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FIXED POINT THEOREMS IN CAT(0) SPACES USING A GENERALIZED Z-TYPE CONDITION

G.S. SALUJA

Department of Mathematics, Govt. Nagarjuna P.G. College of Science, Raipur - 492010 (C.G.), India

Abstract. In this paper, a convergence theorem is proved for a three-step iterative scheme in the framework of $CAT(0)$ spaces by using a generalized Z-type condition which is more general than the Zamfirescu operator.

Keywords: strong convergence, three-step iteration scheme, fixed point, CAT(0) space.

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1. Introduction-preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces [1], $\mathbb{R}$-trees [2], Euclidean buildings [3], the complex Hilbert ball with a hyperbolic metric [4], and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [1].

E-mail address: saluja1963@gmail.com

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Fixed point theory in CAT(0) spaces was first studied by Kirk; see [5,6] and the references therein. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since, then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared; see, e.g., [7]-[18] and the references therein. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$; see, *e.g.*, [1].

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and let $d(c(t),c(t')) = |t-t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry, and $d(x,y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$, which we will denoted by $[x,y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i},\overline{x_j}) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. Such a triangle always exists; see [1].

A geodesic metric space is said to be a *CAT*(0) space if all geodesic triangles of appropriate size satisfy the following *CAT*(0) comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the *CAT*(0) inequality if for all $x,y \in \triangle$ and all comparison points $\bar{x},\bar{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x},\bar{y}). \tag{1}$$

Complete *CAT*(0) spaces are often called *Hadamard spaces*; see [19]. If x,y_1,y_2 are points of a *CAT*(0) space and y_0 is the mid point of the segment $[y_1,y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the *CAT*(0) inequality implies

$$d^2\Big(x,\frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2}\, d^2(x,y_1) + \frac{1}{2}\, d^2(x,y_2) - \frac{1}{4}\, d^2(y_1,y_2). \tag{2}$$

The inequality (2) is the (CN) inequality of Bruhat and Tits [20]. The above inequality has been extended in [21] as

$$d^2(z,\alpha x\oplus(1-\alpha)y) \leq \alpha d^2(z,x)+(1-\alpha)d^2(z,y) -\alpha(1-\alpha)d^2(x,y) \tag{3}$$

for any $\alpha\in[0,1]$ and $x,y,z\in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ s pace if and only if it satisfies the (CN) inequality; see [1, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x,y\in X$, then for any $\alpha\in[0,1]$, there exists a unique point $\alpha x\oplus(1-\alpha)y\in[x,y]$ such that

$$d(z,\alpha x\oplus(1-\alpha)y) \leq \alpha d(z,x)+(1-\alpha)d(z,y), \tag{4}$$

for any $z\in X$ and $[x,y]=\{\alpha x\oplus(1-\alpha)y:\alpha\in[0,1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x,y\in C$, we have $[x,y]\subset C$.

We recall the following definitions in a metric space (X,d). A mapping $T\colon X\to X$ is called an a-contraction if

$$d(Tx,Ty) \leq a\,d(x,y) \quad \text{for all } x,\,y\in X, \tag{5}$$

where $a\in(0,1)$.

The mapping T is called Kannan mapping [22] if there exists $b\in(0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq b\,[d(x,Tx)+d(y,Ty)] \quad \text{for all } x,\,y\in X. \tag{6}$$

Following definition is due to Chatterjea [23]: there exists $c\in(0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq c\,[d(x,Ty)+d(y,Tx)] \quad \text{for all } x,\,y\in X. \tag{7}$$

Combining these three definitions, Zamfirescu [24] proved the following important result.

Theorem Z. *Let (X,d) be a complete metric space and $T\colon X\to X$ a mapping for which there exists the real numbers a,b and c satisfying $a\in(0,1)$, $b,c\in(0,\frac{1}{2})$ such that for any pair $x,\,y\in X$, at least one of the following conditions holds:*

(z_1) $d(Tx,Ty)\leq a\,d(x,y)$,

(z_2) $d(Tx,Ty)\leq b\,[d(x,Tx)+d(y,Ty)]$,

(z_3) $d(Tx,Ty) \le c\,[d(x,Ty)+d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0,1,2,\ldots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

An operator T which satisfies at least one of the contractive conditions (z_1), (z_2) and (z_3) is called a *Zamfirescu operator* or a *Z-operator*.

In 2004, Berinde [25] proved the strong convergence of Ishikawa iterative process defined by: for $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n,\ n \ge 0, \end{aligned} \tag{8}$$

to approximate fixed points of Zamfirescu operators in an arbitrary Banach space E. While proving the theorem, he made use of the condition,

$$\|Tx - Ty\| \le \delta\,\|x-y\| + 2\delta\,\|x - Tx\|, \tag{9}$$

which holds for any $x, y \in E$ where $0 \le \delta < 1$.

Iteration procedures in fixed point theory are lead by the considerations in summability theory. For example, if a given sequence converges, then we don't look for the convergence of the sequence of its arithmetic means. Similarly, if the sequence of Picard iterates of any mapping T converges, then we don't look for the convergence of other iteration procedures.

In 2002, Xu and Noor [26] introduced a three-step iterative scheme as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n,\ n \ge 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$.

The three-step iterative approximation problems were studied extensively by Noor [27,28], Glowinsky and Le Tallec [29], and Haubruge *et al.*, [30]. It has been shown [29] that three-step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences.

In this paper, inspired and motivated [24,25], we employ a condition introduced in [31] which is more general than condition (9) and establish fixed point theorem of three-step iteration scheme in the framework of CAT(0) spaces. The condition is defined as follows:

Let C be a nonempty, closed, convex subset of a CAT(0) space X and $T: C \to C$ a self map of C. There exists a constant $L \geq 0$ such that for all $x,\, y \in C$, we have

$$d(Tx,Ty) \;\leq\; e^{Ld(x,Tx)}\Big(\delta\, d(x,y) + 2\delta\, d(x,Tx)\Big), \tag{10}$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in C$.

Throughout this paper, we call this condition as the generalized Z-type condition.

Remark 1.1. If $L = 0$, in the above condition, we obtain

$$d(Tx,Ty) \leq \delta\, d(x,y) + 2\delta\, d(x,Tx),$$

which is the Zamfirescu condition used by Berinde [25] where

$$\delta = \max\Big\{a, \frac{b}{1-b}, \frac{c}{1-c}\Big\},\ 0 \leq \delta < 1,$$

while constants a, b and c are as defined in Theorem Z.

Example 1.2. Let X be the real line with the usual norm $\|.\|$ and suppose $K = [0,1]$. Define $T: K \to K$ by $\mathrm{Tx} = \frac{\mathrm{x+1}}{2}$ for all $x,y \in K$. Obviously T is self-mapping with a unique fixed point 1. Now we check that condition (10) is true. If $x,y \in [0,1]$, then $\|Tx - Ty\| \leq e^{L\|x-Tx\|}\big[\delta\,\|x - y\| + 2\delta\,\|x-Tx\|\big]$ where $0 \leq \delta < 1$. In fact

$$\|Tx - Ty\| \;=\; \|\frac{x-y}{2}\|$$

and

$$e^{L\|x-Tx\|}\big[\delta\,\|x-y\| + 2\delta\,\|x-Tx\|\big] = e^{L\|\frac{x-1}{2}\|}\big[\delta\,\|x-y\| + \delta\,\|x-1\|\big].$$

Clearly, if we chose $x = 0$ and $y = 1$, then contractive condition (10) is satisfied since

$$\|Tx - Ty\| = \|\frac{x-y}{2}\| = \frac{1}{2},$$

and for $L \geq 0$, we chose $L = 0$, then

$$e^{L\|x-Tx\|}\left[\delta\|x-y\| + 2\delta\|x-Tx\|\right] = e^{L\|\frac{x-1}{2}\|}\left[\delta\|x-y\| + \delta\|x-1\|\right]$$

$$= e^{0(1/2)}(2\delta) = 2\delta, \quad \text{where} \quad 0 < \delta < 1.$$

Therefore, one has

$$\|Tx - Ty\| \leq e^{L\|x-Tx\|}\left[\delta\|x-y\| + 2\delta\|x-Tx\|\right].$$

Hence T is a self mapping with unique fixed point satisfying the contractive condition (10).

Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T : C \to C$ be a given operator. Then for a given $x_1 = x_0 \in C$, compute the sequence $\{x_n\}$ by the iterative scheme as follows:

$$\begin{aligned} z_n &= \gamma_n T x_n \oplus (1-\gamma_n)x_n, \\ y_n &= \beta_n T z_n \oplus (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T y_n \oplus (1-\alpha_n)x_n, \quad n \geq 0, \end{aligned} \tag{11}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are appropriate sequences in [0,1]. If $\gamma_n = 0$ for all $n \geq 0$, then (11) reduces to Ishikawa iteration scheme in $CAT(0)$ spaces:

$$\begin{aligned} y_n &= \beta_n T x_n \oplus (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T y_n \oplus (1-\alpha_n)x_n, \quad n \geq 0, \end{aligned} \tag{12}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are appropriate sequences in [0,1].

We note if $\beta_n = 0$ for all $n \geq 0$, then (12) reduces to Mann iteration scheme in $CAT(0)$ spaces:

$$x_{n+1} = \alpha_n T x_n \oplus (1-\alpha_n)x_n, \quad n \geq 0, \tag{13}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in (0,1).

We need the following useful lemmas to prove our main result in this paper.

Lemma 1.3. [27] *Let X be a CAT(0) space.*

(i) For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = t\, d(x, y) \quad \textit{and} \quad d(y, z) = (1-t)\, d(x, y). \quad (A)$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x, y \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 1.4. [32] *Let $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying the following condition:*

$$p_{n+1} \leq (1-s_n)p_n + q_n + r_n, \quad \forall n \geq 0,$$

where $\{s_n\}_{n=0}^{\infty} \subset [0,1]$. If $\sum_{n=0}^{\infty} s_n = \infty$, $\lim_{n\to\infty} q_n = O(s_n)$ and $\sum_{n=0}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n = 0$.

2. Convergence theorems in CAT(0) Spaces

In this section, we establish strong convergence result of the three-step iteration scheme (11) for fixed points of generalized Z-type condition in the framework of CAT(0) spaces.

Theorem 2.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon C \to C$ be a self mapping satisfying generalized Z-type condition given by (10) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (11). If $\sum_{n=0}^{\infty} \alpha_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Proof. From the assumption $F(T) \neq \emptyset$, it follows that T has a fixed point in C, say u. Since T satisfies the generalized Z-type condition given by (10). Taking $x = u$ and $y = x_n$, we see from

(10) that

$$
\begin{aligned}
d(Tu,Tx_n) &\leq e^{Ld(u,Tu)}\Big(\delta\, d(u,x_n)+2\delta\, d(u,Tu)\Big)\\
&= e^{Ld(u,u)}\Big(\delta\, d(u,x_n)+2\delta\, d(u,u)\Big)\\
&= e^{L(0)}\Big(\delta\, d(u,x_n)+2\delta\,(0)\Big),
\end{aligned}
$$

which implies that

$$d(Tx_n,u) \leq \delta\, d(x_n,u). \tag{14}$$

Similarly by taking $x=u$ and $y=y_n$, z_n in (10), we have

$$d(Ty_n,u) \leq \delta\, d(y_n,u), \tag{15}$$

and

$$d(Tz_n,u) \leq \delta\, d(z_n,u). \tag{16}$$

Now using (11), (14) and Lemma 1.3 (ii), we have

$$
\begin{aligned}
d(z_n,u) &= d(\gamma_n Tx_n \oplus (1-\gamma_n)x_n,u)\\
&\leq \gamma_n d(Tx_n,u)+(1-\gamma_n)d(x_n,u)\\
&\leq \gamma_n \delta\, d(x_n,u)+(1-\gamma_n)d(x_n,u)\\
&\leq [1-\gamma_n+\delta\,\gamma_n]d(x_n,u). \qquad (17)
\end{aligned}
$$

By using (11), (16), (17) and Lemma 1.3 (ii), we have

$$
\begin{aligned}
d(y_n,u) &= d(\beta_n Tz_n \oplus (1-\beta_n)x_n,u)\\
&\leq \beta_n d(Tz_n,u)+(1-\beta_n)d(x_n,u)\\
&\leq \beta_n \delta\, d(z_n,u)+(1-\beta_n)d(x_n,u)\\
&\leq \beta_n \delta[1-\gamma_n+\delta\,\gamma_n]d(x_n,u)+(1-\beta_n)d(x_n,u)\\
&= [1-\beta_n+\delta\,\beta_n(1-\gamma_n+\delta\,\gamma_n)]d(x_n,u). \qquad (18)
\end{aligned}
$$

Note that

$$1-\beta_n+\delta\,\beta_n(1-\gamma_n+\delta\,\gamma_n)=1-[\beta_n(1-\delta)(1+\delta\,\gamma_n)].$$

Since $(1+\delta\gamma_n) \geq (1-\delta)$, we have

$$1-\beta_n+\delta\,\beta_n(1-\gamma_n+\delta\,\gamma_n) \;\; \leq \;\; 1-(1-\delta)^2\beta_n. \tag{19}$$

It follows that

$$d(y_n,u) \;\; \leq \;\; [1-(1-\delta)^2\beta_n]d(x_n,u). \tag{20}$$

By using (11), (15), (20) and Lemma 1.3 (ii), we have

$$\begin{aligned} d(x_{n+1},u) &= d(\alpha_n Ty_n \oplus (1-\alpha_n)x_n,u) \\ &\leq \alpha_n d(Ty_n,u)+(1-\alpha_n)d(x_n,u) \\ &\leq \alpha_n\delta\, d(y_n,u)+(1-\alpha_n)d(x_n,u) \\ &\leq \alpha_n\delta[1-(1-\delta)^2\beta_n]d(x_n,u)+(1-\alpha_n)d(x_n,u) \\ &\leq [1-\alpha_n+\delta\,\alpha_n-\delta(1-\delta)^2\alpha_n\beta_n]d(x_n,u) \\ &= [1-(1-\delta)\alpha_n-\delta(1-\delta)^2\alpha_n\beta_n]d(x_n,u) \\ &= [1-(1-\delta)\alpha_n\{1+\delta(1-\delta)\beta_n\}]d(x_n,u). \qquad (21) \end{aligned}$$

Since $(1+\delta(1-\delta)\beta_n) \geq (1-\delta)^2$, we have

$$1-(1-\delta)\alpha_n\{1+\delta(1-\delta)\beta_n\} \;\; \leq \;\; 1-(1-\delta)^3\alpha_n. \tag{22}$$

Hence, we find that

$$\begin{aligned} d(x_{n+1},u) &\leq [1-(1-\delta)^3\alpha_n]d(x_n,u) \\ &\leq (1-B_n)d(x_n,u), \qquad (23) \end{aligned}$$

where $B_n = (1-\delta)^3\alpha_n$, since $0 \leq \delta < 1$ and by assumption of the theorem $\sum_{n=0}^{\infty}\alpha_n = \infty$, it follows that $\sum_{n=0}^{\infty}B_n = \infty$, therefore by Lemma 1.4, we get that $\lim_{n\to\infty} d(x_n,u) = 0$. Thus $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.

To show uniqueness of the fixed point u, assume that $u_1, u_2 \in F(T)$ and $u_1 \neq u_2$.

Applying the generalized Z-type condition given by (10) and using the fact that $0 \leq \delta < 1$, we obtain

$$\begin{aligned} d(u_1,u_2) &= d(Tu_1,Tu_2) \\ &\leq e^{Ld(u_1,Tu_1)}\Big\{\delta\, d(u_1,u_2)+2\delta\, d(u_1,Tu_1)\Big\} \\ &= e^{Ld(u_1,u_1)}\Big\{\delta\, d(u_1,u_2)+2\delta\, d(u_1,u_1)\Big\} \\ &= e^{L(0)}\Big\{\delta\, d(u_1,u_2)+2\delta\,(0)\Big\} \\ &= \delta\, d(u_1,u_2) \\ &< d(u_1,u_2), \end{aligned}$$

which is a contradiction. Therefore $u_1 = u_2$. Thus $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

Corollary 2.2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space and let $T: C \to C$ be a self mapping satisfying generalized Z-type condition given by (10) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (12). If $\sum_{n=0}^{\infty}\alpha_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of Corollary 2.2 follows by taking $\gamma_n = 0$ for all $n \geq 0$ in Theorem 2.1. This completes the proof.

Corollary 2.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space and let $T: C \to C$ be a self mapping satisfying generalized Z-type condition given by (10) with $F(T) \neq \emptyset$. For any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (13). If $\sum_{n=0}^{\infty}\alpha_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. The proof of Corollary 2.3 follows by taking $\beta_n = \gamma_n = 0$ for all $n \geq 0$ in Theorem 2.1. This completes the proof.

The contraction condition (5) makes T continuous function on X while this is not the case with contractive conditions (6), (7) and (10).

Remark 2.4. Our result extends and improves the corresponding result proved by Berinde [33], Yildirim *et al.* [34] and Bosede [31] to the case of three-step iteration schemes and from Banach spaces or normed linear space to the setting of CAT(0) spaces.

3. Conclusion

The generalized Z-type condition is more general than Zamfirescu operators and the Noor iterative scheme is more general than iterative schemes comparing with Mann, Ishikawa and Picard iterative schemes. Thus the result presented in this paper is an extension and generalization of corresponding result proved in [31, 33, 34] and some others given in the existing literature.

Conflict of Interests

The author declares that there is no conflict of interests.

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