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Advances in Fixed Point Theory, 3 (2013), No. 1, 1-8

ISSN: 1927-6303

A COMMON FIXED POINT THEOREM IN INTUITIONISTIC FUZZY METRIC SPACE USING OCCASIONALLY CONVERSE COMMUTING MAPS AND IMPLICIT RELATION

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Abstract. In this paper, we use the notion of occasionally converse commuting (occ) and occasionally weakly compatible mappings in intuitionistic fuzzy metric space. By using this concept, we prove two common fixed point results for a quadruple of self-mappings which satisfy an implicit relation. Our result generalizes the results of Pathak et al. [16] in intuitionistic fuzzy metric space.

Keywords: Intuitionistic Fuzzy metric space; occasionally converse commuting (occ); occasionally weakly compatible.

2000 AMS Subject Classification: 47H10; 54H25;

1. Introduction

Atanassov [2] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [18] and later there has been much progress in the study of intuitionistic fuzzy sets [8]. In 2004, Park [12] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [4]. Fixed point theory has important applications in diverse disciplines of mathematics, statistics, engineering, and economics in dealing with problems arising in: Approximation theory, potential theory, game theory, mathematical economics, etc. Lu [10] presented the concept of converse commuting mappings and proved some common fixed point results. Liu and Hu [9] used this concept for multi-valued mappings. Popa [17]

Received January 18, 2012

extended his result for the mappings satisfying an implicit relation. Recently, Pathak et al. [16] introduce the notion of occasionally converse commuting (occ) mappings and prove some fixed point theorems on four self maps by using this concept.

In this paper, we use the notion of occasionally converse commuting (occ) and occasionally weakly compatible mappings in intuitionistic fuzzy metric space. By using this concept, we prove two common fixed point results for a quadruple of self-mappings which satisfy an implicit relation. Our result generalizes the results of Pathak et al. [16] in intuitionistic fuzzy metric space.

2. Preliminaries

The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skelton that we use are characterization fuzzy intersections and union respectively. These concepts were originally introduced by Menger [11] in study of statistical metric spaces.

Definition 2.1[13]. A binary operation * : [0,1]×[0,1] → [0,1] is continuous t-norm if * satisfies the following conditions:

(i) * is commutative and associative;

(ii) * is continuous;

(iii) $a * 1 = a$ for all $a \in [0, 1]$;

(iv) $a * b \leq c * d$ whenever a$\leq$ c and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2[13]. A binary operation ⋄ : [0,1]×[0,1] → [0,1] is continuous t-conorm if ⋄ satisfies the following conditions:

(i) ⋄ is commutative and associative;

(ii) ⋄ is continuous;

(iii) $a \diamond 0 = a$ for all $a \in [0, 1]$;

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Alaca et al. [1] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [7] as :

Definition 2.3[1]. A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;

(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;

(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if x = y;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(xi) $N(x, y, t)\diamond N(y, z, s) = N(x, z, t + s)$ for all $x, y \in X$ and $s, t > 0$;

(xii) for all $x, y \in X$, N(x, y, .) : $[0, \infty) \to [0, 1]$ is right continuous;

(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$.

Then (M, N) is called an intuitionistic fuzzy metric space on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t respectively.

Remark 2.1.[1] Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that t-norm * and t-conorm $\diamond$ are associated as $x \diamond y = 1 - ((1 - x) * (1 - y))$ for all $x, y \in X$.

Remark 2.2.[1] In intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, $M(x, y, *)$ is non-decreasing and $N(x, y,\diamond)$ is non-increasing for all $x, y \in X$.

Alaca, Turkoglu and Yildiz [1] introduced the following notions:

Definition 2.4[1]. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$and $p > 0$,

$\lim_{n\to\infty}$ M(x_{n+p},x_n, t) = 1 and $\lim_{n\to\infty}$ N(x_{n+p},x_n, t) = 0,

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$\lim_{n\to\infty}$ M(x_n,x,t)= 1 and $\lim_{n\to\infty}$ N(x_n,x,t) = 0.

Definition 2.5[1]. An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Lu [10] presented the concept of converse commuting mappings

Definition 2.6[10]. A pair of self mappings (A, S) of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be conversely commuting if for all $x \in X$, the $ASx = SAx$ implies $Ax = Sx$.

In the literature, many results have been proved using conversely commuting maps in various abstract spaces. [14-16].

Pathak et al. [16] introduce the notion of occasionally converse commuting (occ) mappings

Definition 2.7[16]. A pair of self mappings (A, S) of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be occasionally conversely commuting (occ), if for some $x \in X$, the $ASx = SAx$ implies $Ax = Sx$.

Every conversely commuting mappings is occ but the reverse need not be true.(see[16]).

In 1996, Jungck [5] introduced the notion of weakly compatible maps as follows:

Definition 2.8[5]. A pair of self mappings (A, S) of a metric space is said to be weakly compatible if they commute at the coincidence points i.e. $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

The concept of weakly compatible mapping was generalised to occasionally weakly compatible.

Definition 2.9[6]. A pair of self mappings (A, S) of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be occasionally weakly compatible (owc) if they commute at coincidence points i.e. $ASx = SAx$ whenever $Ax = Sx$ for some $x \in X$.

Every weakly compatible mapping is owc but not conversely (see [6]).

3. Main results

Theorem 3.1. Let A, B, S and T be self mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ satisfying the following: (3.1) for any $x, y \in X$, and for all $t > 0$

$\triangle(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M(Sx, By, t), M(Ax, Ty, t)) \geq 0$

$\bigtriangledown(N(Ax, By, t), N(Sx, Ty, t), N(Sx, Ax, t), N(Ty, By, t), N(Sx, By, t), N(Ax, Ty, t)) \leq 0$

where $\triangle$, $\bigtriangledown$:$[0, 1]^6 \to [0, 1]$ is in the class of all continuous mappings satisfying

$\triangle(t, t, 1, 1, t, t) < 0$

and

$\nabla(t, t, 0, 0, t, t) > 0$ for all $t \in (0, 1)$

If one of the following conditions holds:

(3.2) the pair (A, S) is occ and the pair (B, T) is owc, or

(3.3) the pair (B, T) is occ and the pair (A, S) is owc.

Then A, B, S and T have a unique common fixed point.

Proof. Suppose condition (3.2) holds, i.e. the pair (A, S) is occ and the pair (B, T) is owc. Then, as (B, T) is owc, there exist some point $p \in X$ such that $BTp = TBp$ whenever $Bp = Tp = z$ (say) in X. So that for a given $p \in X$, $Bz = Tz$ whenever $Bp = Tp = z$. Next, since (A, S) is occasionally converse commuting (occ). Then, by definition, there exist some such that $ASu = SAu$ implies $Au = Su = w$ (say). So that for a given u, $Aw = Sw$ implies that $Au = Su = w$ (say) in X. We claim that $AAu = Bz$. If not, then putting $x = Au$ and $y = z$ in (3.1), and using $ASu = SAu = AAu$ and $Tz = Bz$, we obtain

$$\triangle(M(AAu, Bz, t), M(SAu, Tz, t), M(SAu, AAu, t), M(Tz, Bz, t), M(SAu, Bz, t), M(AAu, Tz, t)) \geq 0$$

$$\triangle(M(AAu, Bz, t), M(AAu, Bz, t), 1, 1, M(AAu, Bz, t), M(AAu, Bz, t)) \geq 0$$

and

$$\bigtriangledown(N(AAu, Bz, t), N(SAu, Tz, t), N(SAu, AAu, t), N(Tz, Bz, t), N(SAu, Bz, t), N(AAu, Tz, t)) \leq 0$$

$$\bigtriangledown(N(AAu, Bz, t), N(AAu, Bz, t), 0, 0, N(AAu, Bz, t), N(AAu, Bz, t)) \leq 0$$

a contradiction to definition of $\triangle$ and $\bigtriangledown$. Thus $AAu = Bz$. Therefore $Aw = Bz = Sw = Tz$. We claim $Au = Bz$. If not, then putting $x = u$ and $y = z$ in (3.1), we get

$$\triangle(M(Au, Bz, t), M(Su, Tz, t), M(Su, Au, t), M(Tz, Bz, t), M(Su, Bz, t), M(Au, Tz, t)) \geq 0$$

$$\triangle(M(Au, Bz, t), M(Au, Bz, t), 1, 1, M(Au, Bz, t), M(Au, Bz, t)) \geq 0$$

and

$$\bigtriangledown(N(Au, Bz, t), N(Su, Tz, t), N(Su, Au, t), N(Tz, Bz, t), N(Su, Bz, t), N(Au, Tz, t)) \leq 0$$

$$\bigtriangledown(N(Au, Bz, t), N(Au, Bz, t), 0, 0, N(Au, Bz, t), N(Au, Bz, t)) \leq 0$$

a contradiction to definition of $\triangle$ and $\bigtriangledown$. Thus $Au = Bz$. Therefore, $Au = Bz = Tz = Su = AAu = SAu$. It follows that Au is a common fixed point of A and S. Next, we claim that $Bz = z$. If not, take $x = u$ and $y = p$ in (3.1), we obtain

$$\triangle(M(Au, Bp, t), M(Su, Tp, t), M(Su, Au, t), M(Tp, Bp, t), M(Su, Bp, t), M(Au, Tp, t)) \geq 0$$

$$\triangle(M(Bz, z, t), M(Bz, z, t), 1, 1, M(Bz, z, t), M(Bz, z, t)) \geq 0$$

and

$\bigtriangledown(N(Au, Bp, t), N(Su, Tp, t), N(Su, Au, t), N(Tp, Bp, t), N(Su, Bp, t), N(Au, Tp, t)) \leq 0$

$\bigtriangledown(N(Bz, z, t), N(Bz, z, t), 0, 0, N(Bz, z, t), N(Bz, z, t)) \leq 0$

a contradiction to definition of $\triangle$ and $\bigtriangledown$. Thus $Bz = z$. Therefore, $Bz = z = Tz = Au = Su = AAu = SAu$. Hence z is a common fixed point of A, B, S and T. For uniqueness, let w be another common fixed point of A, B, S and T. We show that $w = z$, suppose not, then by (3.1) take $x = z, y = w$, we obtain

$\triangle(M(Az, Bw, t), M(Sz, Tw, t), M(Sz, Az, t), M(Tw, Bw, t), M(Sz, Bw, t), M(Az, Tw, t)) \geq 0$

$\triangle(M(z, w, t), M(z, w, t), 1, 1, M(z, w, t), M(z, w, t)) \geq 0$

and

$\bigtriangledown(N(Az, Bw, t), N(Sz, Tw, t), N(Sz, Az, t), N(Tw, Bw, t), N(Sz, Bw, t), N(Az, Tw, t)) \leq 0$

$\bigtriangledown(N(z, w, t), N(z, w, t), 0, 0, N(z, w, t), N(z, w, t)) \leq 0$

a contradiction to definition of $\triangle$ and $\bigtriangledown$. Thus A, B, S and T have a unique common fixed point. The proof is same if condition (3.3) holds.

Next, we prove the following result for both pairs occasionally converse commuting:

Theorem 3.2. Let A, B, S and T be self mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ satisfying the condition (3.1). If both the pairs (A, S) and (B, T) are occasionally converse commuting (occ), then A, B, S and T have a unique common fixed point in X.

Proof. As the pair (A, S) is occasionally converse commuting, by definition, there exist some such that $ASu = SAu$ implies $Au = Su$. It follows that $AAu = ASu = SAu$. Also, the occasionally converse commuting for the pair (B, T) implies that there exist such that $BTv = TBv$ implies $Bv = Tv$. Hence $BBv = BTv = TBv$. First, we show that $Au = Bv$. If not, then putting $x = u$ and $y = v$ in (3.1), we obtain

$\triangle(M(Au, Bv, t), M(Su, Tv, t), M(Su, Au, t), M(Tv, Bv, t), M(Su, Bv, t), M(Au, Tv, t)) \geq 0$

$\triangle(M(Au, Bv, t), M(Au, Bv, t), 1, 1, M(Au, Bv, t), M(Au, Bv, t)) \geq 0$

and

$\bigtriangledown(N(Au, Bv, t), N(Su, Tv, t), N(Su, Au, t), N(Tv, Bv, t), N(Su, Bv, t), N(Au, Tv, t)) \leq 0$

$\bigtriangledown(N(Au, Bv, t), N(Au, Bv, t), 0, 0, N(Au, Bv, t), N(Au, Bv, t)) \leq 0$

a contradiction to definition of $\triangle$ and $\bigtriangledown$. Thus, $Au = Bv$. Next, we show that $AAu = Au$. Suppose not, then, by putting $x = Au$ and $y = v$ in (3.1), we have

$\triangle(M(AAu, Bv, t), M(SAu, Tv, t), M(SAu, AAu, t), M(Tv, Bv, t), M(SAu, Bv, t), M(AAu, Tv, t)) \geq 0$

$\triangle(M(AAu, Au, t), M(AAu, Au, t), 1, 1, M(AAu, Au, t), M(AAu, Au, t)) \geq 0$

and

$\bigtriangledown(N(AAu, Bv, t), N(SAu, Tv, t), N(SAu, AAu, t), N(Tv, Bv, t), N(SAu, Bv, t), N(AAu, Tv, t)) \leq 0$

$\bigtriangledown(N(AAu, Au, t), N(AAu, Au, t), 0, 0, N(AAu, Au, t), N(AAu, Au, t)) \leq 0$

a contradiction to definition of $\triangle$ and $\bigtriangledown$. Thus $Au = AAu$. Similarly, $Bv = BBv$. Since $Au = Bv$, we have $Au = Bv = AAu = ASu = SAu = BBv = BTv = TBv$. Therefore $Au = z$ (say), is a common fixed point of A, B, S and T.

For uniqueness, let w be another common fixed point of A, B, S and T. We show that $w = z$, suppose not, then by (3.1) take $x = z, y = w$, we obtain

$\triangle(M(Az, Bw, t), M(Sz, Tw, t), M(Sz, Az, t), M(Tw, Bw, t), M(Sz, Bw, t), M(Az, Tw, t)) \geq 0$

$\triangle(M(z, w, t), M(z, w, t), 1, 1, M(z, w, t), M(z, w, t)) \geq 0$

and

$\bigtriangledown(N(Az, Bw, t), N(Sz, Tw, t), N(Sz, Az, t), N(Tw, Bw, t), N(Sz, Bw, t), N(Az, Tw, t)) \leq 0$

$\bigtriangledown(N(z, w, t), N(z, w, t), 0, 0, N(z, w, t), N(z, w, t)) \leq 0$

a contradiction to definition of $\triangle$ and $\bigtriangledown$. Therefore $z = w = Au$. Hence, Au is a unique common fixed point of A, B, S and T. This completes the proof.

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