Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 1, 25-40
ISSN: 1927-6303

TOPOLOGICAL SEQUENCE ENTROPY OF CONTINUOUS MAPS ON TOPOLOGICAL SPACES

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Abstract. In this paper we propose a new definition of topological sequence entropy for continuous maps on arbitrary topological spaces (compactness, metrizability, even axioms of separation not necessarily required), investigate fundamental properties of the new sequence entropy, and compare the new sequence entropy with the existing ones. The defined sequence entropy generates that of Goodman. Yet, it holds various basic properties of Goodman's sequence entropy, e.g., the sequence entropy of a subsystem is bounded by that of the original system, topologically conjugated systems have a same sequence entropy, the sequence entropy of the induced hyperspace system is larger than or equal to that of the original system, and in particular this new sequence entropy coincides with Goodman's sequence entropy for compact systems.

Keywords: topological sequence entropy; locally compact space; Alexandroff compactification; hyperspace dynamical system.

2000 AMS Subject Classification: 11N32.

1. Introduction

The concepts of entropy are useful for studying topological and measure-theoretic structures of dynamical systems, i.e., topological entropy (see [1, 3, 4]) and measure-theoretic entropy

Received August 13, 2013

(see [8, 16]). For instance, two conjugate systems have a same entropy and thus entropy is a numerical invariant of the class of conjugated dynamical systems. The theory of expansive dynamical systems has been closely related to the theory of topological entropy [5, 15, 24]. Entropy and chaos are closely related, e.g., a continuous map of interval is chaotic if and only if it has a positive topological entropy [2].

In [18], Kushnirenko extended the definition of the entropy of a measure preserving transformation to give a new invariant: sequence entropy. Newton [23] and Goodman [11] further extended the concept of topological entropy of a continuous map and gave the concept of topological sequence entropy for compact dynamical systems. Franzová and Smítal [10] studied the relationship between Li-Yorke chaos [20] and topological sequence entropy for the continuous map of the interval and gave "the map is Li-Yorke chaos if and only if there exists an increasing sequence of nonnegative integers such that it has the positive topological sequence entropy". Hric [12, 13] discussed topological sequence entropy for maps of the interval and the circle. Cánovas [6] discussed topological sequence entropy of piecewise monotonic mappings and gave a full classification of piecewise monotonic maps from the point of view of the topological sequence entropy. Li [19] studied the characterisation of topologically weak mixing by using the topological sequence entropy.

This paper investigates a more general definition of topological sequence entropy for continuous maps defined on arbitrary topological spaces (compactness, metrizability, even axioms of separation not necessarily required), and explore the properties of such a topological sequence entropy. This definition generalizes that of Goodman's. Moreover, we have proved that the topological sequence entropy defined in this paper holds most properties of the topological sequence entropy under Goodman's definition, e.g., for compact systems, this new sequence entropy coincides with the sequence entropy defined by Goodman's, the defined sequence entropy (over arbitrary topological spaces) either retains the fundamental properties of sequence entropy (over compact or metric spaces) or has similar properties, the topological sequence entropy of a subsystem is bounded by that of the original system, topologically conjugated systems have a same topological sequence entropy, the topological sequence entropy of an autohomeomorphism from R onto itself is 0 if the sequence is an increasing sequence of nonnegative integers, and the sequence entropy of the induced hyperspace map is at least that of the original mapping.

2. The new definition of topological sequence entropy and its relations with other definitions

2.1. The new definition of topological sequence entropy

Let X be an arbitrary topological space and $f : X \to X$ be a continuous mapping. Then the pair (X, f) is said to be a topological dynamical system. If X is compact, (X, f) is called a compact dynamical system.

For compact topological dynamical systems, Goodman introduced the concept of topological sequence entropy and studied its properties [11]. Their definition is as follows:

Definition 2.1. Let X be a compact topological space and $f : X \to X$ be a continuous map. For any open cover α of X, denote by $N_X(\alpha)$ the smallest cardinality of all subcovers of α. Let $H_X(\alpha) = \log N_X(\alpha)$ and $T = (t_i : i = 1, 2, \cdots)$ be a sequence of nonnegative integers. Then $h_T(f, \alpha, X) = \lim_{n\to\infty} \sup \frac{1}{n} H_X(\bigvee_{i=0}^{n} f^{-t_i}(\alpha))$ is called the topological sequence entropy of f relative to the cover α with respect to T, and $h_T(f) = h_T(f, X) = \sup_{\alpha}\{h_T(f, \alpha, X)\}$, where the supremum is taken over α of X, is called the topological sequence entropy of f with respect to T.

Now, we begin our process to introduce our new definition of topological sequence entropy. Let (X, f) be an arbitrary topological dynamical system, i.e., X is an arbitrary topological space and f is a continuous map from X to itself. Let α be an open cover of X and F be a nonempty compact subset of X invariant under f, i.e., $f(F) \subseteq F$. Denote $N_F^*(\alpha)$ the smallest cardinality of all subcovers (for F) of α, i.e., $N_F^*(\alpha) = \min\{card(\beta) : \beta$ is a subset of α and β covers $F\}$. As F is compact, $N_F^*(\alpha)$ is a positive integer. Let $H_F^*(\alpha) = \log N_F^*(\alpha)$.

Let α and β be two open covers of X. Define their join by $\alpha \vee \beta = \{U \cap V : U \in \alpha, V \in \beta\}$. Clearly, the join $\alpha \vee \beta$ remains an open cover of X. If for every element V of β, there exists an element U of α satisfying $V \subseteq U$, then β is called a refinement of α, denoted by $\alpha \prec \beta$. Proofs for the four properties below are straightforward from the definition of H_F^*.

For convenience, we will take the following convention. If α is a cover of X, F is a subset of X, and a subcollection $\{U_s : s \in S\}$ of α forms a cover of F, then we will say that $\{U_s : s \in S\}$ is a subcover (for F) of α. Alternatively, we say that for F, $\{U_s \in \alpha : s \in S\}$ is a subcover of α. Recall that a compact Hausdorff space is of course locally compact; but a compact non-Hausdorff space need not be locally compact. When the Hausdorff property is not assumed, the image of a compact subset under a continuous map is not necessarily a closed subset.

Property 2.1: $H_F^*(\alpha) \geq 0$.

Property 2.2: If $\alpha \prec \beta$, then $H_F^*(\alpha) \leq H_F^*(\beta)$.

Property 2.3: $H_F^*(\alpha \vee \beta) \leq H_F^*(\alpha) + H_F^*(\beta)$.

Property 2.4: $H_F^*(f^{-1}(\alpha)) \leq H_F^*(\alpha)$. When $f(F) = F$, the equality holds.

Denote by $K(X,f)$ the set of all $f-$invariant nonempty compact subsets of X, i.e., $K(X,f) = \{F \subseteq X : F \neq \emptyset, F \text{ is compact and} f(F) \subseteq F\}$. If X is compact, it follows from $f(X) \subseteq X$ that $K(X,f) \neq \emptyset$. However, when X is noncompact, $K(X,f)$ could be empty. The translation $f : R \to R$ defined by $x \mapsto x+1$ is such an example. Another example is $f : (0,\infty) \to (0,\infty)$ where $f(x) = 2x$ and $(0,\infty)$ has the subspace topology of R.

Definition 2.2. Let (X,f) be a topological dynamical system and $T = (t_i : i = 1,2,\cdots)$ be a sequence of nonnegative integers. For $F \in K(X,f)$ and any cover α of X, $h_T^*(f,\alpha,F) = \limsup\limits_{n\to\infty} \frac{1}{n} H_F^*(\bigvee\limits_{i=0}^{n} f^{-t_i}(\alpha))$ is called the topological sequence entropy of f on F relative to α with respect to T. $h_T^*(f,F) = \sup\limits_{\alpha}\{h_T^*(f,\alpha,F)\}$, where the supremum is taken over α of X, is called the topological sequence entropy of f on F with respect to T.

Lemma 2.1. *Let (X,f) be a topological dynamical system and $T = (t_i : i = 1,2,\cdots)$ be a sequence of nonnegative integers. Let F be a nonempty compact subset of X invariant under f and α be any open cover of X. Then $h_T^*(f,\alpha,F) = h_T(f|_F, \alpha|_F, F)$, where $\alpha|_F = \{U \cap F : U \in \alpha\}$ and $f|_F : F \to F$ is the induced map of f, i.e., for any $x \in F$, $f|_F(x) = f(x)$.*

Proof. Since $h_T^*(f,\alpha,F) = \limsup\limits_{n\to\infty} \frac{1}{n} H_F^*(\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha))$, $h_T(f|_F, \alpha|_F, F) = \limsup\limits_{n\to\infty} \frac{1}{n} H_F(\bigvee\limits_{j=1}^{n}$ $(f|_F)^{-t_j}(\alpha|_F))$, by Definitions 2.1 and 2.2, it suffices to show $N_F^*(\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha)) = N_F(\bigvee\limits_{j=1}^{n} (f|_F)^{-t_j}$

$(\alpha|_F))$. Let $N_F(\bigvee_{j=1}^{n}(f|_F)^{-t_j}(\alpha|_F))=m$. Suppose that $\{U_1,U_2,\cdots,U_m\}$ is a finite subcover (for F) of $\bigvee_{j=1}^{n}(f|_F)^{-t_j}(\alpha|_F)$. Without loss of generality, assume $U_i=\bigcap_{j=1}^{n}(f|_F)^{-t_j}(U_{ij}\cap F), U_{ij}\in\alpha$, $(i=1,2,\cdots,m)$. Denote $V_i=\bigcap_{j=1}^{n}f^{-t_j}(U_{ij})$. Then $U_i\subseteq V_i$, $(i=1,2,\cdots,m)$. Hence, $\{V_1,V_2,\cdots,V_m\}$ is a finite subcover (for F) of $\bigvee_{j=1}^{n}f^{-t_j}(\alpha)$. Therefore, $N_F^*(\bigvee_{j=1}^{n}f^{-t_j}(\alpha))\leq N_F(\bigvee_{j=1}^{n}(f|_F)^{-t_j}(\alpha|_F))$.

In order to complete the proof. we now show the reversed inequality $N_F^*(\bigvee_{j=1}^{n}f^{-t_j}(\alpha))\geq N_F(\bigvee_{j=1}^{n}(f|_F)^{-t_j}(\alpha|_F))$. Let $N_F^*(\bigvee_{j=1}^{n}f^{-t_j}(\alpha))=m$. Suppose that $\{U_1,U_2,\cdots,U_m\}$ is a finite subcover (for F) of $\bigvee_{j=1}^{n}f^{-t_j}(\alpha)$. Denote $U_i=\bigcap_{j=1}^{n}f^{-t_j}(U_{ij}), U_{ij}\in\alpha$, $(i=1,2,\cdots,m)$. As $F\subseteq\bigcup_{i=1}^{m}U_i$, we have $\bigcup_{i=1}^{m}(U_i\cap F)=(\bigcup_{i=1}^{m}U_i)\cap F=F$. On the other hand, it follows from $f(F)\subseteq F$ that $F\subseteq f^{-1}(F)$ and consequently $f\subseteq f^{-j}(F)$, $(j=0,1,2,\cdots)$. Since

$$U_i\cap F=(\bigcap_{j=1}^{n}f^{-t_j}(U_{ij}))\cap F\subseteq\bigcap_{j=1}^{n}(f^{-t_j}(U_{ij})\cap f^{-t_j}(F))=\bigcap_{j=1}^{n}f^{-t_j}(U_{ij}\cap F),$$

we proved $U_i\cap F=\bigcap_{j=1}^{n}(f^{-t_j}(U_{ij}\cap F)\cap F)=\bigcap_{j=1}^{n}(f|_F)^{-t_j}(U_{ij}\cap F)$, which implies $F=\bigcup_{i=1}^{m}\bigcap_{j=1}^{n}(f|_F)^{-t_j}(U_{ij}\cap F)$. Hence, $N_F(\bigvee_{j=1}^{n}(f|_F)^{-t_j}(\alpha|_F))\leq N_F^*(\bigvee_{j=1}^{n}f^{-t_j}(\alpha))$.

Theorem 2.1. *Let (X,f) be a topological dynamical system and $T=(t_i:i=1,2,\cdots)$ be a sequence of nonnegative integers. Assume that X is Hausdorff. If $F\in K(X,f)$, then $h_T^*(f,F)=h_T(f|_F,F)$.*

Proof. Let α_F be an open cover of F, where α_F consists of open subsets of F. Since X is Hausdorff and F is a compact subset of X, F is a closed subset of X. For every $A\in\alpha_F$, there exists an open subset U_A of X satisfying $A=U_A\cap F$. Denote $\alpha'=\{U_A:A\in\alpha_F\}$. Clearly, $\alpha=\alpha'\cup(X\setminus F)$ is an open cover of X satisfying $\alpha|_F=\alpha_F\cup\{\emptyset\}$. Hence, every open cover α_F of F is a restriction $\alpha|_F$ of some special open cover α of X that includes open subset $X\setminus F$ of X. From Definition 2.1, we have

$$h_T(f,F)=\sup_{\alpha_F}h_T(f|_F,\alpha_F,F)=\sup_{\alpha}h_T(f|_F,\alpha|_F,F).$$

From Definition 2.2, we have $h_T^*(f,F)=\sup\limits_{\beta} h_T^*(f,\beta,F)$, where β runs over all open covers of X. It follows from Lemma 2.1 that $h_T^*(f,\alpha,F)=h_T(f|_F,\alpha|_F,F)$, thus

$$h_T(f|_F,F)=\sup_{\alpha_F} h_T(f|_F,\alpha|_F,F)=\sup_{\alpha} h_T^*(f,\alpha,F).$$

Recall that the open covers α are some special open covers of X. Hence, we have $\sup\limits_{\alpha} h_T(f,\alpha,F)$ $\leq h_T^*(f,F)$, implying $h_T(f|_F,F)\leq h_T^*(f,F)$.

Next, we show the reversed inequality $h_T(f|_F,F)\geq h_T^*(f,F)$. Let α be any open cover of X. From Lemma 2.1, we have $h_T^*(f,\alpha,F)=h_T(f|_F,\alpha|_F,F)$ and thus from Definition 2.2, $h_T^*(f,F)=\sup\limits_{\alpha} h_T^*(f,\alpha,F)=\sup\limits_{\alpha} h_T(f|_F,\alpha|_F,F)$. As $\alpha|_F$ are only some special open covers of F, we have $\sup\limits_{\alpha} h_T(f|_F,\alpha|_F,F)\leq h_T(f|_F,F)$, which implies $h_T^*(f,F)\leq h_T(f|_F,F)$.

Theorem 2.2. *Let (X,f) be a topological dynamical system and $T=(t_i:i=1,2,\cdots)$ be a sequence of nonnegative integers. For $F_1,F_2\in K(X,f)$ with $F_1\subseteq F_2$, an open cover α of X, the inequalities $h_T^*(f,\alpha,F_1)\leq h_T^*(f,\alpha,F_2)$ and $h_T^*(f,F_1)\leq h_T^*(f,F_2)$ hold.*

Proof. Let α be any open cover of X. Denote $N_{F_2}^*(\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha))=m$. Let $\{U_1,U_2,\cdots,U_m\}$ be such a subcover (for F_2) of $\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha)$ with the smallest cardinality m. As F_1 is a subset of F_2, $\{U_1,U_2,\cdots,U_m\}$ is also a subcover (for F_1) of $\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha)$ implying $N_{F_1}^*(\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha))\leq m$, i.e., $N_{F_1}^*(\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha))\leq N_{F_2}^*(\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha))$ which implies $H_{F_1}^*(\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha))\leq H_{F_2}^*(\bigvee\limits_{j=1}^{n} f^{-t_j}(\alpha))$. Hence,

$$h_T^*(f,\alpha,F_1)=\lim_{n\to\infty}\sup\frac{1}{n}H_{F_1}^*(\bigvee_{j=1}^{n} f^{-t_j}(\alpha))\leq\lim_{n\to\infty}\sup\frac{1}{n}H_{F_2}^*(\bigvee_{j=1}^{n} f^{-t_j}(\alpha))=h_T^*(f,\alpha,F_2),$$

i.e., $h_T^*(f,\alpha,F_1)\leq h_T^*(f,\alpha,F_2)$. The first inequality is thus proved. It follows from $h_T^*(f,F_1)=\sup\limits_{\alpha}\{h_T^*(f,\alpha,F_1)\}$ and $h_T^*(f,F_2)=\sup\limits_{\alpha}\{h_T^*(f,\alpha,F_2)\}$ that the second inequality $h_T^*(f,F_1)\leq h_T^*(f,F_2)$ holds.

Definition 2.3. Let (X,f) be a topological dynamical system and $T=(t_i:i=1,2,\cdots)$ be a sequence of nonnegative integers. When $K(X,f)\neq\emptyset$, define $h_T^*(f)=\sup\limits_{F\in K(X,f)}\{h_T^*(f,F)\}$. When $K(X,f)=\emptyset$, define $h_T^*(f)=0$. $h_T^*(f)$ is said to be the topological sequence entropy of f with respect to T.

2.2. The relations with other definitions

The next theorem indicates the concept of topological sequence entropy $h_T^*(f)$ with respect to T defined above generates that of Goodman, i.e., $h_T^*(f)$ coincides with $h_T(f)$ with respect to T when X is compact. Recall that $h_T(f)$ with respect to T is defined for compact dynamical systems only while in the preceding section $h_T^*(f)$ with respect to T is defined for arbitrary topological spaces.

Theorem 2.3. *If (X,f) is a compact topological dynamical system and $T=(t_i : i=1,2,\cdots)$ is a sequence of nonnegative integers, then $h_T^*(f)=h_T(f,X)$.*

Proof. Since X is compact and $f(X)\subseteq X$, we have $X\in K(X,f)$ implying $K(X,f)\neq\emptyset$. Thus from Definition 2.3, $h_T^*(f)=\sup\limits_{F\in K(X,f)}\{h_T^*(f,F)\}$. By Theorem 2.2, for any $F\in K(X,f)$, it holds $h_T^*(f,F)\leq h_T^*(f,X)$, i.e., the supremum is achieved when $F=X$. By Lemma 2.1, $h_T^*(f,\alpha,X)=h_T(f,\alpha,X)$, where α is any open cover of X. Recall the definitions of $h_T^*(f,X)$ and $h_T(f,X)$, i.e., $h_T^*(f,X)=\sup\limits_{\alpha}\{h_T^*(f,\alpha,X)\}$ and $h_T(f,X)=\sup\limits_{\alpha}\{h_T(f,\alpha,X)\}$. Hence, we have $h_T^*(f,X)=h_T(f,X)$. So, from the previous proved equality $h_T^*(f)=h_T^*(f,X)$ we conclude $h_T^*(f)=h_T(f,X)$.

In [21] Liu, Wang and Wei gave the definition topological entropy for mappings on general topological spaces. To compare the relationship between our definition and that given by [21], we recall their definition.

Definition 2.4. [21] Let (X,f) be a topological dynamical system. For $F\in K(X,f)$ and any open cover α of X, $ent^*(f,\alpha,F)=\lim\limits_{n\to\infty}\frac{1}{n}H_F(\bigvee\limits_{i=0}^{n-1}f^{-i}(\alpha))$ is called the topological entropy of f on F relative to α, $ent^*(f,F)=\sup\limits_{\alpha}\{ent^*(f,\alpha,F)\}$, where the supremum is taken over all open covers α of X. When $K(X,f)\neq\emptyset$, define $ent^*(f)=\sup\limits_{F\in K(X,f)}\{ent^*(f,F)\}$. When $K(X,f)=\emptyset$, define $ent^*(f)=0$. $ent^*(f)$ is said to be the entropy of f.

Clearly, if $t_i=i-1$, then $h_T^*(f)=ent^*(f)$. Further, if (X,f) is a compact topological dynamical system and $t_i=i-1$, then by Theorem 2.3, $h_T^*(f)$ is the topological entropy of Adler, Konheim and McAndrew [1]. If (X,f) is a topological dynamical system where X is a metric space and $t_i = i-1$, then $h_T^*(f)$ is the topological entropy which given by Canovas and Rodriguez [7] from [21].

3. Fundamental properties and main results of the topological sequence entropy

Proposition 3.1. *Let X be any topological space and id be the identity map from X onto itself. Then for the dynamical system (X,id) and a nonnegative integers sequence $T=(t_i : i = 1,2,\cdots)$, we have $h_T^*(id)=0$.*

Proof. For any $F \in K(X,id)$ and any open cover α of X, $\bigvee_{j=1}^{n} (id)^{-t_j}(\alpha) \prec \alpha$ implies $H_F^*(\bigvee_{j=1}^{n} (id)^{-t_j}(\alpha)) \leq H_F^*(\alpha)$. Hence,

$$h_T^*(id,\alpha,F) = \lim_{n\to\infty}\sup \frac{1}{n} H_T^*(\bigvee_{j=1}^{n} (id)^{-t_j}(\alpha)) \leq \lim_{n\to\infty}\sup \frac{1}{n} H_T^*(\alpha) = 0,$$

and subsequently $h_T^*(id,\alpha,F)=0$. It follows from Definitions 2.2 and 2.3 that $h_T^*(id,F) = \sup_{\alpha} h_T^*(id,\alpha,F) = 0$, which gives $h_T^*(id) = \sup_{F\in K(X,f)} \{h_T^*(id,F)\} = 0$.

Let (X,f) and (Y,g) be two topological dynamical system. For the product space $X\times Y$, define a map $f\times g : X\times Y \to X\times Y$ by $(f\times g)(x,y) = (f(x),g(y))$. This map $f\times g$ is continuous and $(X\times Y, f\times g)$ forms a topological dynamical dynamical system. If α and β are open covers of X and Y, respectively, then $\alpha\times\beta$ is an open cover of $X\times Y$.

Lemma 3.1. [21] *Let (X,f) and (Y,g) be two topological dynamical system. Let $P_x : X\times Y \to X$ and $P_y : X\times Y \to Y$ be the projections on X and Y, respectively. If $F\in K(X\times Y, f\times g)$, then $P_x(F)\in K(X,f)$, $P_y(F)\in K(Y,g)$ and $F\subseteq P_x(F)\times P_y(F)$.*

Lemma 3.2. [21] *Let (X,f) and (Y,g) be two compact topological dynamical system. If γ is an open cover of $X\times Y$, then there exists an open cover α of X and an open cover β of Y satisfying $\gamma \prec \alpha\times\beta$, i.e., $\alpha\times\beta$ refines γ.*

Proposition 3.2. *Let (X,f) and (Y,g) be two topological dynamical systems, where X and Y are hausdorff, and $T=(t_i : i=1,2,\cdots)$ be a sequence of nonnegative integers. Then $h_T^*(f\times g) \leq h_T^*(f) + h_T^*(g)$.*

Proof. If $K(X\times Y, f\times g)=\emptyset$, by Definition 2.3 we have $h_T^*(f\times g)=0$, thus $h_T^*(f\times g)\le h_T^*(f)+h_T^*(g)$.

Now, consider $K(X\times Y, f\times g)\neq\emptyset$. Recall the projections $P_x : X\times Y\to X$ and $P_y : X\times Y\to Y$. For $F\in K(X\times Y, f\times g)$ and any open cover γ of $X\times Y$, by Lemma 3.1, $P_x(F)\in K(X,f)$, $P_y(F)\in K(Y,g)$ and $F\subseteq P_x(F)\times P_y(F)$. As X and Y are Hausdorff, $P_x(F)$ is a closed subset of X and $P_y(F)$ is a closed subset of Y. Denote $P_x(F)$ by F_x and $P_y(F)$ by F_y.

By Theorem 2.2, $h_T^*(f\times g,\gamma,F)\le h_T^*(f\times g,\gamma,F_x\times F_y)$. From Lemma 2.1, we have $h_T^*(f\times g,\gamma,F_x\times F_y)=h_T(f\times g|_{F_x\times F_y},\gamma|_{F_x\times F_y},F_x\times F_y)$. As $\gamma|_{F_x\times F_y}$ is an open cover of $F_x\times F_y$, by Lemma 3.2, there exist an open cover $\alpha'=\{U_i : i=1,\cdots,n\}$ of F_x (U_i's are open subsets of F_x) and an open cover $\beta'=\{V_j : j=1,\cdots,m\}$ of F_y (V_j's are open subsets of F_y) satisfying $\gamma|_{F_x\times F_y}\prec\alpha'\times\beta'$. For every U_i, there exists an open subset A_i of X satisfying $U_i=A_i\cap F_x$, $(i=1,\cdots,n)$. Denote $\alpha=\{A_1,\cdots,A_n,X\setminus F_x\}$. Then α is an open cover of X and $\alpha|_{F_x}=\alpha'\cup\{\emptyset\}$. Similarly, there exists an open cover β of Y satisfying $\beta|_{F_y}=\beta'\cup\{\emptyset\}$. Hence, from $\gamma|_{F_x\times F_y}\prec\alpha'\times\beta'$, we have $\gamma|_{F_x\times F_y}\prec(\alpha'\cup\{\emptyset\})\times(\beta'\cup\{\emptyset\})$. Therefore, $\gamma|_{F_x\times F_y}\prec\alpha|_{F_x}\times\beta|_{F_y}$. Moreover, we also have $\alpha\times\beta|_{F_x\times F_y}=\alpha|_{F_x}\times\beta|_{F_y}$. It follows from Goodman result [11] that

$$h_T(f\times g|_{F_x\times F_y},\gamma|_{F_x\times F_y},F_x\times F_y)\le h_T(f\times g|_{F_x\times F_y},\alpha|_{F_x}\times\beta|_{F_y},F_x\times F_y)$$

$$=h_T(f\times g|_{F_x\times F_y},\alpha\times\beta|_{F_x\times F_y},F_x\times F_y)\le h_T(f|_{F_x},\alpha|_{F_x},F_x)+h_T(g|_{F_y},\beta|_{F_y},F_y).$$

Moreover, from Lemma 2.1, $h_T^*(f\times g,\alpha\times\beta,F_x\times F_y)=h_T(f\times g|_{F_x\times F_y},\alpha\times\beta|_{F_x\times F_y},F_x\times F_y)$, $h_T^*(f,\alpha,F_x)=h_T(f|_{F_x},F_x)$ and $h_T^*(g,\beta,F_y)=h_T(g|_{F_y},\beta|_{F_y},F_y)$, thus

$$h_T^*(f\times g,\alpha\times\beta,F_x\times F_y)\le h_T^*(f,\alpha,F_x)+h_T^*(g,\beta,F_y).$$

By Theorem 2.2,

$$h_T^*(f\times g,\gamma,F)\le h_T^*(f\times g,\gamma,F_x\times F_y)=h_T(f\times g|_{F_x\times F_y},\alpha\times\beta|_{F_x\times F_y})$$

$$\le h_T(f\times g|_{F_x\times F_y},\alpha|_{F_x}\times\beta|_{F_y},F_x\times F_y).$$

Hence, $h_T^*(f\times g,\gamma,F)\le h_T^*(f)+h_T^*(g)$. Finally from Definitions 2.2 and 2.3, we have $h_T^*(f\times g)=\sup\limits_{F\in K(X\times Y,f\times g)}\sup\limits_{\gamma}h_T^*(f\times g,\gamma,F)$, thus $h_T^*(f\times g)\le h_T^*(f)+h_T^*(g)$.

Lemma 3.3. (Goodman [11]) *Let (X,f) be a compact dynamical system where X is Hausdorff and $T=(t_i : i=1,2,\cdots)$ be a sequence of nonnegative integers. Then $h_T(f\times f)=2h_T(f)$.*

Proposition 3.3. Let (X,f) be a topological dynamical system where X is Haudorff and $T=(t_i : i=1,2,\cdots)$ be a sequence of nonnegative integers. Then $h_T^*(f\times f)=2h_T^*(f)$.

Proof. By Proposition 3.2, we have $h_T^*(f\times f)\le 2h_T^*(f)$.

Next, we prove $h_T^*(f\times f)\ge 2h_T^*(f)$. If $K(X,f)=\emptyset$, then $h_T^*(f)=0$, thus $h_T^*(f\times f)\ge 2h_T^*(f)$.

Now consider $K(X,f)\neq\emptyset$. For any $F\in K(X,f)$, by Lemma 3.3, we have $h_T(f\times f|_{F\times F},F\times F)=2h_T(f|_F,F)$. By Theorem 2.1, $h_T(f\times f|_{F\times F},F\times F)=h_T^*(f\times f,F\times F)$ and $h_T(f|_F,F)=h_T^*(f,F)$, thus we have $h_T^*(f\times f,F\times F)=2h_T^*(f,F)$. Further, $2h_T^*(f,F)\le h_T^*(f\times f)$. Hence, we have $2h_T^*(f)=2\sup\limits_{F\in K(X,f)}h_T^*(f,F)\le h_T^*(f\times f)$. Therefore, $h_T^*(f\times f)\ge 2h_T^*(f)$.

Definition 3.1. Let (X,f) be a topological dynamical system. If $\Lambda\subseteq X$ and $f(\Lambda)\subseteq\Lambda$, then $(\Lambda,f|_\Lambda)$ is said to be a topological subsystem of (X,f), or simply a subsystem of (X,f).

Remark 3.1. In above definition, Λ is not necessarily compact or closed. In the literature of dynamics, many authors assume subsystems to be compact or closed.

Theorem 3.1. *Let $(\Lambda,f|_\Lambda)$ be a subsystem of (X,f), where X is Hausdorff and $T=(t_i : i=1,2,\cdots)$ be a sequence of nonnegative integers. Then $h_T^*(f|_\Lambda)\le h_T^*(f)$.*

Proof. If $K(\Lambda,f|_\Lambda)=\emptyset$, it follows from Definition 2.3 that $h_T^*(f|_\Lambda)=0$, thus $h_T^*(f|_\Lambda)\le h_T^*(f)$. If $K(\Lambda,f|_\Lambda)\neq\emptyset$, then $K(\Lambda,f|_\Lambda)\subseteq K(X,f)$. For any $F\in K(\Lambda,f|_\Lambda)$, as X is Hausdorff, by Theorem 2.1, we have $h_T^*(f|_\Lambda,F)=h_T(f|_F,F)$ and $h_T^*(f,F)=h_T(f|_F,F)$. Hence, $h_T^*(f|_\Lambda,F)=h_T^*(f,F)$, which implies $h_T^*(f|_\Lambda)=\sup\limits_{F\in K(\Lambda,f|_\Lambda)}h_T^*(f|_\Lambda,F)\le\sup\limits_{F\in K(X,f)}h_T^*(f,F)=h_T^*(f)$. Therefore, $h_T^*(f|_\Lambda)\le h_T^*(f)$.

Let (X,f) and (Y,g) be two topological dynamical systems. Then, (X,f) is an extension of (Y,g), or (Y,g) is a factor of (X,f) if there exists a surjective continuous map $\pi : X\to Y$ (called a factor map) such that $\pi\circ f(x)=g\circ\pi(x)$ for every $x\in X$. If further, π is a homeomorphism, then (X,f) and (Y,g) are said to be topologically conjugate and the homeomorphism π is called a conjugate map.

Lemma 3.4. ([11]) *Let (X,f) and (Y,g) be two compact topological dynamical systems, where X, Y are two metric spaces, and $T=(t_i : i=1,2,\cdots)$ be an increasing sequence of nonnegative integers. If (X,f) and (Y,g) are topologically conjugate, then $h_T(f)=h_T(g)$.*

Theorem 3.2. *Let (X,f) and (Y,g) be two topological dynamical systems, where X, Y are Hausdorff, and $T=(t_i : i=1,2,\cdots)$ be an increasing sequence of nonnegative integers. If (X,f) and (Y,g) are topologically conjugate, i.e., there exists a continuous map $\pi : X \to Y$ satisfying $\pi\circ f = g\circ\pi$, then $h_T^*(f)=h_T^*(g)$.*

Proof. Consider two cases.

Case 1 $K(X,f)=\emptyset$. We claim $K(Y,g)=\emptyset$. If not, assume $K(Y,g)\neq\emptyset$. Then there exists some $F\in K(Y,g)\neq\emptyset$ satisfying $g(F)\subseteq F$. As $\pi : X\to Y$ is a conjugate map, i.e., $\pi\circ f=g\circ\pi$, the inverse π^{-1} is a conjugate map from (Y,g) and (X,f), i.e., $\pi^{-1}\circ g = f\circ\pi^{-1}$. Note that $\pi^{-1}(F)$ is a nonempty compact subset of X and $f(\pi^{-1}(F))=\pi^{-1}(g(F))\subseteq\pi^{-1}(F)$. Hence, $\pi^{-1}(F)\in K(X,f)$, which contradicts $K(X,f)=\emptyset$. Therefore, $K(X,f)=\emptyset$ implies $K(Y,g)=\emptyset$. Similarly, we can prove that $K(Y,g)=\emptyset$ implies $K(X,f)=\emptyset$. So we have proved that $K(X,f)=\emptyset$ if and only if $K(Y,g)=\emptyset$, and thus by Definition 2.3, $h_T^*(f)=h_T^*(g)$.

Case 2 $K(X,f)\neq\emptyset$. We prove that $2^\pi : K(X,f)\to K(Y,g), 2^\pi(F)=\pi(F)$ for every $F\in K(X,f)$ is a one-to-one correspondence between $K(X,f)$ and $K(Y,g)$. Recall $\pi : X\to Y$ is a conjugate map, i.e., $\pi\circ f=g\circ\pi$. Since $2^\pi(F)=\pi(F)$ and $g(\pi(F))=\pi(f(F))\subseteq\pi(F)$, so we have $\pi(F)\in K(Y,g)$. Hence, 2^π is well definite. Further, for any $F_1,F_2\in K(X,f)$ and $F_1\neq F_2$, we have $2^\pi(F_1)=\pi(F_1)$, $2^\pi(F_2)=\pi(F_2)$ and $\pi(F_1)\neq\pi(F_2)$, thus $2^\pi(F_1)\neq 2^\pi(F_2)$. Moreover, for any $F\in K(Y,g)$, we have $\pi^{-1}(F)\in K(X,f)$ and $2^\pi(\pi^{-1}(F))=\pi(\pi^{-1}(F))=F$. Therefore, $2^\pi : K(X,f)\to K(Y,g)$ is bijective. We consider $F\in K(X,f)$, then $\pi : F\to\pi(F)$ is a conjugate map, i.e., $\pi\circ f|_F = g|_{\pi(F)}\circ\pi$. By Lemma 3.4, we have $h_T(f|_F,F)=h_T(g|_{\pi(F)},\pi(F))$. As X and Y are Hausdorff, so by Theorem 2.1, $h_T^*(f,F)=h_T(f|_F,F)$ and $h_T^*(g,\pi(F))=h_T(g|_{\pi(F)},\pi(F))$, further, $h_T^*(f,F)=h_T^*(g,\pi(F))$. Hence,

$$h_T^*(f)=\sup_{F\in K(X,f)} h_T^*(f,F)=\sup_{F\in K(X,f)} h_T^*(g,\pi(F)).$$

Since $2^{\pi} : K(X,f) \to K(Y,g)$ is a one-to-one correspondence, we have

$$\sup_{F \in K(X,f)} h_T^*(g, \pi(F)) = \sup_{F' \in K(Y,g)} h_T^*(g, F') = h_T^*(g).$$

Therefore, $h_T^*(f) = h_T^*(g)$.

4. Topological sequence entropies of locally compact spaces and induced hyperspaces

Let R denotes the one-dimensional Euclidean space and X denotes a (noncompact) locally compact metrizable space, if not indicated otherwise. From Kelley's result [14], the Alexandroff compactification (i.e., one-point compactification) $\omega X = X \cup \{\omega\}$ of X is also metrizable.

Definition 4.1. [21] Let $f : X \to X$ be a continuous map.

- **(1):** If there exists an $a \in X$ such that for every sequence x_n of points of X, $\lim_{n\to\infty} f(x_n) = a$ holds whenever x_n does not have any convergent subsequence in X, then f is said to be convergent to a at infinity.
- **(2):** If for every sequence x_n of points of X that does not have any convergent subsequence in X, $f(x_n)$ dose not have any convergent subsequence, then f is said to be convergent to infinity at the infinity.
- **(3):** If (1) or (2) holds, f is said to be convergent at the infinity.

Theorem 4.1. [25] *A continuous map $f : X \to X$ is convergent at the infinity if and only if f can be extended to a continuous map $\bar{f}$ on the Alexandroff compactification ωX.*

Theorem 4.2. *Let (X, f) be a dynamical system and $T = (t_i : i = 1, 2, \cdots)$ be a sequence of nonnegative integers. If f can be extended to a continuous map on the Alexandroff compactification ωX, i.e., f is convergent at the infinity and $\bar{f}(\omega) = a$ or $\bar{f}(\omega) = \omega$ (refer to Definition 4.1), then $h_T^*(f) \le h_T^*(\bar{f})$.*

Proof. By the assumption, $(\omega X, \bar{f})$ is topological dynamical system and (X, f) is a subsystem of $(\omega X, \bar{f})$ (by a clear embedding). Hence, from Theorem 3.1, $h_T^*(f) \le h_T^*(\bar{f})$.

Example 4.1. *Let $T = (t_i : i = 1,2,\cdots)$ be a sequence of nonnegative integers. Consider $f : R \to R$, $f(x) = 2x$, $x \in R$. Then $h_T^*(f) = 0$.*

From assumption, the only invariant compact subset of f is $\{0\}$, i.e., $K(R, f) = \{\{0\}\}$. Denote $F = \{0\}$. We prove $h_T^*(f) = 0$. In fact, for any open cover α of R, any subcover (for $\{0\}$) of α with the smallest cardinality contains a single elements of α. Hence, $N_{\{0\}}^*(\bigvee_{j=1}^{n})f^{-t_j}(\alpha) = 1$, further, $H_{\{0\}}^*(\bigvee_{j=1}^{n})f^{-t_j}(\alpha)) = 0$, which implies $h_T^*(f, \alpha, \{0\}) = \lim_{n\to\infty}\sup \frac{1}{n} H_{\{0\}}^*(\bigvee_{j=1}^{n})f^{-t_j}(\alpha)) = 0$. Therefore, by Definition 2.3, we have $h_T^*(f) = 0$.

If R is replaced by $(0,\infty)$ which is equipped with the subspace topology of R, $K((0,\infty), f) = \emptyset$. It follows from Definition 2.3 that $h_T^*(f) = 0$.

Theorem 4.3. *If $f : R \to R$ is an autohomeomorphism and $T = (t_i : i = 1,2,\cdots)$ be an increasing sequence of nonnegative integers, then $h_T^*(f) = 0$.*

Proof. Let x_n be a sequence of points of R that dose not have any convergent subsequence in R. As f is a homeomorphism, the sequence $f(x_n)$ dose not have any convergent subsequence in R neither. By Theorem 4.1, f can be extended to a continuous map $\bar{f} : \omega R \to \omega R$ and $\bar{f}(\omega) = \omega$. Clearly, $\bar{f}$ is also an autohomeomorphism . On the other hand, ωR is homeomorphic to the unit circle S^1. Let $\pi : \omega R \to S^1$ be such a homeomorphism. Define $g : S^1 \to S^1$ by $g = \pi \circ \bar{f} \circ \pi^{-1}$. Then, g is a homeomorphism and π gives the conjugace between $(\omega R, \bar{f})$ and (S^1, g). Hence, by Theorem 3.2, $h_T^*(\bar{f}) = h_T^*(g)$. If g is a homeomorphism then $h_T^*(g) = 0$ from Ref [17], $h_T(g) = 0$. Since S^1 is a compact space, by Theorem 2.3, we have $h_T^*(g) = h_T(g)$, i.e., $h_T^*(g) = 0$. Hence, $h_T^*(\bar{f}) = 0$. From Theorem 4.2, $h_T^*(f) \le h_T^*(\bar{f})$. Therefore, $h_T^*(f) = 0$.

We investigate the sequence entropy relation between a topological dynamical system and its induced hyperspace topological dynamical system. The hyperspace is employed with the Vietoris topology. Notice that if X is a noncompact metric space, the Vietoris topology nonmetrizable [22].

The Vietoris topology on 2^X, the family of all nonempty closed subsets of X, is generated by the base

$$\upsilon(U_1, U_2, \cdots, U_n) = \{F \in 2^X : F \subseteq \bigcup_{i=1}^{n} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i \le n\}$$

where $U_1, U_2, \cdots, U_n$ are open subsets of X [9].

Let (X, f) be a topological dynamical system, where $f : X \to X$ is a closed mapping. The hyperspace map $2^f : 2^X \to 2^X$ is induced by f as follows: for every $F \in 2^X$, $2^f(F) = f(F)$. When f is a closed and continuous map, 2^f is well defined and it is continuous [14, 22], thus ensuring that $(2^X, 2^f)$ forms a topological dynamical system, i.e., the induced hyperspace topological dynamical system of (X, f).

By Michael's results [22], we have the following facts.

Fact 1: If X is compact, then 2^X is compact.

Fact 2: If X is compact and Hausdorff, then 2^X is compact and Hausdorff.

Fact 3: $\pi : X \to 2^X$ defined by $\pi(x) = \{x\}$ for $x \in X$ is continuous. If X is compact and Hausdorff, then π is homeomorphic embedding and (X, f) and $(\pi(X), 2^f)$ are topologically conjugate.

Theorem 4.4. [21] *Let (X, f) be a topological dynamical system, where X is Hausdorff and f is a closed mapping. If $F \in K(X, f)$, then $2^F \in K(2^X, 2^f)$. Hence, $(2^F, 2^f)$ is a topological dynamical subsystem of $(2^X, 2^f)$.*

Theorem 4.5. *Let (X, f) be a topological dynamical system, where X is Hausdorff and f is a closed mapping and $T = (t_i : i = 1, 2, \cdots)$ be a sequence of nonnegative integers. Then the topological sequence entropy of $(2^X, 2^f)$ is at least that of (X, f), i.e., $h_T^*(2^f) \geq h_T^*(f)$.*

Proof. Case 1. $K(X, f) = \emptyset$. By Definition 2.3, we have $h_T^*(f) = 0$. Hence, $h_T^*(2^f) \geq h_T^*(f)$.

Case 2. $K(X, f) \neq \emptyset$. For $F \in K(X, f)$, it follows from Theorem 4.4 that $2^F \in K(2^X, 2^f)$. Define $\pi : F \to 2^F$ by $\pi(x) = \{x\}$, $x \in F$. From Fact 3 in the preceding paragraph of Theorem 4.4, (F, f) and $(\pi(F), 2^f)$ are topologically conjugate. From Goodman's result [11], $h_T(f|_F, F) = h_T(2^f|_{h(F)}, h(F))$. By Theorem 2.1, $h_T^*(f, F) = h_T(f|_F, F)$ and $h_T^*(2^f, \pi(F)) = h_T(2^f|_{\pi(F)}, \pi(F))$, which implies $h_T^*(f, F) = h_T^*(2^f, \pi(F))$. Again, by the Fact 3, $\pi(F)$ is a compact subset of 2^X. On the other hand, from $2^f(\pi(F)) = \pi(f(F))$ and $f(F) \subseteq F$, we have $2^f(\pi(F)) = \pi(f(F)) \subseteq \pi(F)$, thus $\pi(F) \in K(2^X, 2^f)$. Furthermore, it follows from Definition

2.3 that $h_T^*(2^f,\pi(F)) \le h_T^*(2^f)$ implying $h_T^*(f,F) \le h_T^*(2^f)$. Therefore,

$$h_T^*(f) = \sup_{F\in K(X,f)} \le h_T^*(2^f).$$

Conflict of Interests

The author declares that there is no conflict of interests.

Acknowledgements

The work was supported by the Natural Science Foundation of Henan Province (122300410420, 122300410148), PR China

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