Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 2, 280-309
ISSN: 1927-6303

A FIXED POINT APPROACH TO THE STABILITY OF AN ADDITIVE CUBIC FUNCTIONAL EQUATION IN PARANORMED SPACES

K. RAVI[1,*], J.M. RASSIAS[2], R. JAMUNA[3]

[1]Department of Mathematics, Sacred Heart College, Tirupattur - 635601, Tamilnadu, India
[2]Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrain University of Athens, 4, Agamemnonos str, Aghia Paraskevi, Athens 15342, Greece
[3]Department of Mathematics, R.M.K. College of Engineering and Technology, R.S.M. Nagar, Puduvoyal, Gummidipoondi Taluk, Tiruvallur Dist., Tamilnadu, India - 601206

Abstract. In this paper, using fixed point methods, we prove the generalized Hyers-Ulam stability of the additive cubic functional equation

$$f(x+ky)+f(x-ky)=k^2[f(x+y)+f(x-y)]+2(1-k^2)f(x)$$

for fixed integers k with $k \neq 0, \pm 1$ in paranormed spaces.

Keywords: paranormed space; generalized Hyers-Ulam stability; fixed point; additive cubic functional equation.

2010 AMS Subject Classification: 39B52, 39B82.

1. Introduction-preliminaries

*Corresponding author
E-mail address: schkravi@yahoo.co.in (K. Ravi)
Received December 17, 2013

A basic question in the theory of functional equations araised as follows: When is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation? If the problem accepts a unique solution, we say the equation is stable. The first stability problem concerning group homomorphisms is related to a question of Ulam [30] in 1940.

"Let G be a group and G' be a metric group with metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists homomorphism $H : G \to G'$ with $d(f(x), H(x)) < \varepsilon$ for all $x \in G$?"

In 1941, Hyers [11] gave the first affirmative partial answer to the question of Ulam for Banach spaces. To be more precise, he proved the following celebrated theorem.

Theorem 1.1. [11] *Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \tag{1}$$

for all $x, y \in X$. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{2}$$

exists for all $x \in X$ and $a : X \to Y$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \le \varepsilon \tag{3}$$

for all $x \in X$.

Aoki [2] generalized Hyers theorem for additive mappings. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Rassias [24]. He proved the following:

Theorem 1.2. [24] *Let X be a normed vector space and Y be a Banach space. If a function $f : X \to Y$ satisfies the inequality.*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p) \text{ for all } x, y \in X, \tag{4}$$

where θ and p are constants and $\theta > 0$ and $0 \le p < 1$, then the limit

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{5}$$

exists for all $x \in X$ and $T : X \to Y$ is the unique additive mapping which satisfies

$$\|f(x) - T(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p \tag{6}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each $x \in X$, then T is linear.

The theorem of Rassias was later extended to all $p \neq 1$ and generalized by many mathematicians; see, e.g., [9, 10, 12, 17]. This concept is known as the Hyers-Ulam-Rassias stability.

In 1982, Rassias [25] provided a stability result from the innovative approach of Rassias [24] for the unbounded Cauchy difference, in which he replaced the factor $(\|x\|^p + \|y\|^p)$ by $(\|x\|^p\|y\|^q)$ for $p, q \in R$ with $p+q \neq 1$.

In 1994, a generalization of Rassias' theorem was obtained by Gavruta [9], who replaced $\theta(\|x\|^p + \|y\|^p)$ and $\theta(\|x\|^p\|y\|^p)$ by a general control function $\phi(x,y)$. Recently Rassias [27] generalized the Hyers stability by replacing the bound $\theta(||x||^p||y||^q)$ in [9], by a mixed one involving the product and sum of powers of norms, that is, $\theta\{||x||^p||y||^p + (||x||^{2p} + ||y||^{2p})\}$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{7}$$

is said to be a quadratic functional equation because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (7). In fact every solution of the quadratic functional equation is said to be a quadratic mapping. A quadratic functional equation was used to characterize inner product spaces; see [1, 14].

In [13], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{8}$$

It is easy to show $f(x) = x^3$ satisfies the function equation (8) and therefore every solution of the cubic functional equation is said to be a cubic mapping.

In [18], Lee *et al.* considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{9}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (9), and every solution of the quartic functional equation is said to be a quartic mapping.

The functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y) \\ &\quad -4f(y)-4f(-y) \end{aligned} \tag{10}$$

is called additive-quadratic-cubic-quartic functional equation (briefly, AQCQ - functional equation). The generalized Hyers-Ulam stability of the AQCQ functional equation was proved by Park in Non-Archimedean spaces [21] and Paranormed spaces [22].

In [26], Ravi *et al.* investigated the AQCQ - functional equation

$$\begin{aligned} f(x+ky)+f(x-ky) &= k^2f(x+y)+k^2f(x-y)+2(1-k^2)f(x) \\ &\quad +\frac{(k^4-k^2)}{12}[f(2y)+f(-2y)-4f(y)-4f(-y)], \end{aligned} \tag{11}$$

which is a generalized form of AQCQ - functional equation (10) and obtained its general solution and generalized Hyers-Ulam stability for a fixed integer k with $k \neq 0, \pm 1$ in Banach spaces.

In this paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of the additive cubic functional equation

$$f(x+ky)+f(x-ky) = k^2[f(x+y)+f(x-y)]+2(1-k^2)f(x), \tag{12}$$

which is obtained from (11) when $f(2y)+f(-2y)-4f(y)-4f(-y)=0$, that is, when f is odd, in Paranormed spaces. Before giving the main results, we present here some basic facts concerning Paranormed spaces and some preliminary results.

The concept of statistical convergence for sequence of real numbers was introduced by Fast [7] and Steinhaus [29] independently, and since then several generalizations and applications of this notion have been investigated by various authors (see [8, 15, 28]). This notion was defined in normed spaces by Kolk [16].

Definition 1.2. [31] Let X be a vector space. A paranorm $P : X \to [0,\infty)$ is a function such that

(1) $P(0)=0$;

(2) $P(-x)=P(x)$;

(3) $P(x+y) \leq P(x)+P(y)$ (triangle inequality);

(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_0 - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a paranormed space if P is a paranorm on X.

The Paranorm is called total if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A Fréchet space is a total and complete - Paranormed space.

Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.4. ([3, 5]) *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{13}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, for all $n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \left(\frac{1}{(1-L)}\right) d(y, Jy)$ for all $y \in Y$.

By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 20, 23]*).*

Throughout this paper, assume that (X, P) is a Fréchet space and $(Y, || \cdot ||)$ is a Banach space.

One can easily show that an odd mapping $f : X \to Y$ satisfies (11) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping. It was shown in [6, Lemma 2.2] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive respectively and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

2. Generalized Hyers-Ulam Stability of the Functional Equation (12)

For a given mapping f, we define

$$Df(x,y) := f(x+ky) + f(x-ky) - k^2[f(x+y) + f(x-y)] - 2(1-k^2)f(x).$$

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

Theorem 2.1. *Let $\phi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x,y) \le 8L\phi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{14}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping with $f(0) = 0$ which satisfies the inequality

$$||Df(x,y)|| \le \phi(x,y). \tag{15}$$

Then there exists a unique cubic mapping $C : X \to Y$ such that

$$||f(2x) - 2f(x) - C(x)|| \le \frac{1}{8-8L}\Phi_a(x) \tag{16}$$

for all $x \in X$, where

$$\begin{aligned}\Phi_a(x) = \frac{1}{k^4-k^2}&\left[(5-4k^2)\phi(x,x) + k^2\phi(2x,2x) + 2k^2\phi(2x,x)\right.\\ &+ (4-2k^2)\phi(x,2x) + \phi(x,3x) + 2\phi((1+k)x,x) + 2\phi((1-k)x,x)\\ &\left.+\phi((1+2k)x,x) + \phi((1-2k)x,x)\right].\end{aligned} \tag{17}$$

Proof. Replacing y by x in (15), we obtain

$$\left\|f((1+k)x) + f((1-k)x)) - k^2 f(2x) - 2(1-k^2)f(x)\right\| \le \phi(x,x) \tag{18}$$

for all $x, y \in X$. Replacing x by $2x$ in (18), we get

$$\left\|f(2(1+k)x) + f(2(1-k)x) - k^2 f(4x) - 2(1-k^2)f(2x)\right\| \le \phi(2x,2x) \tag{19}$$

for all $x \in X$. Again replacing (x,y) by $(2x,x)$ in (15), we obtain

$$\left\|f((2+k)x) + f((2-k)x) - k^2 f(3x) - k^2 f(x) - 2(1-k^2)f(2x)\right\| \le \phi(2x,x) \tag{20}$$

for all $x \in X$. Replacing y by $2x$ in (15), we get

$$\left\|f((1+2k)x)+f((1-2k)x)-k^2f(3x)+k^2f(x)-2(1-k^2)f(x)\right\| \leq \phi(x,2x) \tag{21}$$

for all $x \in X$. Replacing y by $3x$ in (15), we obtain

$$\left\|f((1+3k)x)+f((1-3k)x)-k^2f(4x)+k^2f(2x)-2(1-k^2)f(x)\right\| \leq \phi(x,3x) \tag{22}$$

for all $x \in X$. Replacing (x,y) by $((1+k)x,x)$ in (15), we get

$$\|f((1+2k)x)+f(x)-k^2f((2+k)x)-k^2f(kx)-2(1-k^2)f((1+k)x)\| \leq \phi((1+k)x,x) \tag{23}$$

for all $x \in X$. Again replacing (x,y) by $((1-k)x,x)$ in (15), we obtain

$$\|f((1-2k)x)+f(x)-k^2f((2-k)x)+k^2f(kx)-2(1-k^2)f((1-k)x)\| \leq \phi((1-k)x,x) \tag{24}$$

for all $x \in X$. Adding (23) and (24), we arrive at

$$\begin{aligned}\|f((1+2k)x)+&f((1-2k)x)+2f(x)-k^2f((2+k)x)-k^2f((2-k)x)\\&-2(1-k^2)f((1+k)x)-2(1-k^2)f((1-k)x)\|\\&\leq \phi((1+k)x,x)+\phi((1-k)x,x)\end{aligned} \tag{25}$$

for all $x \in X$. Replacing (x,y) by $((1+2k)x,x)$ in (15), we get

$$\|f((1+3k)x)+f((1+k)x)-k^2f(2(1+k)x)-k^2f(2kx)-2(1-k^2)f((1+2k)x)\| \leq \phi((1+2k)x,x) \tag{26}$$

for all $x \in X$. Again replacing (x,y) by $((1-2k)x,x)$ in (15), we obtain

$$\|f((1-3k)x)+f((1-k)x)-k^2f(2(1-k)x)+k^2f(2kx)-2(1-k^2)f((1-2k)x)\| \leq \phi((1-2k)x,x) \tag{27}$$

for all $x \in X$. Adding (26) and (27), we arrive at

$$\begin{aligned}\|f((1+3k)x)+&f((1-3k)x)+f((1+k)x)+f((1-k)x)-k^2f(2(1+k)x)\\&-k^2f(2(1-k)x)-2(1-k^2)f((1+2k)x)-2(1-k^2)f((1-2k)x\|\\&\leq \phi((1+2k)x,x)+\phi((1-2k)x,x)\end{aligned} \tag{28}$$

for all $x \in X$. Now multiplying (18) by $2(1-k^2)$, (20) by k^2 and adding (21) and (25), we have

$$\begin{aligned}
&(k^4-k^2)\|f(3x)-4f(2x)+5f(x)\| \\
&\quad = \|\{2(1-k^2)f((1+k)x)+2(1-k^2)f((1-k)x)-2k^2(1-k^2)f(2x)-4(1-k^2)^2f(x)\} \\
&\qquad +\{k^2f((2+k)x)+k^2f((2-k)x)-k^4f(3x)-k^4f(x)-2k^2(1-k^2)f(2x)\} \\
&\qquad +\{-f((1+2k)x)-f((1-2k)x)+k^2f(3x)-k^2f(x)-2(1-k^2)f(x)\} \\
&\qquad +\{f((1+2k)x)+f((1-2k)x)+2f(x)-k^2f((2+k)x)-k^2f((2-k)x) \\
&\qquad\qquad -2(1-k^2)f((1+k)x)-2(1-k^2)f((1-k)x)\}\| \\
&\quad \le 2(1-k^2)\phi(x,x)+k^2\phi(2x,x)+\phi(x,2x)+\phi((1+k)x,x)+\phi((1-k)x,x)
\end{aligned}$$

for all $x \in X$. Hence from the above inequality, we get

$$\begin{aligned}
\|f(3x)-4f(2x)+5f(x)\| \le \frac{1}{(k^4-k^2)}\Big[&2(1-k^2)\phi(x,x)+k^2\phi(2x,x) \\
&+\phi(x,2x)+\phi((1+k)x,x)+\phi((1-k)x,x)\Big]
\end{aligned} \tag{29}$$

for all $x \in X$. Now multiplying (19) by k^2, (21) by $2(1-k^2)$ and adding (18), (22) and (28), we have

$$\begin{aligned}
&(k^4-k^2)\|f(4x)-2f(3x)-2f(2x)+6f(x)\| \\
&\quad = \|\{-f((1+k)x)-f((1-k)x)+k^2f(2x)+2(1-k^2)f(x)\} \\
&\qquad +\{k^2f(2(1+k)x)+k^2f(2(1-k)x)-k^4f(4x)-2k^2(1-k^2)f(2x)\} \\
&\qquad +\{2(1-k^2)f((1+2k)x)+2(1-k^2)f((1-2k)x)-2k^2(1-k^2)f(3x) \\
&\qquad +2k^2(1-k^2)f(x)-4(1-k^2)^2f(x)\} \\
&\qquad +\{-f((1+3k)x)-f((1-3k)x)+k^2f(4x)-k^2f(2x)+2(1-k^2)f(x)\}\| \\
&\qquad +\{f((1+3k)x)+f((1-3k)x)+f((1+k)x)+f((1-k)x)-k^2f(2(1+k)x) \\
&\qquad -k^2f(2(1-k)x)-2(1-k^2)f((1+2k)x)-2(1-k^2)f((1-2k)x)\}\| \\
&\quad \le \phi(x,x)+k^2\phi(2x,2x)+2(1-k^2)\phi(x,2x)+\phi(x,3x)+\phi((1+2k)x,x) \\
&\qquad +\phi((1-2k)x,x)
\end{aligned}$$

for all $x \in X$. Hence from the above inequality, we get

$$\|f(4x)-2f(3x)-2f(2x)+6f(x)\| \leq \frac{1}{(k^4-k^2)}\Big[\phi(x,x)+k^2\phi(2x,2x) + 2(1-k^2)\phi(x,2x)+\phi(x,3x)+\phi((1+2k)x,x)+\phi((1-2k)x,x)\Big] \tag{30}$$

for all $x \in X$. From (29) and (30), we arrive at

$$\begin{aligned}
&\|f(4x)-10f(2x)+16f(x)\| \\
&= \|2f(3x)-8f(2x)+10f(x)+f(4x)-2f(3x)-2f(2x)+6f(x)\| \\
&\leq 2\|f(3x)-4f(2x)+5f(x)\|+\|f(4x)-2f(3x)-2f(2x)+6f(x)\| \\
&\leq \frac{1}{(k^4-k^2)}\Big[(5-4k^2)\phi(x,x)+k^2\phi(2x,2x)+2k^2\phi(2x,x) \\
&+(4-2k^2)\phi(x,2x)+\phi(x,3x)+2\phi((1+k)x,x)+2\phi((1-k)x,x) \\
&+\phi((1+2k)x,x)+\phi((1-2k)x,x)\Big]
\end{aligned} \tag{31}$$

for all $x \in X$. From (31), we have

$$\|f(4x)-10f(2x)+16f(x)\| \leq \Phi_a(x), \tag{32}$$

where

$$\begin{aligned}
\Phi_a(x) = \frac{1}{(k^4-k^2)}\Big[&(5-4k^2)\phi(x,x)+k^2\phi(2x,2x)+2k^2\phi(2x,x) \\
&+(4-2k^2)\phi(x,2x)+\phi(x,3x)+2\phi((1+k)x,x)+2\phi((1-k)x,x) \\
&+\phi((1+2k)x,x)+\phi((1-2k)x,x)\Big]
\end{aligned}$$

for all $x \in X$. It is easy to see from (32),

$$\|f(4x)-2f(2x)-8(f(2x)-2f(x)) \leq \Phi_a(x) \tag{33}$$

for all $x \in X$. Now, define $g : X \to Y$ by

$$g(x) := f(2x)-2f(x) \tag{34}$$

for all $x \in X$. Using (34) in (33), we obtain

$$\|g(2x) - 8g(x)\| \leq \Phi_a(x) \tag{35}$$

for all $x \in X$. From (35), we have

$$\left\|g(x) - \frac{g(2x)}{8}\right\| \leq \frac{\Phi_a(x)}{8} \tag{36}$$

for all $x \in X$. Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S :

$$d(k,h) = \inf\{\mu \in \mathbb{R}_+ : \|k(x) - h(x)\| \leq \mu\Phi_a(x),\ \ \forall\, x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is a complete generalized metric space (See [19, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$J\, h(x) := \frac{1}{8}h(2x)$$

for all $x \in X$. Let $k, h \in S$ be given such that $d(k,h) = \beta$; by the definition

$$\|k(x) - h(x)\| \leq \beta\Phi_a(x),\ \ \forall\, x \in X.$$

Hence, we have

$$\begin{aligned} \|J\, k(x) - J\, h(x)\| = \frac{1}{8}\|k(2x) - h(2x)\| &\leq \frac{1}{8}\beta\Phi_a(2x) \\ &\leq \frac{1}{8}\beta 8L\Phi_a(x) \\ &= \beta L\Phi_a(x) \end{aligned}$$

for all $x \in X$. By definition $d(Jk, Jh) \leq L\beta$. Therefore,

$$d(Jk, Jh) \leq Ld(k,h),\ \ \forall\, k,h \in S.$$

This means that J is a strictly contractive self-mapping of S with a Lipschitz constant L. It follows from (36) that $d(g, Jg) \leq 1$ and therefore, by Theorem 1.3, there exists a mapping

$C : X \to Y$ satisfying the following:

(1) C is a unique fixed point of J in the set

$$\Delta = \{k \in S : d(h,k) < \infty\}, i.e.,$$

$$C(2x) = 8C(x). \tag{37}$$

This implies that C is a unique mapping satisfying (37) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|g(x) - C(x)\| \leq \mu \frac{\Phi_a(x)}{8}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{8^n} g(2^n x) = C(x)$$

for all $x \in X$;

(3) Moreover, $d(g,C) \leq \frac{1}{1-L} d(g,Jg) \leq \frac{1}{1-L}$.

This implies that the inequality (16), holds true. It follows from (15) and (14) that

$$\begin{aligned}
\|DC(x,y)\| &= \lim_{n\to\infty} \left\| \frac{1}{8^n} Dg(2^n x, 2^n y) \right\| \\
&\leq \lim_{n\to\infty} \frac{1}{8^n} \left(\phi(2^n.2x, 2^n.2y) + 2\phi(2^n x, 2^n y) \right) \\
&\leq \lim_{n\to\infty} \left(\frac{8^n L^n}{8^n} \phi(2x,2y) + \frac{2}{8^n} 8^n L^n \phi(x,y) \right) \\
&= 0
\end{aligned}$$

for all $x \in X$ and $n \in N$. So $DC(x,y) = 0$ for all $x, y \in X$. By [6, Lemma 2.2], the function $x \to C(2x) - 2C(x)$ is cubic. Hence $C(2x) = 8C(x)$ implies that C is an cubic function.

To prove the uniqueness assertion, let us assume that there exists an cubic function $T : X \to Y$ which satisfies (16). Then, T is a fixed point of J in Δ. However, by Theorem 1.3, J has only one fixed point in Δ, and hence $C = T$. This completes the proof.

The following corollaries are immediate consequence of Theorem 2.1 concerning stability of (12).

Corollary 2.2. *Let r be a positive real number with $r < 3$, and let $f : X \to Y$ be an odd mapping with $f(0) = 0$ such that*

$$\|Df(x,y)\| \leq P(x)^r + P(y)^r \tag{38}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{\lambda_1 P(x)^r}{8 - 2^r}, \tag{39}$$

where

(40)

$$\lambda_1 = \frac{1}{(k^4 - k^2)}\Big\{21 - 8k^2 + 2^r(2k^2+4) + 3^r + 2(1+k)^r + 2(1-k)^r + (1+2k)^r + (1-2k)^r\Big\}$$

for all $x \in X$.

Proof. Taking $\phi(x,y) = P(x)^r + P(y)^r$ for all $x, y \in X$ and choosing $L = \frac{2^r}{8}$ in Theorem 2.1, we get the desired result.

Corollary 2.3. *Let $f : X \to Y$ be an odd mapping with $f(0) = 0$ satisfying*

$$\|Df(x,y)\| \leq P(x)^r P(y)^s + (P(x)^{r+s} + P(y)^{r+s}) \tag{41}$$

for all $x, y \in X$, where r, s are non negative real numbers such that $\lambda := r + s \in (0,3)$. Then, there exists a unique cubic function $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{\lambda_2 P(x)^\lambda}{8 - 2^\lambda}, \tag{42}$$

where

$$\begin{aligned} \lambda_2 = \frac{1}{(k^4-k^2)}\{&26 - 12k^2 + 2^\lambda(3k^2+4) + 3^\lambda + 2(1+k)^\lambda + 2(1-k)^\lambda \\ &+ (1+2k)^\lambda + (1-2k)^\lambda + 4(2^s) + 3^s + 2(1+k)^r + 2(1-k)^r \\ &+ (1+2k)^r + (1-2k)^r + 2k^2[2^r - 2^s]\} \end{aligned} \tag{43}$$

for all $x \in X$.

Proof. Let $\phi : X^2 \to [0,\infty)$ be defined by $\phi(x,y) = P(x)^r P(y)^s + (P(x)^{r+s} + P(y)^{r+s})$ for all $x, y \in X$. Then, the corollary is followed from Theorem 2.1 by $L = \frac{2^\lambda}{8} < 1$.

Theorem 2.4. *Let $\phi : Y^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x,y) \le \frac{L}{8}\phi(2x,2y) \tag{44}$$

for all $x,y \in Y$. Let $f : Y \to X$ be an odd mapping with $f(0) = 0$ such that

$$P(Df(x,y)) \le \phi(x,y) \tag{45}$$

for all $x,y \in Y$. Then there exists a unique cubic mapping $C : Y \to X$ such that

$$P(f(2x) - 2f(x) - C(x)) \le \frac{L}{8-8L}\Phi_a(x) \tag{46}$$

for all $x \in Y$.

Proof. Along similar lines to those in the proof of Theorem 2.1, from (45), we have

$$P(f(4x) - 10f(2x) + 16f(x)) \le \Phi_a(x), \tag{47}$$

where

$$\begin{aligned}\Phi_a(x) = \frac{1}{k^4-k^2}\Big[&(5-4k^2)\phi(x,x) + k^2\phi(2x,2x) + 2k^2\phi(2x,x)\\ &+ (4-2k^2)\phi(x,2x) + \phi(x,3x) + 2\phi((1+k)x,x) + 2\phi((1-k)x,x)\\ &+ \phi((1+2k)x,x) + \phi((1-2k)x,x)\Big]\end{aligned}$$

for all $x \in Y$. It is easy to see from (47),

$$P(f(4x) - 2f(2x) - 8(f(2x) - 2f(x))) \le \Phi_a(x) \tag{48}$$

for all $x \in Y$. Replacing x by $\frac{x}{2}$ and letting $g(x) := f(2x) - 2f(x)$ in (48), we get

$$P\left(g(x) - 8g\left(\frac{x}{2}\right)\right) \le \Phi_a\left(\frac{x}{2}\right) \le \frac{L}{8}\Phi_a(x) \tag{49}$$

for all $x \in Y$. Consider the set

$$S := \{h : Y \to X\}$$

and introduce the generalized metric on S:

$$d(k,h) = \inf\{\mu \in \mathbb{R}_+ : P(k(x) - h(x)) \le \mu\Phi_a(x),\ \ \forall\, x \in Y\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d) is complete (see [19, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 8h\left(\frac{x}{2}\right), \quad \forall x \in Y.$$

Let $k, h \in S$ be given such that $d(k,h) = \beta$; by the definition

$$P(k(x) - h(x)) \leq \beta \Phi_a(x), \ \forall\, x \in Y.$$

Hence, we have

$$\begin{aligned} P(Jk(x) - Jh(x)) &= P\left(8k\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right)\right) \\ &\leq 8\beta\Phi_a\left(\frac{x}{2}\right) \\ &\leq 8\beta\frac{L}{8}\Phi_a(x) \\ &= \beta L \Phi_a(x) \end{aligned}$$

for all $x \in Y$. By definition $d(Jk, Jh) \leq L\beta$. Therefore,

$$d(Jk, Jh) \leq Ld(k,h), \ \forall\, k,h \in S.$$

This means that J is a strictly contractive self-mapping of S with a Lipschitz constant L.

It follows from (49) that $d(g, Jg) \leq L$. By Theorem 1.3, there exists a mapping $C : Y \to X$ satisfying the following:

(1) C is a unique fixed point of J in the set

$$\Delta = \{k \in S : d(h,k) < \infty\}, i.e.,$$

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{50}$$

for all $x \in Y$. This implies that C is a unique mapping satisfying (50) such that there exists a $\mu \in (0, \infty)$ satisfying

$$P(g(x) - C(x)) \leq \mu \frac{\Phi_a(x)}{8}$$

for all $x \in Y$. Since $g : Y \to X$ is odd, $C : Y \to X$ is an odd mapping;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in Y$;

(3) Moreover, $d(g,C) \leq \frac{1}{1-L} d(g,Jg) \leq \frac{1}{1-L}$.

This implies that the inequality (46) holds true. It follows from (45) and (44) that

$$\begin{aligned} P(DC(x,y)) &= \lim_{n\to\infty} P\left(8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \\ &\leq \lim_{n\to\infty} 8^n \left(\phi\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) + 2\phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \\ &\leq \lim_{n\to\infty} \left(\frac{8^n L^n}{8^n}\phi(2x,2y) + 2\frac{8^n L^n}{8^n}\phi(x,y)\right) \\ &= 0 \end{aligned}$$

for all $x,y \in Y$ and $n \in \mathbb{N}$. So $DC(x,y) = 0$ for all $x,y \in Y$. By [6, Lemma 2.2], the function $x \to C(2x) - 2C(x)$ is cubic. Hence $C(x) = 8C\left(\frac{x}{2}\right)$ implies that C is a cubic function, as desired. The uniqueness assertion follows in the same way as in Theorem 2.1.

As applications of Theorem 2.4, one can get the following Corollaries 2.5 and 2.6 concerning the stability of (12).

Corollary 2.5. *Let r,θ be a positive real numbers with $r > 3$, and let $f : Y \to X$ be an odd mapping with $f(0) = 0$ such that*

$$P(Df(x,y)) \leq \theta(\|x\|^r + \|y\|^r) \tag{51}$$

for all $x,y \in Y$. Then there exists a unique cubic mapping $C : Y \to X$ such that

$$P(f(2x) - 2f(x) - C(x)) \leq \frac{\lambda_1 \theta \|x\|^r}{2^r - 8} \tag{52}$$

for all $x \in Y$, where λ_1 is given in Corollary 2.2.

Proof. Taking $\phi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x,y \in Y$ and choosing $L = \frac{8}{2^r}$ in Theorem 2.4, we get the desired result.

Corollary 2.6. *Let $f : Y \to X$ be an even mapping with $f(0) = 0$ satisfying*

$$P(Df(x,y)) \leq \theta(\|x\|^r \|y\|^r) \tag{53}$$

for all $x, y \in Y$, where r, θ are non negative real numbers such that $r > \frac{3}{2}$. Then, there exists a unique cubic function $C : Y \to X$ such that

$$P(f(2x) - 2f(x) - C(x)) \leq \frac{\lambda_3 \theta \|x\|^{2r}}{2^{2r} - 8}, \tag{54}$$

where

$$\begin{aligned}\lambda_3 = \frac{1}{(k^4 - k^2)} \Big\{ & 5 - 4k^2 + k^2 2^{2r} + 4(2^r) + 3^r + 2(1+k)^r + 2(1-k)^r \\ & + (1+2k)^r + (1-2k)^r \Big\}\end{aligned} \tag{55}$$

for all $x \in Y$.

Proof. Let $\phi : Y^2 \to [0, \infty)$ be defined by $\phi(x, y) = \theta(\|x\|^r \|y\|^r)$ for all $x, y \in Y$. Then, the corollary is followed from Theorem 2.4 by choosing $L = \frac{8}{2^{2r}} < 1$.

Theorem 2.7. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x, y) \leq 2L\phi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{56}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping with $f(0) = 0$, satisfying the inequality (15) for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2 - 2L} \Phi_a(x) \tag{57}$$

for all $x, y \in X$.

Proof. Following along similar lines to those in the proof of Theorem 2.1, from (32), we arrive

$$\|(f(4x) - 8f(2x)) - 2(f(2x) - 8f(x))\| \leq \Phi_a(x) \tag{58}$$

for all $x \in X$. Let $g : X \to Y$ by

$$g(x) := f(2x) - 8f(x) \tag{59}$$

for all $x \in X$. Using (59) in (58), we obtain

$$\|g(2x) - 2g(x)\| \leq \Phi_a(x) \tag{60}$$

for all $x \in X$. From (60), we have

$$\left\| g(x) - \frac{1}{2} g(2x) \right\| \leq \frac{\Phi_a(x)}{2} \tag{61}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S

$$d(k,h) = \inf\{\mu \in \mathbb{R}_+ : \|k(x) - h(x)\| \leq \mu \Phi_a(x), \ \forall\, x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is a complete generalized metric space (See [19, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2} h(2x)$$

for all $x \in X$. Let $k,h \in S$ be given such that $d(k,h) = \beta$; by the definition

$$\|k(x) - h(x)\| \leq \beta \Phi_a(x), \ \forall\, x \in X.$$

Hence,

$$\begin{aligned} \|Jk(x) - Jh(x)\| &= \frac{1}{2} \|k(2x) - h(2x)\| \\ &\leq \frac{1}{2} \beta \Phi_a(2x) \\ &\leq \frac{1}{2} \beta 2L \Phi_a(x) \\ &= \beta L \Phi_a(x) \end{aligned}$$

for all $x \in X$. By definition $d(Jk, Jh) \leq L\beta$. Therefore,

$$d(Jk, Jh) \leq L d(k,h), \ \forall\, k,h \in S.$$

This means that J is a strictly contractive self-mapping of S with a Lipschitz constant L. It follows from (61) that $d(g, Jg) \leq 1$. By Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a unique fixed point of J in the set

$$\Delta = \{k \in S : d(h,k) < \infty\}, i.e.,$$

$$A(2x) = 2A(x). \tag{62}$$

This implies that A is a unique mapping satisfying (62) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|g(x) - A(x)\| \le \mu \frac{\Phi_a(x)}{2}.$$

for all $x \in X$. Since $g : X \to Y$ is odd, $A : X \to Y$ is an odd mapping;

(2) $d(J^n g, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} g(2^n x) = A(x)$$

for all $x \in X$;

(3) Moreover, $d(g,A) \le \frac{1}{1-L} d(g,Jg) \le \frac{1}{1-L}$.

This implies that the inequality (57), holds true. It follows from (15) and (56) that

$$\begin{aligned}
\|DA(x,y)\| &= \lim_{n\to\infty} \left\| \frac{1}{2^n} Dg(2^n x, 2^n y) \right\| \\
&\le \lim_{n\to\infty} \frac{1}{2^n} (\phi(2^n.2x, 2^n.2y) + 8\phi(2^n x, 2^n y)) \\
&\le \lim_{n\to\infty} \left(\frac{2^n L^n}{2^n} \phi(2x,2y) + 8\frac{2^n L^n}{2^n} \phi(x,y) \right) \\
&= 0
\end{aligned}$$

for all $x, y \in X$. So $DA(x,y) = 0$ for all $x, y \in X$. By [6, Lemma 2.2], the function $x \to f(2x) - 8f(x)$ is additive. Hence $A(2x) = 2A(x)$ implies that A is an additive function. The uniqueness assertion follows in the same way as in Theorem 2.1.

The following corollaries are immediate consequence of Theorem 2.7 concerning the stability of (12).

Corollary 2.8. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping satisfying $f(0) = 0$ and (38) for all $x, y \in X$. Then there exists a unique additive mapping*

$A : X \to Y$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{\lambda_1 P(x)^r}{2 - 2^r} \tag{63}$$

for all $x \in X$, where λ_1 is given in Corollary 2.2.

Proof. Taking $\phi(x,y) = P(x)^r + P(y)^r$ for all $x, y \in X$ and choosing $L = \frac{2^r}{2}$ in Theorem 2.7, we get the desired result.

Corollary 2.9. *Let $f : X \to Y$ be an even mapping with $f(0) = 0$ satisfying (41) for all $x, y \in X$, where r, s are non negative real numbers such that $\lambda := r + s \in (0,1)$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{\lambda_2 P(x)^\lambda}{2 - 2^\lambda} \tag{64}$$

for all $x \in X$, where λ_2 is given in Corollary 2.3.

Proof. Let $\phi : X^2 \to [0,\infty)$ be defined by $\phi(x,y) = P(x)^r P(y)^s + (P(x)^{r+s} + P(y)^{r+s})$ for all $x, y \in X$. Then, the corollary is followed from Theorem 2.7 by $L = \frac{2^\lambda}{2} < 1$.

Theorem 2.10. *Let $\phi : Y^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x,y) \leq \frac{L}{2}\phi(2x,2y) \tag{65}$$

for all $x, y \in Y$. Let $f : Y \to X$ be an odd mapping with $f(0) = 0$ satisfying (45) for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(2x) - 8f(x) - A(x)) \leq \frac{L}{2 - 2L}\Phi_a(x) \tag{66}$$

for all $x \in Y$.

Proof. Following along similar lines to those in the proof of Theorem 2.1, from (47), we obtain

$$P((f(4x) - 8f(2x)) - 2(f(2x) - 8f(x))) \leq \Phi_a(x) \tag{67}$$

for all $x \in Y$. Replacing x by $\frac{x}{2}$ and letting $g(x) := f(2x) - 8f(x)$ in (67), we get

$$P\left(g(x) - 2g\left(\frac{x}{2}\right)\right) \leq \Phi_a\left(\frac{x}{2}\right) \leq \frac{L}{2}\Phi_a(x) \tag{68}$$

for all $x \in Y$.

Consider the set

$$S := \{h : Y \to X\}$$

and introduce the generalized metric on S

$$d(k,h) = \inf\{\mu \in \mathbb{R}_+ : P(k(x) - h(x)) \leq \mu \Phi_a(x), \ \forall\, x \in Y\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [19, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right)$$

for all $x \in Y$. Let $k,h \in S$ be given such that $d(k,h) = \beta$; by the definition

$$P(k(x) - h(x)) \leq \beta \Phi_a(x), \ \ \forall x \in Y.$$

Hence,

$$\begin{aligned} P(Jk(x) - Jh(x)) &= P\left(2k\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right) \\ &\leq 2\beta \Phi_a\left(\frac{x}{2}\right) \\ &\leq 2\beta \frac{L}{2} \Phi_a(x) \\ &= \beta L \phi_a(x) \end{aligned}$$

for all $x \in Y$. By definition $d(Jk, Jh) \leq L\beta$. Therefore,

$$d(Jk,Jh) \leq Ld(k,h), \ \ \forall\, k,h \in S.$$

This means that J is a strictly contractive self-mapping of S with a Lipschitz constant L. It follows from (68) that $d(g,Jg) \leq 1$. By Theorem 1.3, there exists a mapping $A : Y \to X$ satisfying the following:

(1) A is a unique fixed point of J in the set

$$\Delta = \{k \in S : d(h,k) < \infty\}, i.e.,$$

(69)
$$A\left(\frac{x}{2}\right) = \frac{1}{2} A(x)$$

for all $x \in Y$. This implies that A is a unique mapping satisfying (69) such that there exists a $\mu \in (0,\infty)$ satisfying

$$P(g(x) - A(x)) \leq \mu \frac{\Phi_a(x)}{2}$$

for all $x \in Y$. Since $g : Y \to X$ is odd, $C : Y \to X$ is an odd mapping;

(2) $d(J^n g, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in Y$;

(3) $d(g,A) \leq \frac{1}{1-L} d(g,Jg) \leq \frac{1}{1-L}$.

This implies that the inequality (66) holds true. It follows from (45) and (65) that

$$\begin{aligned} P(DA(x,y)) &= \lim_{n\to\infty} P\left(2^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \\ &\leq \lim_{n\to\infty} 2^n P\left(Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \\ &\leq \lim_{n\to\infty} 2^n \left(\phi\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) + 8\phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \\ &\leq \lim_{n\to\infty} \left(\frac{2^n L^n}{2^n}\phi(2x,2y) + 8\frac{2^n L^n}{2^n}\phi(x,y)\right) \\ &= 0 \end{aligned}$$

for all $x, y \in Y$. So $DA(x,y) = 0$ for all $x, y \in Y$. By [6, Lemma 2.2], $A : Y \to X$ is an additive mapping, as desired. The rest of the proof is similar to Theorem 2.1.

The following corollaries are immediate consequence of Theorem 2.10 concerning the stability of (12).

Corollary 2.11. *Let r, θ be a positive real number with $r > 1$, and let $f : Y \to X$ be an odd mapping with $f(0) = 0$ satisfying (51) for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that*

$$P(f(2x) - 8f(x) - A(x)) \leq \frac{\lambda_1 \theta \|x\|^r}{2^r - 2} \tag{70}$$

for all $x \in Y$, where λ_1 is given in Corollary 2.2.

Proof. Taking $\phi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x,y \in Y$ and choosing $L = \frac{2}{2^r}$ in Theorem 2.10, we get the desired result.

Corollary 2.12. *Let $f : Y \to X$ be an odd mapping with $f(0) = 0$ satisfying (53) for all $x,y \in Y$, where r,θ are non-negative real numbers such that $r > \frac{1}{2}$. Then, there exists a unique additive mapping $A : Y \to X$ such that*

$$P(f(2x) - 8f(x) - A(x)) \leq \frac{\lambda_3 \theta \|x\|^{2r}}{2^{2r} - 2} \tag{71}$$

for all $x \in Y$, where λ_3 is given in Corollary 2.6.

Proof. Let $\phi : Y^2 \to [0,\infty)$ be defined by $\phi(x,y) = \theta(\|x\|^r \|y\|^r)$ for all $x,y \in Y$. Then, the corollary is followed from Theorem 2.10 by choosing $L = \frac{2}{2^{2r}} < 1$.

The generalized Hyers-Ulam stability problem for the case $r = 3$ and $r = 1$ was excluded in Corollaries 2.2, 2.5, 2.8 and 2.11. The following example illustrates the fact that the functional equation (12) is not stable for $r = 3$ in Corollaries 2.2 and 2.5.

Example 2.13. Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x^3, & |x| < 1, \\ 1, & |x| \geq 1. \end{cases}$$

Consider the function $g(x) = f(2x) - 2f(x) : \mathbb{C} \to \mathbb{C}$ be defined by

$$g(x) = \sum_{j=0}^{\infty} \frac{\phi(\alpha^j x)}{\alpha^{3j}} \tag{72}$$

for all $x \in \mathbb{C}$, where $\alpha > \max\{|k|, 1\}$. Then the mapping g satisfies the functional inequality

$$|Dg(x,y)| \leq \frac{4\alpha^3}{\alpha^3 - 1} \left(|x|^3 + |y|^3\right) \tag{73}$$

for all $x,y \in \mathbb{C}$, where

$$Dg(x,y) = g(x+ky) + g(x-ky) - k^2[g(x+y) + g(x-y)] - 2(1-k^2)g(x)$$

but there do not exist a cubic mapping $C : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|g(x) - C(x)| \leq d|x|^3$ for all $x \in C$.

Proof. Consider

$$|g(x)| \leq \sum_{j=0}^{\infty} \frac{|\phi(\alpha^j x)|}{|\alpha^{3j}|} = \sum_{j=0}^{\infty} \frac{1}{\alpha^{3j}} = \left(1 - \frac{1}{\alpha^3}\right)^{-1} = \frac{\alpha^3}{\alpha^3 - 1}.$$

Hence g is bounded by $\frac{\alpha^3}{\alpha^3-1}$ on $\mathbb{C}$. If $|x|^3 + |y|^3 = 0$ or $|x|^3 + |y|^3 \geq \frac{1}{\alpha^3}$, then

$$Dg(x,y) \leq \frac{4\alpha^3}{\alpha^3 - 1}\left(|x|^3 + |y|^3\right).$$

Now suppose that $0 < |x|^3 + |y|^3 < \frac{1}{\alpha^3}$. Then there exists an integer $n \geq 1$ such that

$$\frac{1}{\alpha^{3(n+1)}} < |x|^3 + |y|^3 < \frac{1}{\alpha^{3n}}. \tag{74}$$

Hence

$$\alpha^{3n}(|x|^3 + |y|^3) < 1$$

or

$$\alpha^{3n}|x|^3 < 1, \;\; \alpha^{3n}|y|^3 < 1.$$

Consequently, we find

$$\alpha^n x < 1, \;\; \alpha^n y < 1.$$

Hence $\alpha^j|x+ky|<1$, $\alpha^j|x-ky|<1$, $\alpha^j|x+y|<1$, $\alpha^j|x-y|<1$, $\alpha^j|2y|<1$, $\alpha^j|y|<1$, for all $j=0,1,\dots,n-1$. From the definition of g and (74), we obtain that

$$\begin{aligned}
\frac{|Dg(x,y)|}{|x|^3+|y|^3} &\le \Big|\sum_{j=n}^{\infty}\frac{\alpha^{-3j}}{|x|^3+|y|^3}\left[\phi(\alpha^j(x+ky))+\phi(\alpha^j(x-ky))\right.\\
&\quad \left.-k^2\left\{\phi(\alpha^j(x+y))+\phi(\alpha^j(x-y))\right\}-2(1-k^2)\phi(\alpha^j x)\right]\Big|\\
&\le \sum_{j=n}^{\infty}\frac{1}{\alpha^{3j}(|x|^3+|y|^3)}\left[1+1+k^2(1+1)+2(1-k^2)\right]\\
&\le \sum_{j=n}^{\infty}\frac{4}{\alpha^{3j}(|x|^3+|y|^3)}\\
&\le \sum_{\ell=0}^{\infty}\frac{4}{\alpha^{3\ell}\alpha^{3n}(|x|^3+|y|^3)}\\
&\le \sum_{\ell=0}^{\infty}\frac{4}{\alpha^{2\ell}}\\
&\le \frac{4\alpha^3}{\alpha^3-1}.
\end{aligned}$$

Therefore, g satisfies (73). Now, we claim that the functional equation (12) is not stable for $r=3$ in Corollaries 2.2 and 2.5. Suppose, on the contrary, that there exist a cubic mapping $C:\mathbb{C}\to\mathbb{C}$ and a constant $d>0$ such that $|g(x)-C(x)|\le d|x|^3$ for all $x\in C$. Then there exists a constant $c\in\mathbb{C}$ such that $C(x)=cx^3$ for all rational numbers x. So, we obtain

$$|g(x)|\le(d+|c|)|x|^3 \tag{75}$$

for all rational numbers x. Let $s\in\mathbb{N}$ with $s+1>d+|c|$. If x is a rational number in $(0,\alpha^{-s})$, then $\alpha^j x\in(0,1)$ for all $j=0,1,\dots,s$ and for this x, we get

$$g(x)=\sum_{j=0}^{\infty}\frac{\phi(\alpha^j x)}{\alpha^{3j}}\ge\sum_{j=0}^{s}\frac{\phi(\alpha^j x)}{\alpha^{3j}}=(s+1)x^3>(d+|c|)x^3,$$

which contradicts (75).

The following example illustrates the fact that the functional equation (12) is not stable for $r=1$ in Corollaries 2.8 and 2.11.

Example 2.14. Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x, & |x| < 1, \\ 1, & |x| \geq 1. \end{cases}$$

Consider the function $g(x) = f(2x) - 8f(x) : \mathbb{C} \to \mathbb{C}$ be defined by

$$g(x) = \sum_{j=0}^{\infty} \frac{\phi(\alpha^j x)}{\alpha^j} \tag{76}$$

for all $x \in \mathbb{C}$, where $\alpha > max\{|k|, 1\}$. Then the mapping g satisfies the functional inequality

$$|Dg(x,y)| \leq \frac{4\alpha}{\alpha - 1}\left(|x| + |y|\right) \tag{77}$$

for all $x, y \in \mathbb{C}$, where

$$Dg(x,y) = g(x+ky) + g(x-ky) - k^2[g(x+y) + g(x-y)] - 2(1-k^2)g(x)$$

but there do not exist an additive mapping $A : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|g(x) - A(x)| \leq d|x|$ for all $x \in \mathbb{C}$.

Proof. Consider

$$|g(x)| \leq \sum_{j=0}^{\infty} \frac{|\phi(\alpha^j x)|}{|\alpha^j|} = \sum_{j=0}^{\infty} \frac{1}{\alpha^j} = \left(1 - \frac{1}{\alpha}\right)^{-1} = \frac{\alpha}{\alpha - 1}.$$

Hence g is bounded by $\frac{\alpha}{\alpha-1}$ on $\mathbb{C}$. If $|x| + |y| = 0$ or $|x| + |y| \geq \frac{1}{\alpha}$, then

$$|Dg(x,y)| \leq \frac{4\alpha}{\alpha - 1}\left(|x| + |y|\right).$$

Now suppose that $0 < |x| + |y| < \frac{1}{\alpha}$. Then there exists an integer $n \geq 1$ such that

$$\frac{1}{\alpha^{(n+1)}} < |x| + |y| < \frac{1}{\alpha^n}. \tag{78}$$

Hence

$$\alpha^n(|x| + |y|) < 1$$

or

$$\alpha^n x < 1, \ \ \alpha^n y < 1.$$

Hence $\alpha^j|x+ky|<1$, $\alpha^j|x-ky|<1$, $\alpha^j|x+y|<1$, $\alpha^j|x-y|<1$, $\alpha^j|2y|<1$, $\alpha^j|y|<1$, for all $j=0,1,\ldots,n-1$. From the definition of g and (78), we obtain that

$$\begin{aligned}
\frac{|Dg(x,y)|}{|x|+|y|} &\leq \Bigg|\sum_{j=n}^{\infty}\frac{\alpha^{-j}}{|x|+|y|}\left[\phi(\alpha^j(x+ky))+\phi(\alpha^j(x-ky))\right.\\
&\quad \left.-k^2\left\{\phi(\alpha^j(x+y))+\phi(\alpha^j(x-y))\right\}-2(1-k^2)\phi(\alpha^j x)\right]\Bigg|\\
&\leq \sum_{j=n}^{\infty}\frac{1}{\alpha^j(|x|+|y|)}\left[1+1+k^2(1+1)+2(1-k^2)\right]\\
&\leq \sum_{j=n}^{\infty}\frac{4}{\alpha^j(|x|+|y|)}\\
&\leq \sum_{\ell=0}^{\infty}\frac{4}{\alpha^\ell\alpha^n(|x|+|y|)}\\
&\leq \sum_{\ell=0}^{\infty}\frac{4}{\alpha^\ell}\\
&\leq \frac{4\alpha}{\alpha-1}.
\end{aligned}$$

Therefore, g satisfies (77). Now, we claim that the functional equation (12) is not stable for $r=1$ in Corollaries 2.8 and 2.11. Suppose, on the contrary, that there exist a additive mapping $A:\mathbb{C}\to\mathbb{C}$ and a constant $d>0$ such that $|g(x)-A(x)|\leq d|x|$ for all $x\in\mathbb{C}$. Then there exists a constant $c\in\mathbb{C}$ such that $A(x)=cx$ for all rational numbers x.
So, we obtain

$$|g(x)|\leq(d+|c|)|x| \tag{79}$$

for all rational numbers x. Let $s\in\mathbb{N}$ with $s+1>d+|c|$. If x is a rational number in $(0,\alpha^{-s})$, then $\alpha^j x\in(0,1)$ for all $j=0,1,\ldots,s$ and for this x, we get

$$g(x)=\sum_{j=0}^{\infty}\frac{\phi(\alpha^j x)}{\alpha^j}\geq\sum_{j=0}^{s}\frac{\phi(\alpha^j x)}{\alpha^j}=(s+1)x>(d+|c|)x$$

which contradicts (79). Hence, the above theorems and corollaries can be summarized as follows.

3. Generalised Hyers-Ulam Stability of the Functional Equation (12) : General Mapping Case

Theorem 3.1. *Let $\phi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x,y) \le 2L\phi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x,y \in X$. Let $f : X \to Y$ be an odd mapping satisfying $f(0) = 0$ and (15). Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{6}C(x) \right\| \le \left(\frac{1}{12-12L} + \frac{1}{48-48L} \right) \Phi_a(x) \tag{80}$$

for all $x \in X$, where $\Phi_a(x)$ is given in Theorem 2.1.

Proof. Since $\phi(x,y) \le 2L\phi\left(\frac{x}{2},\frac{y}{2}\right)$, $\phi(x,y) \le 8L\phi\left(\frac{x}{2},\frac{y}{2}\right)$.

The result follows from Theorems 2.1 and 2.7.

Corollary 3.2. *Let r be a positive real number with $r < 1$. Let $f : X \to Y$ be an odd mapping satisfying $f(0) = 0$ and (38). Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that*

$$\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{6}C(x) \right\| \le \left\{ \frac{\lambda_1}{6(2-2^r)} + \frac{\lambda_1}{6(8-2^r)} \right) P(x)^r \tag{81}$$

for all $x \in X$, where λ_1 is given in Corollary 2.2.

Corollary 3.3. *Let $f : X \to Y$ be an odd mapping satisfying $f(0) = 0$ and (41) where $\lambda = r+s \in (0,1)$. Then there exists a unique additive mapping $A : X \to Y$, and a unique cubic mapping $C : X \to Y$ such that*

$$\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{6}C(x) \right\| \le \left\{ \frac{\lambda_2}{6(2-2^\lambda)} + \frac{\lambda_2}{6(8-2^\lambda)} \right) P(x)^\lambda \tag{82}$$

for all $x \in X$, where λ_2 is given in Corollary 2.3.

Theorem 3.4. *Let $\phi : Y^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\phi(x,y) \le \frac{L}{8}\phi(2x,2y)$$

for all $x,y \in Y$. Let $f: Y \to X$ be an odd mapping satisfying $f(0)=0$ and (45). Then there exists a unique additive mapping $A: Y \to X$ and a unique cubic mapping $C: Y \to X$ such that

$$P\left(f(x)-\frac{1}{6}A(x)-\frac{1}{6}C(x)\right) \leq \left(\frac{L}{12-12L}+\frac{L}{48-48L}\right)\Phi_a(x) \tag{83}$$

for all $x \in Y$.

Proof. Since $\phi(x,y) \leq \frac{L}{8}\phi(2x,2y)$, $\phi(x,y) \leq \frac{L}{2}\phi(2x,2y)$. The result follows from Theorems 2.4 and 2.10.

Corollary 3.5. *Let r,θ be positive real numbers with $r>3$. Let $f: Y \to X$ be an odd mapping satisfying $f(0)=0$ and (51). Then there exists an unique additive mapping $A: Y \to X$ and a unique cubic mapping $C: Y \to X$ such that*

$$P\left(f(x)-\frac{1}{6}A(x)-\frac{1}{6}C(x)\right) \leq \left(\frac{\lambda_1}{6(2^r-2)}+\frac{\lambda_1}{6(2^r-8)}\right)\theta\|x\|^r \tag{84}$$

for all $x \in Y$, where λ_1 is given in Corollary 2.2.

Corollary 3.6. *Let $f: Y \to X$ be an odd mapping satisfying $f(0)=0$ and (53) for all $x,y \in Y$, where r,θ are non-negative real numbers such that $r>\frac{3}{2}$. Then there exists an unique additive mapping $A: Y \to X$ and a unique cubic mapping $C: Y \to X$ such that*

$$P\left(f(x)-\frac{1}{6}A(x)-\frac{1}{6}C(x)\right) \leq \left(\frac{\lambda_3}{6(2^{2r}-2)}+\frac{\lambda_3}{6(2^{2r}-8)}\right)\theta\|x\|^{2r} \tag{85}$$

for all $x \in Y$, where λ_3 is given in Corollary 2.6.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] J. Aczel, J. Dhombres, Functional equations in several variables, Cambridge University Press, Cambridge (1989).

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Japan Math. Soc. 2 (1951), 64-66.

[3] L. Cădariu, V. Radu, Fixed points and the stability of Jensen's functional equation, J. Inequal. Pure Appl. Math. 4 (2003), Article ID 4.

[4] L. Cădariu, V. Radu, Fixed point methods for the generalized stability of functional equations in a single variable, Fixed Point Theory Appl. 2008 (2008), Article ID 749392.

[5] J. Diaz, B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968), 305-309.

[6] M. Ezhaghi Gordji, S. Kaboli-Gharetapeh, C. Park, S. Zolfaghari, Stability of an additive-cubic-quartic functional equation, Adv. Difference Equ. 2009 (2009), Article ID 395693.

[7] H. Fast, Sur la Convergence statistique, Colloq. Math., 2 (1951), 241-244.

[8] J. Fridy, On statistical convergence, Anal. 5 (1985), 301-313.

[9] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431-436.

[10] A. Grabeic, The generalized Hyers-Ulam stability of a class of functional equations, Publ. Math. Debrecen, 48 (1996), 217-235.

[11] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA, 27 (1941), 222-224.

[12] S.M. Jung, On the Hyer-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 204 (1996), 221-226.

[13] K.W. Jun, H.M. Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, J. Math. Anal. Appl. 274 (2002), 867-878.

[14] P. Kannappan, Quadratic functional equation inner product spaces, Results Math. 27 (1995), 368-372.

[15] S. Karakus, Statistical convergence on probabilistic normed spaces, Math. Commun. 12 (2007), 11-23.

[16] E. Kolk, The statistical convergence in Banach spaces, Tartu Ul. Toime, 928 (1991), 41-52.

[17] Y.W. Lee, The stability of derivatives on Banach Algebra, Bull. Inst. Math. Acad. Sinica, 28 (2000), 113-116.

[18] S.H. Lee, S.M. Im, I.S. Hwang, Quartic functional equations, J. Math. Anal. Appl. 307 (2005), 387-394.

[19] D. Mihet, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl. 343 (2008), 567-572.

[20] C. Park, Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equation in Banach algebra, Fixed Point Theory Appl. 2007 (2007), Article ID 50175.

[21] C. Park, Fixed points and the stability of an AQCQ - functional equation in non-archimedean normed spaces, Abst. Appl. Anal. 2010 (2010), Article ID 849543.

[22] C. Park, Stability of an AQCQ - functional equation in paranormed space, Adv. Difference Equ. 2012 (2012), Article ID 148.

[23] V. Radu, The fixed point alternative and the stability of functional equations, Fixed Point Theory 4 (2003), 91-96.

[24] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72 (1978), 297–300.

[25] J.M. Rassias, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal. 46 (1982), 126-130.

[26] K. Ravi, J.M. Rassias, M. Arun Kumar, R. Kodandan, Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation, J. Inequal. Pure Appl. Math. 10 (2009), Article 114.

[27] K. Ravi, M. Arunkumar, J.M. Rassias, Ulam Stability for the orthogonally general Euler-Lagrange type functional equation, Int. J. Math. Stat. 3 (2008), 36-46.

[28] T. Salat, On the statistically convergent sequence of real numbers, Math. Slovaca. 30 (1980), 139-150.

[29] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951), 73-34.

[30] S.M. Ulam, Problems in Modern mathematics, Science Edition, Wiley, New York (1960).

[31] A. Wilansky, Modern methods in Topological vector space, McGraw-Hill International Book Co., New York (1978).