c} x^* \\ y^* \end{array}\right), \left(\begin{array}{c} x' \\ y' \end{array}\right)\right) = d\left(\left(\begin{array}{c} T^n(x^*, y^*) \\ T^n(y^*, x^* \end{array}\right), \left(\begin{array}{c} T^n(x', y') \\ T^n(y', x') \end{array}\right)\right)$$

$$\preceq d\left(\left(\begin{array}{c} T^n(x^*,y^*) \\ T^n(y^*,x^* \end{array}\right),\left(\begin{array}{c} T^n(w,v) \\ T^n(v,w) \end{array}\right)\right) + d\left(\left(\begin{array}{c} T^n(w,v) \\ T^n(v,w \end{array}\right),\left(\begin{array}{c} T^n(x',y') \\ T^n(y',x') \end{array}\right)\right)$$

$$\preceq d(T^n(x^*,y^*),T^n(w,v)) + d(T^n(y^*,x^*),T^n(v,w)) + d(T^n(w,v),T^n(x',y'))$$

$$+ d(T^n(v,w),T^n(y',x'))$$

$$\preceq \tfrac{\alpha^n\cdot d(x^*,T^n(x^*,y^*))\cdot d(w,T^n(w,v))}{d(x^*,w)} + \beta^n\cdot d(x^*,w) + \tfrac{\alpha^n\cdot d(y^*,T^n(y^*,x^*))\cdot d(v,T^n(v,w))}{d(y^*,w)} + \beta^n\cdot d(y^*,v)$$

$$+ \tfrac{\alpha^n\cdot d(x',T^n(x',y'))\cdot d(w,T^n(w,v))}{d(w,x')} + \beta^n\cdot d(w,x') + \tfrac{\alpha^n\cdot d(v,T^n(v,w))\cdot d(y',T^n(y',x'))}{d(v,y')} + \beta^n\cdot d(v,y')$$

$$= \beta^n\left[d(x^*,w) + d(y^*,v) + d(x',w) + d(y',v)\right] \to\ 0 \text{ as n} \to \infty.$$

Hence, T has a unique coupled fixed point.

Remark 2.1: Our results extend the corresponding results of Harjani et al [5] from fixed point setting to coupled fixed point sense and also a generalization of Theorem 2.1 of [2].

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