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A FIXED POINT APPROACH TO THE NON-ARCHIMEDEAN RANDOM STABILITY OF GENERALIZED MIXED TYPE AQCQ-FUNCTIONAL EQUATIONS

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Abstract. Using the fixed point methods, we prove the generalized Hyers-Ulam stability of a general mixed additive-quadratic-cubic-quartic functional equation.

Keywords: non-Archimedean normed spaces; random Banach spaces; fixed point method, quadratic function; generalized Hyers-Ulam stability.

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1. Introduction

In Ulam [1], the following question concerning the stability of homomorphisms is studied: Let G be a group and let H be a metric group with metric $d(.,.)$. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that if the function $f : G \to H$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with $d(f(x), a(x)) < \varepsilon$ for all $x \in G$. In 1941, Hyers [2] gave the first affirmative partial answer to the question of Ulam in Banach

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spaces. In 1950, Aoki [3] generalized the Hyers theorem for additive mappings. In 1978, Rassias [4] provided a generalized version of the Hyers theorem which permits the Cauchy difference to become unbounded. In 1942, Menger [5] introduced the notion of probabilistic metric spaces. Since then, the theory of probabilistic metric spaces has been developed by many authors in many directions; see [6], [7] and the references therein. The idea of Menger was to use the distribution functions instead of non-negative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be interested in the investigation of physical quantities, physiological thresholds and some other fields. It is also of fundamental importance in probabilistic functional analysis. In 1962, Serstnev [8] introduced the concept of a probabilistic normed space introduced by means of a definition that was closely modeled on the theory of (classical) normed spaces and used to study the problems of best approximation in statistics. On the other hand, Isac and Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. The stability problems of several various functional equations have been extensively investigated by a number of authors using fixed point methods; see [10-12] and the references therein.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is said to be a quadratic functional equation because the quadratic function $f(x)=ax^2$ is a solution of the functional equation (1.1). A quadratic functional equation was used to characterize inner product spaces [13,14]. It is well known that a function f is a solution of (1.1) if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all x; see [14]. The biadditive function B is given by

$$B(x,y)=\frac{1}{4}[f(x+y)+f(x-y)]. \tag{1.2}$$

The functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

is called a cubic functional equation, because the cubic function $f(x)=cx^3$ is a solution of the equation (1.3). The general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3) was discussed by Jun and Kim [15]. They proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique function $C: X^3 \to Y$ such that $f(x)=C(x,x,x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables. The quartic functional equation

$$f(x+2y)+f(x-2y)-6f(x)=4[f(x+y)+f(x-y)]+24f(y) \tag{1.4}$$

was introduced by Rassias [16]. Later Lee *et al.* [17] remodified Rassias equation and obtained a new quartic functional equation of the form

$$f(2x+y)+f(2x-y)=4[f(x+y)+f(x-y)]+24f(x)-6f(y) \tag{1.5}$$

and discussed its general solution. In fact Lee *et al.* [17] proved that a function f between vector spaces X and Y is a solution of (1.5) if and only if there exists a unique symmetric multi - additive function $Q: X^4 \to Y$ such that $f(x)=Q(x,x,x,x)$ for all $x \in X$. It is easy to show that the function $f(x)=kx^4$ is the solution of (1.4) and (1.5).

A function

$$f(x)=Q(x_1,x_2,x_3,x_4) \tag{1.6}$$

is called symmetric multi additive if Q is additive with respect to each variable $x_i, i=1,2,3,4$ in (1.6). A function f is defined as $f(x)=\frac{\beta(x)-\alpha(x)}{12}$, where $\alpha(x)=f(2x)-16f(x)$, $\beta(x)=f(2x)-4f(x)$. Further, f satisfies $f(2x)=4f(x)$ and $f(2x)=16f(x)$ is said to be a quadratic - quartic function.

Jun and Kim [18] introduced the following generalized quadratic and additive type functional equation

$$f(\sum_{i=1}^{n} x_i)+(n-2)\sum_{i=1}^{n} f(x_i)=\sum_{1\le i<j\le j} f(x_i+x_j) \tag{1.7}$$

in the class of functions between real vector spaces. For $n=3$, Kannappan proved that a function f satisfies the functional equation (1.7) if and only if there exists a symmetric bi-additive function B and an additive function A such that $f(x)=B(x,x)+A(x)$ for all x; see [14]. The Hyers-Ulam stability for the equation (1.7) when $n=3$ was proved by Jung [19]. The

Hyers-Ulam-Rassias stability for the equation (1.7) when $n=4$ was also investigated by Chang *et al.* [20].

The general solution and the generalized Hyers-Ulam stability for the quadratic and additive type functional equation

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y) \tag{1.8}$$

for any positive integer a with $a\neq -1,0,1$ was discussed by Jun and Kim [21]. Recently Najati and Moghimi [22] investigated the generalized Hyers-Ulam-Rassias stability for a quadratic and additive type functional equation of the form

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{1.9}$$

Very recently, the authors [23,24] investigated a mixed type functional equation of cubic and quartic type and obtained its general solution. The stability of generalized mixed type functional equations of the form

$$f(x+ky)+f(x-ky)=k^2[f(x+y)+f(x-y)]+2(1-k^2)f(x) \tag{1.10}$$

for fixed integers k with $k\neq 0,\pm 1$ in quasi -Banach spaces was investigated by Gordji and Khodaie [25]. The mixed type functional equation (1.10) is additive, quadratic and cubic. Park *et al.* [26] proved the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation (briefly, AQCQ-functional equation):

$$\begin{aligned} f(x+2y)+f(x-2y)= &\ 4f(x+y)+4f(x-y)-6f(x)+f(2y) \\ & +f(-2y)-4f(y)-4f(-y) \end{aligned} \tag{1.11}$$

in random normed spaces .

Throughout this paper, we assume that X be a vector space over a non-Archimedean field $\mathbb{K}$, (Y,μ,T) is a non-Archimedean random Banach space over $\mathbb{K}$. Based on fixed point methods, we prove the generalized stability of the following equation:

$$\begin{aligned} f(x+ay)+f(x-ay)= &\ a^2[f(x+y)+f(x-y)]+2(1-a^2)f(x)+ \\ & \frac{a^4-a^2}{12}[f(2y)+f(-2y)-4f(y) \\ & -f(-y)] \end{aligned} \tag{1.12}$$

for fixed integers a with $a \neq 0, \pm 1$ in random normed spaces. In the sequel, we shall adopt the usual terminologies, notions, and conventions of the theory of non-Archimedean random normed spaces (non-ARN-spaces) as in [27, 28, 7]. In this paper, the space of all probability distribution functions is denoted by Δ^+. Elements of Δ^+ are functions $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$, such that F is left continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0$, $F(+\infty) = 1$. It is clear that the subset $D^+ := \{F \in \Delta^+ : l^- F(+\infty) = 1\}$, where $l^- f(x) = lim_{t \to x^-} f(t)$ is a subset of Δ^+. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, *i.e.*, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 1, & \text{if } t > 0. \\ 0, & \text{if } t \leq 0. \end{cases}$$

2. Preliminaries

In this section, we give the definition and theorems that are important in this paper.

Theorem 2.1. [29] *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strict contractive mapping with a Lipschitz constant $0 < L < 1$. If there exists a nonnegative integer k such that $d(J^{k+1}x, J^k x) < \infty$ for some $x \in X$, then the followings are true:*

(1) the sequence $\{J^n x\}$ converge to a fixed point x^* for J,

(2) x^* is the unique fixed point of J in $X^* = \{y \in X, d(J^k x, y) \prec \infty\}$,

(3) if $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(Jy, y)$.

Definition 2.2. [7] A mapping $T : [0,1]^2 \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(1) T is commutative and associative;

(2) T is continuous;

(3) $T(a,1) = a$ for all $a \in [0,1]$;

(4) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_p(a,b) = ab$, $T_M(a,b) = min(a,b)$ and $T_L(a,b) = max(a+b-1,0)$ (the Lukasiewicz t-norm). Recall (see [5], [30]) that if T is a t-norm and

$\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, ..., x_n)$ for $n \geq 1$. $T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$. It is known ([31]) that for the Lukasiewicz t-norm the following holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

Definition 2.3. By a non-Archimedean field, we mean a field IK equipped with a function(valuation) $|.| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$;

(2) $|rs| = |r||s|$;

(3) $|r + s| \leq max(|r|, |s|)$ for all $r, s \in \mathbb{K}$.

Clearly,$|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. The function $|.|$ is called the trivial valuation if $|r| = 1$, $\forall r \in \mathbb{K}$, $r \neq 0$, and $|0| = 0$.

Definition 2.4. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $\|.\| : X \to \mathbb{R}$ is non-Archimedean norm (valuation) if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(3) $\|x + y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in X$.

Then, $(X, \|.\|)$ is called a non-Archimedean space. Due to the fact that

$$\|x_m - x_n\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\},$$

in which $n > m$, the sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. In a complete non-Archimedean space, every Cauchy sequence is convergent.

Definition 2.5. A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple (X, μ, T), where X is a linear space over a non-Archimedean field $\mathbb{K}$, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold:

(1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $t \geq 0$ and $\alpha \neq 0$;

(3) $\mu_{x+y}(\max(t,s)) \geq T(\mu_x(t),\mu_y(s))$ for all $x,y \in X$ and $t,s \geq 0$.

It is easy to see that if (3) holds, then (3'): $\mu_{x+y}(t+s) \geq T(\mu_x(t),\mu_y(s))$ for all $x,y \in X$ and $t,s \geq 0$.

Every non-Archimedean normed linear space $(X,\|.\|)$ defines a non-Archimedean RN-space (X,μ,T_M) where $\mu_x(t) = \frac{t}{t+\|x\|}$ for all $t > 0$ and $x \in X$.

Definition 2.6. Let (X,μ,T) be a non-Archimedean RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for all $t > 0$, $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$;

(2) A sequence $\{x_n\}$ in X is said to be Cauchy sequence in X if for each $\varepsilon > 0$ and $t > 0$, there exist a positive integer n_0 such that for all $n \geq n_0$ and $p > 0$, we have $\mu_{x_{n+p}-x_n}(t) > 1-\varepsilon$;

(3) A non-Archimedean RN-space (X,μ,T) is said to be complete (i.e.,(X,μ,T) is called a non-Archimedean random Banach space) if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.7. [7] *If (X,μ,T) is a non-Archimedean RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Theorem 2.8. [31] *Let $f : X \to Y$ be a function satisfying (1.12) for all $x,y \in X$. If f is even then f is quadratic - quartic.*

Theorem 2.9. [31] *Let $f : X \to Y$ be a function satisfying (1.12) for all $x,y \in X$. If f is odd then f is additive - cubic.*

Theorem 2.10. [31] *Let $f : X \to Y$ be a function satisfying (1.12) for all $x,y \in X$ If and only if there exists functions $A : X \to Y$, $B : X^2 \to Y$, $C : X^3 \to Y$ and $D : X^4 \to Y$ such that*

$$f(x) = A(x) + B(x,x) + C(x,x,x) + D(x,x,x,x) \tag{2.1}$$

for all $x \in X$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is is symmetric multi - additive.

3. Stability of Equation (1.12) in non-Archimedean RN-Spaces

In the rest of the paper let $f : X \to Y$ and we define

$$\Delta f(x,y) = f(x+ay)+f(x-ay)-a^2[f(x+y)+f(x-y)]-2(1-a^2)f(x) - \frac{a^4-a^2}{12}[f(2y)+f(-2y)-4f(y)-f(-y)],$$

where a in $\mathbb{Z}-\{-1,0,1\}$.

Now using fixed point approach to the non-Archimedean RN-space under arbitrary t-norm, we prove the stability of generalized mixed type of AQCQ- functional equations $\Delta f(x,y) = 0$.

Theorem 3.1. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y,μ,T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0<\lambda<4$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.1}$$

for all $x,y \in X$ and $t>0$. If $f : X \to Y$ be an even mapping such that

$$\mu_{\Delta f(x,y)}(t) \geq \varphi_{x,y}(t); \tag{3.2}$$

and $f(0) = 0$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.12) and

$$\mu_{f(2x)-16f(x)-Q(x)}(t) \geq \psi_x(\frac{4-\lambda}{4}t), \tag{3.3}$$

where $\psi_x(t) = T_M(\varphi_{0,x}(\frac{a^2t}{12}), \varphi_{x,x}(\frac{(a^2-1)t}{12}), \varphi_{0,2x}(\frac{(a^4-a^2)t}{6}), \varphi_{ax,x}(\frac{(a^4-a^2)t}{12}))$. Moreover

$$Q(x) = \lim_{n\to\infty} \frac{1}{4^n}(f(2^{n+1}x)-16f(2^nx)).$$

Proof. Using the evenness of f and (3.2), we get

$$\mu_{f(x+ay)+f(x-ay)-a^2(f(x+y)+f(x-y))-2(1-a^2)f(x)-\frac{a^4-a^2}{12}(2f(2y)-8f(y))}(t) \geq \varphi_{x,y}(t). \tag{3.4}$$

for all $x \in X$ and $t>0$. Interchanging x and y in (3.4), we obtain

$$\mu_{f(ax+y)+f(ax-y)-a^2(f(x+y)+f(x-y))-2(1-a^2)f(y)-\frac{a^4-a^2}{12}(2f(2x)-8f(x))}(t) \geq \varphi_{y,x}(t). \tag{3.5}$$

for all $x,y \in X$ and $t>0$. Letting $y=0$ in (3.5), we get

$$\mu_{2f(ax)-2a^2f(x)-\frac{a^4-a^2}{12}(2f(2x)-8f(x))}(t) \geq \varphi_{0,x}(t). \tag{3.6}$$

Putting $y = x$ in (3.5), we obtain

$$\mu_{f((a+1)x)+f((a-1)x)-a^2f(2x)-2(1-a^2)f(x)-\frac{a^4-a^2}{12}(2f(2x)-8f(x))}(t) \geq \varphi_{x,x}(t) \tag{3.7}$$

for $x \in X$ and $t > 0$. Replacing x by $2x$ in (3.6) we get

$$\mu_{2f(2ax)-2a^2f(2x)-\frac{a^4-a^2}{12}(2f(4x)-8f(2x))}(t) \geq \varphi_{0,2x}(t). \tag{3.8}$$

for $x \in X$ and $t > 0$. Setting y by ax in (3.5), we obtain

$$\mu_{f(2ax)-a^2(f((1+a)x)+f((1-a)x))-2(1-a^2)f(ax)-\frac{a^4-a^2}{12}(2f(2x)-8f(x))}(t) \geq \varphi_{ax,x}(t). \tag{3.9}$$

Hence, (3.6), (3.7), (3.8) and (3.9) imply that

$$\begin{aligned}\mu_{64f(x)-20f(2x)+f(4x)}(t) \geq \ & T_M(\varphi_{0,x}(\frac{a^2t}{12}), \varphi_{x,x}(\frac{(a^2-1)t}{12}), \varphi_{0,2x}(\frac{(a^4-a^2)t}{6}), \\ & \varphi_{ax,x}(\frac{(a^4-a^2)t}{12})) = \psi_x(t)\end{aligned} \tag{3.10}$$

for $x \in X$ and $t > 0$. Now we put $F(x) = f(2x) - 16f(x)$. It follows that

$$\mu_{f(4x)-16f(2x)-4(f(2x)-16f(x))}(t) \geq \psi_x(t),$$

which implies that

$$\mu_{F(2x)-4F(x)}(t) \geq \psi_x(t). \tag{3.11}$$

Now, we define the set S by $S := \{F : X \to Y\}$ and introduce a generalized metric on S as follows

$$d_\psi(F,G) = \inf\{\varepsilon \in \mathbb{R}_+ : \mu_{F(x)-G(x)}(\varepsilon t) \geq \psi_x(t), \forall x \in X, \forall t > 0\}. \tag{3.12}$$

Then, it is easy to verify that (S, d_ψ) is complete (see [30]). We define an operator $J : S \to S$ by $JL(x) = \frac{L(2x)}{4}$, for all $x \in X$. Let $F, G \in S$ and $\varepsilon \in \mathbb{R}_+$ be an arbitrary constant with $d_\psi(F,G) \leq \varepsilon$, that is,

$$\mu_{F(x)-G(x)}(\varepsilon t) \geq \psi_x(t) \tag{3.13}$$

for all $x \in X$ and $t > 0$. Then

$$\mu_{JF(x)-JG(x)}(\frac{\lambda\varepsilon t}{4}) = \mu_{\frac{F(2x)}{4}-\frac{G(2x)}{4}}(\frac{\lambda\varepsilon t}{4}) = \mu_{F(2x)-G(2x)}(\lambda\varepsilon t) \geq \psi_{2x,2x}(\lambda t) \geq \psi_x(t) \tag{3.14}$$

for all $x \in X$ and $t > 0$, that is, $d_\psi(JF, JG) \leq \frac{\lambda\varepsilon}{4}$. We hence conclude that $d_\psi(JF, JG) \leq \frac{\lambda}{4} d_\psi(F, G)$ for any $F, G \in S$. As $0 < \lambda < 4$, then operator J is strictly contractive. It follows from (3.11) that

$$\mu_{JF(x)-F(x)}(\frac{\varepsilon t}{4}) = \mu_{\frac{F(2x)}{4}-F(x)}(\frac{\varepsilon t}{4}) = \mu_{F(2x)-4F(x)}(\varepsilon t) \geq \psi_x(t) \tag{3.15}$$

for all $x \in X$ and $t > 0$, that is, $d_\psi(JF, F) \leq \frac{\varepsilon}{4} < \infty$. Using Theorem 2.1, we deduce existence of a fixed point of J, that is, the existence of mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J such that $\lim_{n\to\infty} d_\psi(J^n F, Q) = 0$. This implies the equality

$$Q(x) = \lim_{n\to\infty} J^n F(x) = \lim_{n\to\infty} \frac{F(2^n x)}{4^n} = \lim_{n\to\infty} \frac{1}{4^n}(f(2^{n+1}x) - 16 f(2^n x))$$

and $Q(2x) = 4Q(x)$ for all $x \in X$. Also Q is the unique fixed point of J on the set

$$S^* = \{G \in S : d_\psi(F, G) < \infty\}.$$

(2) $d_\psi(F, Q) \leq \frac{1}{1-L} d_\psi(JF, F)$ implies the inequality $d_\psi(F, Q) \leq \frac{1}{1-\frac{\lambda}{4}}$. Then

$$\mu_{F(x)-Q(x)}(\frac{4t}{4-\lambda}) \geq \psi_x(t),$$

which implies that

$$\mu_{F(x)-Q(x)}(t) \geq \psi_x(\frac{4-\lambda}{4} t). \tag{3.16}$$

It follows from (3.1) and (3.2) that

$$\mu_{\Delta Q(x,y)}(t) \geq \lim_{n\to\infty} T_M(\varphi_{x,y}(\frac{4^n t}{\lambda^{n+1}}), \varphi_{x,y}(\frac{4^n t}{16\lambda^n})) = 1.$$

Hence $\Delta Q(x, y) = 0$ and $\mu_{f(2x)-16f(x)-Q(x)}(t) \geq \psi_x(\frac{4-\lambda}{4} t)$ for all $x \in X$ and $t > 0$. This completes the proof.

Theorem 3.2. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x, y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 16$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.17}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ be an even mapping such that

$$\mu_{\Delta f(x,y)}(t) \geq \varphi_{x,y}(t) \tag{3.18}$$

and $f(0) = 0$, then there exists a unique quartic mapping $D : X \to Y$ satisfying (1.12) and

$$\mu_{f(2x)-4f(x)-D(x)}(t) \geq \psi_x(\frac{16-\lambda}{16}t), \tag{3.19}$$

where $\psi_x(t) = T_M(\varphi_{0,x}(\frac{a^2t}{12}), \varphi_{x,x}(\frac{(a^2-1)t}{12}), \varphi_{0,2x}(\frac{(a^4-a^2)t}{6}), \varphi_{ax,x}(\frac{(a^4-a^2)t}{12}))$, Moreover

$$D(x) = \lim_{n\to\infty} \frac{1}{16^n}(f(2^{n+1}x) - 4f(2^n x)).$$

Proof. Using Theorem 3.1, we have $\mu_{f(4x)-4f(2x)-16(f(2x)-4f(x))}(t) \geq \psi_x(t)$. It follows that

$$\mu_{G(2x)-16G(x)}(t) \geq \psi_x(t) \tag{3.20}$$

where $G(x) = f(2x) - 4f(x)$, for all $x \in X$ and $t > 0$. Now, we define the set S by $S := \{G : X \to Y\}$ and introduce a generalized metric on S as follows

$$d_\psi(F,G) = \inf\{\varepsilon \in \mathbb{R}_+ : \mu_{F(x)-G(x)}(\varepsilon t) \geq \psi_x(t), \forall x \in X, \forall t > 0\}. \tag{3.21}$$

Then, it is easy to verify that (S, d_ψ) is complete (see [30]). We define an operator $J : S \to S$ by $JL(x) = \frac{L(2x)}{16}$, for all $x \in X$. Let $F, G \in S$ and $\varepsilon \in \mathbb{R}_+$ be an arbitrary constant with $d_\psi(F,G) \leq \varepsilon$, that is,

$$\mu_{F(x)-G(x)}(\varepsilon t) \geq \psi_x(t) \tag{3.22}$$

for all $x \in X$ and $t > 0$. Then

$$\mu_{JF(x)-JG(x)}(\frac{\lambda\varepsilon t}{16}) = \mu_{\frac{F(2x)}{16}-\frac{G(2x)}{16}}(\frac{\lambda\varepsilon t}{16}) = \mu_{F(2x)-G(2x)}(\lambda\varepsilon t) \geq \psi_{2x,2x}(\lambda t) \geq \psi_x(t) \tag{3.23}$$

for all $x \in X$ and $t > 0$, that is, $d_\psi(JF, JG) \leq \frac{\lambda\varepsilon}{16}$. We hence conclude that $d_\psi(JF, JG) \leq \frac{\lambda}{16} d_\psi(F,G)$ for any $F, G \in S$. As $0 < \lambda < 16$, then operator J is strictly contractive. It follows from (3.20) that

$$\mu_{JG(x)-G(x)}(\frac{\varepsilon t}{16}) = \mu_{\frac{G(2x)}{16}-G(x)}(\frac{\varepsilon t}{16}) = \mu_{G(2x)-16G(x)}(\varepsilon t) \geq \psi_x(t) \tag{3.24}$$

for all $x \in X$ and $t > 0$, that is, $d_\psi(JG,G) \leq \frac{\varepsilon}{16} < \infty$. By Theorem 2.1, we deduce existence of a fixed point of J, that is, the existence of mapping $D : X \to Y$ satisfying the following:
(1) D is a fixed point of J such that $\lim_{n\to\infty} d_\psi(J^n G, D) = 0$ This implies the equality

$$D(x) = \lim_{n\to\infty} J^n G(x) = \lim_{n\to\infty} \frac{F(2^n x)}{16^n} = \lim_{n\to\infty} \frac{1}{16^n}(f(2^{n+1}x) - 4f(2^n x))$$

and $D(2x) = 16D(x)$ for all $x \in X$. Also D is the unique fixed point of J on the set

$$M := \{G \in S : d_\psi(F,G) < \infty\}.$$

(2) $d_\psi(G,D) \leq \frac{1}{1-L} d_\psi(JG,G)$ implies the inequality $d_\psi(G,D) \leq \frac{1}{1-\frac{\lambda}{16}}$. Then

$$\mu_{G(x)-D(x)}\left(\frac{16t}{16-\lambda}\right) \geq \psi_x(t),$$

which implies that

$$\mu_{G(x)-D(x)}(t) \geq \psi_x\left(\frac{16-\lambda}{16}t\right), \tag{3.25}$$

It follows from (3.18) and (3.19) that

$$\begin{aligned}\mu_{\Delta D(x,y)}(t) &\geq \lim_{n\to\infty} T_M\left(\varphi_{2^{n+1}x,2^{n+1}y}(16^n t), \varphi_{2^n x,2^n y)}\left(\frac{16^n t}{4}\right)\right)\\ &\geq \lim_{n\to\infty} T_M\left(\varphi_{x,y}\left(\frac{16^n t}{\lambda^{n+1}}\right), \varphi_{x,y}\left(\frac{16^n t}{4\lambda^n}\right)\right) = 1.\end{aligned}$$

Then $\Delta D(x,y) = 0$ and $\mu_{f(2x)-4f(x)-D(x)}(t) \geq \psi_x(\frac{16-\lambda}{16}t)$ for all $x \in X$ and $t > 0$. This completes the proof.

Theorem 3.3. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y,μ,T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 4$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.26}$$

for all $x,y \in X$ and $t > 0$. If $f : X \to Y$ be an even mapping such that

$$\mu_{\Delta f(x,y)}(t) \geq \varphi_{x,y}(t); \tag{3.27}$$

and $f(0) = 0$, then there exists a unique quadratic mapping $Q : X \to Y$ and a unique quartic mapping $D : X \to Y$ satisfying (1.12) and

$$\mu_{f(x)-Q(x)-D(x)}(t) \geq T_M\left(\psi_x\left(\frac{3(16-\lambda)}{4}t\right), \psi_x(3(4-\lambda)t)\right) \tag{3.28}$$

where $\psi_x(t) = T_M(\varphi_{0,x}(\frac{a^2t}{12}), \varphi_{x,x}(\frac{(a^2-1)t}{12}), \varphi_{0,2x}(\frac{(a^4-a^2)t}{6}), \varphi_{ax,x}(\frac{(a^4-a^2)t}{12}))$.

Proof. Using Theorems 3.1 and 3.2, we see that there exists a unique quadratic function $Q_1 : X \to Y$ and a unique quartic function $D_1 : X \to Y$ such that $\mu_{f(2x)-4f(x)-D_1(x)}(t) \geq \psi_x(\frac{16-\lambda}{16}t)$ and $\mu_{f(2x)-16f(x)-Q_1(x)}(t) \geq \psi_x(\frac{4-\lambda}{4}t)$. Then

$$\mu_{f(2x)-4f(x)-D_1(x)-[f(2x)-16f(x)-Q_1(x)]}(t) \geq T_M(\mu_{f(2x)-4f(x)-D_1(x)}(t), \mu_{f(2x)-16f(x)-Q_1(x)}(t)),$$

which implies that $\mu_{12f(x)-D_1(x)+Q_1(x)}(t) \geq T_M(\psi_x(\frac{16-\lambda}{16}t), \psi_x(\frac{4-\lambda}{4}t))$. Hence, we have

$$\mu_{f(x)-\frac{1}{12}D_1(x)+\frac{1}{12}Q_1(x)}(\frac{1}{12}t) \geq T_M(\psi_x(\frac{16-\lambda}{16}t), \psi_x(\frac{4-\lambda}{4}t)).$$

It follows that $\mu_{f(x)-Q(x)-D(x)}(t) \geq T_M(\psi_x(\frac{3(16-\lambda)}{4}t), \psi_x(3(4-\lambda)t))$, where $Q(x) = -\frac{1}{12}Q_1(x)$ and $D(x) = \frac{1}{12}D_1(x)$.

Theorem 3.4. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 2$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.29}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ be an odd mapping with $f(0) = 0$ and satisfying

$$\mu_{\Delta f(x,y)}(t) \geq \varphi_{x,y}(t) \tag{3.30}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (1.12) and

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \beta_x(\frac{2-\lambda}{2}t), \tag{3.31}$$

where $\beta_x(t) = T_M(\phi_x(\frac{t}{2}), \alpha_x(t))$,

$$\phi_x(t) = T_M(\varphi_{x,x}(\frac{a^2t}{2}), \varphi_{2x,x}((a^2-1)t), \varphi_{x,2x}((a^4-a^2)t),$$
$$T_M(\varphi_{(1+a)x,x}((a^4-a^2)t), \varphi_{(1-a)x,x}((a^4-a^2)t))),$$

and

$$\alpha_x(t) = T_M(\varphi_{2x,2x}((a^2-1)t), \varphi_{x,2x}(\frac{a^2t}{2}), \varphi_{x,x}((a^4-a^2)t), \varphi_{x,3x}((a^4-a^2)t),$$
$$T_M(\varphi_{(1+2a)x,x}((a^4-a^2)t), \varphi_{(1-2a)x,x}((a^4-a^2)t))),$$

Moreover $A(x) = \lim_{n\to\infty} \frac{1}{2^n}(f(2^{n+1}x) - 8f(2^n x))$.

Proof. Using the oddness of f and (3.30), we get

$$\mu_{f(x+ay)+f(x-ay)-a^2(f(x+y)+f(x-y))-2(1-a^2)f(x)}(t) \geq \varphi_{x,y}(t). \tag{3.32}$$

Replacing y by x in (3.32), we obtain

$$\mu_{f((1+a)x)+f((1-a)x)-a^2f(2x)-2(1-a^2)f(x)}(t) \geq \varphi_{x,x}(t). \tag{3.33}$$

Replacing x by $2x$ in (3.33), we obtain

$$\mu_{f(2(1+a)x)+f(2(1-a)x)-a^2f(4x)-2(1-a^2)f(2x)}(t) \geq \varphi_{2x,2x}(t). \tag{3.34}$$

Again replacing (x,y) by $(2x,x)$ in (3.32), we get

$$\mu_{f((2+a)x)+f((2-a)x)-a^2(f(3x)+f(x))-2(1-a^2)f(2x)}(t) \geq \varphi_{2x,x}(t) \tag{3.35}$$

for all $x \in X$ and $t > 0$. Replacing y by $2x$ in (3.32), we obtain

$$\mu_{f((1+2a)x)+f((1-2a)x)-a^2(f(3x)-f(x))-2(1-a^2)f(x)}(t) \geq \varphi_{x,2x}(t) \tag{3.36}$$

for all $x \in X$ and $t > 0$. Replacing y by $3x$ in (3.32), we get

$$\mu_{f((1+3a)x)+f((1-3a)x)-a^2(f(4x)-f(2x))-2(1-a^2)f(x)}(t) \geq \varphi_{x,3x}(t) \tag{3.37}$$

for all $x \in X$ and $t > 0$. Replacing (x,y) by $((1+a)x,x)$ in (3.32), we obtain

$$\mu_{f((1+2a)x)+f(x)-a^2(f((2+a)x)+f(ax))-2(1-a^2)f((1+a)x)}(t) \geq \varphi_{(1+a)x,x}(t) \tag{3.38}$$

for all $x \in X$ and $t > 0$. Again replacing (x,y) by $((1-a)x,x)$ in (3.32), we obtain

$$\mu_{f((1-2a)x)+f(x)-a^2(f((2-a)x)-f(ax))-2(1-a^2)f((1-a)x)}(t) \geq \varphi_{(1-a)x,x}(t) \tag{3.39}$$

for all $x \in X$ and $t > 0$. By (3.38) and (3.39), we get

$$\begin{aligned} &\mu_{f((1+2a)x)+f((1-2a)x)+2f(x)-a^2(f((2+a)x)+f((2-a)x))-2(1-a^2)(f((1+a)x)+f((1-a)x))}(t) \\ &\qquad \geq T_M(\varphi_{(1+a)x,x}(t), \varphi_{(1-a)x,x}(t)) \end{aligned} \tag{3.40}$$

for all $x \in X$ and $t > 0$. Replacing (x,y) by $((1-2a)x,x)$ in (3.32), we obtain

$$\mu_{f((1-a)x)+f((1-3a)x)-a^2(f((2-2a)x)-f(2ax))-2(1-a^2)f((1-2a)x)}(t) \geq \varphi_{(1-2a)x,x}(t). \tag{3.41}$$

Again replacing (x,y) by $((1+2a)x,x)$ in (3.32), we get

$$\mu_{f((1+3a)x)+f((1+a)x)-a^2(f((2+2a)x)+f(2ax))-2(1-a^2)f((1+2a)x)}(t) \geq \varphi_{(1+2a)x,x}(t) \tag{3.42}$$

By (3.41) and (3.42), we obtain

$$\begin{aligned}&\mu_{f((1+3a)x)+f((1-3a)x)+f((1+a)x)+f((1-a)x)-a^2(f((2+2a)x)+f((2-2a)x))}\\&\quad_{-2(1-a^2)(f((1+2a)x)+f((1-2a)x))}(t) \geq T_M(\varphi_{(1+a)x,x}(t),\varphi_{(1-a)x,x}(t)).\end{aligned} \tag{3.43}$$

Using (3.33), (3.35), (3.36), and (3.40), we see that

$$\begin{aligned}\mu_{[4f(2x)-5f(x)-f(3x)]}(t) &\geq T_M(\varphi_{x,x}(\frac{a^2t}{2}),\varphi_{2x,x}((a^2-1)t),\varphi_{x,2x}((a^4-a^2)t),\\&\quad T_M(\varphi_{(1+a)x,x}((a^4-a^2)t),\varphi_{(1-a)x,x}((a^4-a^2)t)))\\&= \phi_x(t).\end{aligned} \tag{3.44}$$

It follows from (3.34), (3.36), (3.33), (3.37), and (3.43) that

$$\begin{aligned}\mu_{[f(4x)-2f(2x)-2f(3x)+6f(x)]}(t) &\geq T_M(\varphi_{2x,2x}((a^2-1)t),\varphi_{x,2x}(\frac{a^2t}{2}),\varphi_{x,x}((a^4-a^2)t),\\&\quad \varphi_{x,3x}((a^4-a^2)t),T_M(\varphi_{(1+2a)x,x}((a^4-a^2)t),\\&\quad \varphi_{(1-2a)x,x}((a^4-a^2)t)))\\&= \alpha_x(t)\end{aligned} \tag{3.45}$$

for all $x \in X$ and $t > 0$. From (3.44) and (3.45) we have

$$\begin{aligned}&\mu_{[f(4x)-10f(2x)+16f(x)]}(t) = \mu_{2f(3x)-8f(2x)+10f(x)+f(4x)-2f(3x)-2f(2x)+6f(x)}(t)\\&\geq T_M(\mu_{2f(3x)-8f(2x)+10f(x)}(t),\mu_{f(4x)-2f(3x)-2f(2x)+6f(x)}(t))\\&\geq T_M(\phi_x(\frac{t}{2}),\alpha_x(t)) = \beta_x(t),\end{aligned}$$

which implies that

$$\mu_{[f(4x)-8f(2x)-2(f(2x)-8f(x))]}(t) \geq \beta_x(t). \tag{3.46}$$

Putting $H(x) = f(2x) - 8f(x)$, we see that

$$\mu_{H(2x)-2H(x)}(t) \geq \beta_x(t) \tag{3.47}$$

for all $x \in X$ and $t > 0$. New we define the set P by $P := \{H : X \to Y\}$ and introduce a generalized metric on P as follows

$$d_\beta(F,G) = \inf\{\varepsilon \in \mathbb{R}_+ : \mu_{F(x)-G(x)}(\varepsilon t) \geq \beta_x(t), \forall x \in X, \forall t > 0\}. \tag{3.48}$$

Then, it is easy to verify that (P, d_β) is complete (see [30]). We define an operator $J : P \to P$ by $JL(x) = \frac{L(2x)}{2}$, for all $x \in X$. Let $F, H \in P$ and $\varepsilon \in \mathbb{R}_+$ be an arbitrary constant with $d_\beta(F,H) \leq \varepsilon$, that is,

$$\mu_{F(x)-H(x)}(\varepsilon t) \geq \beta_x(t) \tag{3.49}$$

for all $x \in X$ and $t > 0$. Then

$$\mu_{JF(x)-JH(x)}(\frac{\lambda\varepsilon t}{2}) = \mu_{\frac{F(2x)}{2}-\frac{H(2x)}{2}}(\frac{\lambda\varepsilon t}{2}) = \mu_{F(2x)-H(2x)}(\lambda\varepsilon t) \geq \beta_{2x,2x}(\lambda t) \geq \beta_x(t) \tag{3.50}$$

for all $x \in X$ and $t > 0$, that is, $d_\beta(JF,JH) \leq \frac{\lambda\varepsilon}{2}$. We hence conclude that $d_\beta(JF,JH) \leq \frac{\lambda}{2} d_\beta(F,H)$ for any $F, H \in P$. As $0 < \lambda < 2$, then operator J is strictly contractive. It follows from (3.47) that

$$\mu_{JH(x)-H(x)}(\frac{\varepsilon t}{2}) = \mu_{\frac{H(2x)}{2}-H(x)}(\frac{\varepsilon t}{2}) = \mu_{H(2x)-2H(x)}(\varepsilon t) \geq \beta_x(t) \tag{3.51}$$

for all $x \in X$ and $t > 0$, that is, $d_\beta(JH,H) \leq \frac{\varepsilon}{2} < \infty$. Using Theorem 2.1, we deduce existence of a fixed point of J, that is, the existence of mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J such that $\lim_{n\to\infty} d_\beta(J^n H, A) = 0$ This implies the equality

$$A(x) = \lim_{n\to\infty} J^n H(x) = \lim_{n\to\infty} \frac{H(2^n x)}{2^n} = \lim_{n\to\infty} \frac{1}{2^n}(f(2^{n+1}x) - 8f(2^n x))$$

and $A(2x) = 2A(x)$ for all $x \in X$. Also A is the unique fixed point of J on the set

$$P^* := \{H \in S : d_\beta(F,H) < \infty\}.$$

(2) $d_\beta(H,A) \leq \frac{1}{1-L} d_\beta(JH,H)$ implies the inequality $d_\beta(H,A) \leq \frac{1}{1-\frac{\lambda}{2}}$. Then

$$\mu_{H(x)-A(x)}(\frac{2t}{2-\lambda}) \geq \beta_x(t),$$

which implies that

$$\mu_{H(x)-A(x)}(t) \geq \beta_x(\frac{2-\lambda}{2}t). \tag{3.52}$$

It follows from (3.29) and (3.30) that

$$\mu_{\Delta A(x,y)}(t) \geq \lim_{n\to\infty} T_M(\varphi_{x,y}(\frac{2^n t}{\lambda^{n+1}}), \varphi_{x,y}(\frac{2^n t}{8\lambda^n})) = 1.$$

Then $\Delta A(x,y) = 0$ and $\mu_{f(2x)-8f(x)-A(x)}(t) \geq \beta_x(\frac{2-\lambda}{2}t)$ for all $x \in X$ and $t > 0$. This completes the proof.

Theorem 3.5. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 8$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.53}$$

for all $x,y \in X$ and $t > 0$. If $f : X \to Y$ be an odd mapping with $f(0) = 0$ and satisfying

$$\mu_{\Delta f(x,y)}(t) \geq \varphi_{x,y}(t) \tag{3.54}$$

for all $x,y \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (1.12) and

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \beta_x(\frac{8-\lambda}{8}t), \tag{3.55}$$

where $\beta_x(t) = T_M(\phi_x(\frac{t}{2}), \alpha_x(t))$,

$$\phi_x(t) = T_M(\varphi_{x,x}(\frac{a^2 t}{2}), \varphi_{2x,x}((a^2-1)t), \varphi_{x,2x}((a^4-a^2)t),$$
$$T_M(\varphi_{(1+a)x,x}((a^4-a^2)t), \varphi_{(1-a)x,x}((a^4-a^2)t))),$$

and

$$\alpha_x(t) = T_M(\varphi_{2x,2x}((a^2-1)t), \varphi_{x,2x}(\frac{a^2 t}{2}), \varphi_{x,x}((a^4-a^2)t), \varphi_{x,3x}((a^4-a^2)t),$$
$$T_M(\varphi_{(1+2a)x,x}((a^4-a^2)t), \varphi_{(1-2a)x,x}((a^4-a^2)t))),$$

Moreover $C(x) = lim_{n\to\infty} \frac{1}{8^n}(f(2^{n+1}x) - 2f(2^n x))$.

Proof. Using Theorem 3.4, we have

$$\mu_{[f(4x)-2f(2x)-8(f(2x)-2f(x))]}(t) \geq \beta_x(t). \tag{3.56}$$

Putting $K(x) = f(2x) - 2f(x)$, we find that

$$\mu_{[K(2x)-8K(x)]}(t) \geq \beta_x(t) \tag{3.57}$$

for all $x \in X$ and $t > 0$. New we define the set N by $N := \{K : X \to Y\}$ and introduce a generalized metric on N as follows

$$d_\beta(F,G) = inf\{\varepsilon \in \mathbb{R}_+ : \mu_{F(x)-G(x)}(\varepsilon t) \geq \beta_x(t), \forall x \in X, \forall t > 0\}. \tag{3.58}$$

Then, it is easy to verify that (N, d_β) is complete (see [30]). We define an operator $J : N \to N$ by $JL(x) = \frac{L(2x)}{8}$, for all $x \in X$. Let $F, K \in N$ and $\varepsilon \in \mathbb{R}_+$ be an arbitrary constant with $d_\beta(F,K) \leq \varepsilon$, that is,

$$\mu_{F(x)-K(x)}(\varepsilon t) \geq \beta_x(t) \tag{3.59}$$

for all $x \in X$ and $t > 0$. Then

$$\mu_{JF(x)-JK(x)}(\frac{\lambda \varepsilon t}{8}) = \mu_{\frac{F(2x)}{8}-\frac{K(2x)}{8}}(\frac{\lambda \varepsilon t}{8}) = \mu_{F(2x)-K(2x)}(\lambda \varepsilon t) \geq \beta_{2x,2x}(\lambda t) \geq \beta_x(t) \tag{3.60}$$

for all $x \in X$ and $t > 0$, that is, $d_\beta(JF,JK) \leq \frac{\lambda\varepsilon}{8}$. We hence conclude that $d_\beta(JF,JK) \leq \frac{\lambda}{8} d_\beta(F,K)$ for any $F, K \in N$. As $0 < \lambda < 8$, then operator J is strictly contractive. It follows from (3.57) that

$$\mu_{JK(x)-K(x)}(\frac{\varepsilon t}{8}) = \mu_{\frac{K(2x)}{8}-K(x)}(\frac{\varepsilon t}{8}) = \mu_{K(2x)-8K(x)}(\varepsilon t) \geq \beta_x(t) \tag{3.61}$$

for all $x \in X$ and $t > 0$, that is, $d_\beta(JK,K) \leq \frac{\varepsilon}{8} < \infty$. Using Theorem 2.1, we deduce existence of a fixed point of J, that is, the existence of mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J such that $\lim_{n\to\infty} d_\beta(J^n K, C) = 0$. This implies the equality

$$C(x) = \lim_{n\to\infty} J^n K(x) = \lim_{n\to\infty} \frac{K(2^n x)}{2^n} = \lim_{n\to\infty} \frac{1}{8^n}(f(2^{n+1}x) - 2f(2^n x))$$

and $C(2x) = 8C(x)$ for all $x \in X$. Also C is the unique fixed point of J on the set

$$N^* := \{K \in S : d_\beta(F,K) < \infty\}.$$

(2) $d_\beta(K,C) \leq \frac{1}{1-L} d_\beta(JK,K)$ implies the inequality $d_\beta(K,C) \leq \frac{1}{1-\frac{\lambda}{8}}$ Then

$$\mu_{K(x)-C(x)}(\frac{8t}{8-\lambda}) \geq \beta_x(t),$$

which implies that

$$\mu_{K(x)-C(x)}(t) \geq \beta_x(\frac{8-\lambda}{8} t). \tag{3.62}$$

It follows from (3.53) and (3.54) that

$$\mu_{\Delta C(x,y)}(t) \geq \lim_{n\to\infty} T_M(\varphi_{x,y}(\frac{8^n t}{\lambda^{n+1}}), \varphi_{x,y}(\frac{8^n t}{2\lambda^n})) = 1.$$

Then $\Delta C(x,y) = 0$ and $\mu_{f(2x)-2f(x)-C(x)}(t) \geq \beta_x(\frac{8-\lambda}{8}t)$ for all $x \in X$ and $t > 0$. This completes the proof.

Theorem 3.6. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 2$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.63}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ be an odd mapping with $f(0) = 0$ and satisfying

$$\mu_{\Delta f(x,y)}(t) \geq \varphi_{x,y}(t); \tag{3.64}$$

Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ satisfying (1.12) and

$$\mu_{f(x)-A(x)-C(x)}(t) \geq T_M(\beta_x(\frac{3(8-\lambda)}{4}t), \beta_x(3(2-\lambda)t)), \tag{3.65}$$

where $\beta_x(t) = T_M(\phi_x(\frac{t}{2}), \alpha_x(t))$,

$$\phi_x(t) = T_M(\varphi_{x,x}(\frac{a^2 t}{2}), \varphi_{2x,x}((a^2-1)t), \varphi_{x,2x}((a^4-a^2)t),$$
$$T_M(\varphi_{(1+a)x,x}((a^4-a^2)t), \varphi_{(1-a)x,x}((a^4-a^2)t))),$$

and

$$\alpha_x(t) = T_M(\varphi_{2x,2x}((a^2-1)t), \varphi_{x,2x}(\frac{a^2 t}{2}), \varphi_{x,x}((a^4-a^2)t), \varphi_{x,3x}((a^4-a^2)t),$$
$$T_M(\varphi_{(1+2a)x,x}((a^4-a^2)t), \varphi_{(1-2a)x,x}((a^4-a^2)t))).$$

Proof. Using Theorems 3.4 and 3.5, there exists a unique additive function $A_1 : X \to Y$ and a unique cubic function $C_1 : X \to Y$ such that $\mu_{f(2x)-8f(x)-A_1(x)}(t) \geq \beta_x(\frac{2-\lambda}{2}t)$ and

$$\mu_{f(2x)-2f(x)-C_1(x)}(t) \geq \beta_x(\frac{8-\lambda}{8}t)$$

for all $x \in X$ and $t > 0$. Then

$$\mu_{f(2x)-2f(x)-C_1(x)-[f(2x)-8f(x)-A_1(x)]}(t) \geq T_M(\mu_{f(2x)-2f(x)-C_1(x)}(t), \mu_{f(2x)-8f(x)-A_1(x)}(t)),$$

which implies that $\mu_{6f(x)-C_1(x)+A_1(x)}(t) \geq T_M(\beta_x(\frac{8-\lambda}{8}t), \beta_x(\frac{2-\lambda}{2}t))$. Hence

$$\mu_{f(x)-\frac{1}{6}C_1(x)+\frac{1}{6}A_1(x)}(\frac{1}{6}t) \geq T_M(\beta_x(\frac{8-\lambda}{8}t), \beta_x(\frac{2-\lambda}{2}t)).$$

It follows that $\mu_{f(x)-A(x)-C(x)}(t) \geq T_M(\beta_x(\frac{3(8-\lambda)}{4}t), \beta_x(3(2-\lambda)t))$ for all $x \in X$ and $t > 0$, where $A(x) = -\frac{1}{6}A_1(x)$ and $C(x) = \frac{1}{6}C_1(x)$.

Theorem 3.7. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y, μ, T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0 < \lambda < 2$*

$$\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t), \tag{3.66}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying

$$\mu_{\Delta f(x,y)}(t) \geq \varphi_{x,y}(t). \tag{3.67}$$

Then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q : X \to Y$, a unique cubic mapping $C : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that

$$\begin{aligned} \mu_{f(x)-A(x)-Q(x)-C(x)-D(x)}(t) \geq T_M\{&T_M(\widetilde{\psi}_x(\frac{3(16-\lambda)}{4}t), \widetilde{\psi}_x(3(4-\lambda)t)), \\ &T_M(\widetilde{\beta}_x(\frac{3(8-\lambda)}{4}t), \widetilde{\beta}_x(3(2-\lambda)t))\}, \end{aligned} \tag{3.68}$$

where

$$\widetilde{\psi}_x(t) = T_M(\widetilde{\varphi}_{0,x}(\frac{a^2t}{12}), \widetilde{\varphi}_{x,x}(\frac{(a^2-1)t}{12}), \widetilde{\varphi}_{0,2x}(\frac{(a^4-a^2)t}{6}), \widetilde{\varphi}_{ax,x}(\frac{(a^4-a^2)t}{12})),$$

$$\widetilde{\beta}_x(t) = T_M(\widetilde{\phi}_x(\frac{t}{2}), \widetilde{\alpha}_x(t)),$$

$$\begin{aligned} \widetilde{\phi}_x(t) = T_M(&\widetilde{\varphi}_{x,x}(\frac{a^2t}{2}), \widetilde{\varphi}_{2x,x}((a^2-1)t), \widetilde{\varphi}_{x,2x}((a^4-a^2)t), \\ &T_M(\widetilde{\varphi}_{(1+a)x,x}((a^4-a^2)t), \widetilde{\varphi}_{(1-a)x,x}((a^4-a^2)t))), \end{aligned}$$

$$\begin{aligned} \widetilde{\alpha}_x(t) = T_M(&\widetilde{\varphi}_{2x,2x}((a^2-1)t), \widetilde{\varphi}_{x,2x}(\frac{a^2t}{2}), \widetilde{\varphi}_{x,x}((a^4-a^2)t), \widetilde{\varphi}_{x,3x}((a^4-a^2)t), \\ &T_M(\widetilde{\varphi}_{(1+2a)x,x}((a^4-a^2)t), \widetilde{\varphi}_{(1-2a)x,x}((a^4-a^2)t))), \end{aligned}$$

and $\widetilde{\varphi}_{x,y}(t) = T_M(\varphi_{x,y}(2t), \varphi_{-x,-y}(2t))$ *for all* $x \in X$ *and* $t > 0$.

Proof. Let $f^e(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $f^e(0) = 0$, $f^e(-x) = f^e(x)$ and $2\Delta f^e(x,y) = \Delta f(x,y) + \Delta f(-x,-y)$. Hence

$$\mu_{2\Delta f^e(x,y)}(t) = \mu_{\Delta f(x,y)+\Delta f(-x,-y)}(t) \geq T_M(\mu_{\Delta f(x,y)}(t), \mu_{\Delta f(-x,-y)}(t)),$$

which implies that

$$\mu_{\Delta f^e(x,y)}(t) \geq T_M(\mu_{\Delta f(x,y)}(2t), \mu_{\Delta f(-x,-y)}(2t)) \geq T_M(\varphi_{x,y}(2t), \varphi_{-x,-y}(2t)) =: \widetilde{\varphi}_{x,y}(t).$$

Using Theorem 3.3 there exist a unique quadratic mapping $Q : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that

$$\mu_{f^e(x)-Q(x)-D(x)}(t) \geq T_M(\widetilde{\psi}_x(\frac{3(16-\lambda)}{4}t), \widetilde{\psi}_x(3(4-\lambda)t)) \tag{3.69}$$

where $\widetilde{\psi}_x(t) = T_M(\widetilde{\varphi}_{0,x}(\frac{a^2t}{12}), \widetilde{\varphi}_{x,x}(\frac{(a^2-1)t}{12}), \widetilde{\varphi}_{0,2x}(\frac{(a^4-a^2)t}{6}), \widetilde{\varphi}_{ax,x}(\frac{(a^4-a^2)t}{12}))$. for all $x \in X$ and $t > 0$. Again $f^o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $f^o(0) = 0$, $f^o(-x) = -f^o(x)$ and $2\Delta f^o(x,y) = \Delta f(x,y) - \Delta f(-x,-y)$. Hence

$$\mu_{2\Delta f^o(x,y)}(t) = \mu_{\Delta f(x,y)-\Delta f(-x,-y)}(t) \geq T_M(\mu_{\Delta f(x,y)}(t), \mu_{\Delta f(-x,-y)}(t)).$$

Using Theorem 3.6, we see that there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f^o(x)-A(x)-C(x)}(t) \geq T_M(\widetilde{\beta}_x(\frac{3(8-\lambda)}{4}t), \widetilde{\beta}_x(3(2-\lambda)t)), \tag{3.70}$$

where $\widetilde{\beta}_x(t) = T_M(\widetilde{\phi}_x(\frac{t}{2}), \widetilde{\alpha}_x(t))$,

$$\widetilde{\phi}_x(t) = T_M(\widetilde{\varphi}_{x,x}(\frac{a^2t}{2}), \widetilde{\varphi}_{2x,x}((a^2-1)t), \widetilde{\varphi}_{x,2x}((a^4-a^2)t),$$
$$T_M(\widetilde{\varphi}_{(1+a)x,x}((a^4-a^2)t), \widetilde{\varphi}_{(1-a)x,x}((a^4-a^2)t))),$$

and

$$\widetilde{\alpha}_x(t) = T_M(\widetilde{\varphi}_{2x,2x}((a^2-1)t), \widetilde{\varphi}_{x,2x}(\frac{a^2t}{2}), \widetilde{\varphi}_{x,x}((a^4-a^2)t), \widetilde{\varphi}_{x,3x}((a^4-a^2)t),$$
$$T_M(\widetilde{\varphi}_{(1+2a)x,x}((a^4-a^2)t), \widetilde{\varphi}_{(1-2a)x,x}((a^4-a^2)t)))$$

for all $x \in X$ and $t > 0$. Using (3.69) and (3.70), we conclude the desired conclusion immediately.

In the following, we give a corollary which was obtained in [32].

Corollary 3.8. *Let $\mathbb{K}$ be a non-Archimedean field, X be a vector space over $\mathbb{K}$ and (Y,μ,T_M) be a non-Archimedean random Banach space over $\mathbb{K}$. Let $\varphi : X^2 \to D^+$ ($\varphi(x,y)$ is denoted by $\varphi_{x,y}$) be a function such that for some $\lambda \in \mathbb{R}$, $0<\lambda<2$ $\varphi_{2x,2y}(\lambda t) \geq \varphi_{x,y}(t)$, for all $x,y \in X$ and $t>0$. If $f : X \to Y$ be a mapping with $f(0)=0$ and satisfying*

$$\mu_{f(x+2y)+f(x-2y)-4(f(x+y)+f(x-y))+6f(x)-f(2y)-f(-2y)+4f(y)+4f(-y)}(t) \geq \varphi_{x,y}(t).$$

Then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q : X \to Y$, a unique cubic mapping $C : X \to Y$ and a unique quartic mapping $D : X \to Y$ such that

$$\begin{aligned}\mu_{f(x)-A(x)-Q(x)-C(x)-D(x)}(t) \geq T_M\{&T_M(\widetilde{\psi}_x(\frac{3(16-\lambda)}{4}t), \widetilde{\psi}_x(3(4-\lambda)t)),\\ &T_M(\widetilde{\beta}_x(\frac{3(8-\lambda)}{4}t), \widetilde{\beta}_x(3(2-\lambda)t))\},\end{aligned}$$

where $\widetilde{\psi}_x(t) = T_M(\widetilde{\varphi}_{0,x}(\frac{t}{3}), \widetilde{\varphi}_{x,x}(\frac{t}{4}), \widetilde{\varphi}_{0,2x}(2t), \widetilde{\varphi}_{2x,x}(t))$, $\widetilde{\beta}_x(t) = T_M(\widetilde{\phi}_x(\frac{t}{2}), \widetilde{\alpha}_x(t))$,

$$\widetilde{\phi}_x(t) = T_M(\widetilde{\varphi}_{x,x}(2t), \widetilde{\varphi}_{2x,x}(3t), \widetilde{\varphi}_{x,2x}(12t), T_M(\widetilde{\varphi}_{3x,x}(12t), \widetilde{\varphi}_{(-x,x}(12t))),$$

$$\widetilde{\alpha}_x(t) = T_M(\widetilde{\varphi}_{2x,2x}(3t), \widetilde{\varphi}_{x,2x}(2t), \widetilde{\varphi}_{x,x}(12t), \widetilde{\varphi}_{x,3x}(12t), T_M(\widetilde{\varphi}_{5x,x}(12t), \widetilde{\varphi}_{-3x,x}(12t))),$$

and $\widetilde{\varphi}_{x,y}(t) = T_M(\varphi_{x,y}(2t), \varphi_{-x,-y}(2t))$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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