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SPLITTING METHODS FOR TREATING STRICTLY PSEUDOCONTRACTIVE AND MONOTONE OPERATORS IN HILBERT SPACES

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Abstract. In this paper, strictly pseudocontractive and monotone operators are investigated based on a viscosity splitting method. Strong convergence theorems for common solutions are established in the framework of Hilbert spaces.

Keywords: splitting methods; zero point; fixed point; variational inclusion; nonexpansive mapping.

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1. Introduction-preliminaries

In what follows, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H. Let $S : C \to C$ be a mapping. $F(S)$ denoted by the fixed point set of S. S is said to be *contractive* iff there exists a constant $\alpha \in (0,1)$ such that

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

S is said to be *nonexpansive* iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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S is said to be *strictly pseudocontractive* iff there exits a constant $\lambda \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x-y\|^2 + \lambda \|(I-S)x-(I-S)y\|^2, \quad \forall x,y \in C.$$

The class of strictly pseudocontractive mapping was introduced by Browder and Petryshyn [1]. It is clear that the class strictly pseudocontractive mapping includes the class of nonexpansive mappings as a special case.

Let $A : C \to H$ be a mapping. Recall that A is said to be *monotone* iff

$$\langle Ax - Ay, x-y\rangle \ge 0, \quad \forall x,y \in C.$$

Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\kappa > 0$ such that

$$\langle Ax - Ay, x-y\rangle \ge \kappa \|Ax-Ay\|^2, \quad \forall x,y \in C.$$

It is not hard to see that every inverse-strongly monotone mapping is monotone and continuous.

Recall that a set-valued mapping $B : H \rightrightarrows H$ is said to be *monotone* iff, for all $x,y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x-y, f-g\rangle > 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of B. A monotone mapping $B : H \rightrightarrows H$ is *maximal* iff the graph $Graph(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \ge 0$, for all $(y,g) \in Graph(B)$ implies $f \in Bx$. For a maximal monotone operator B on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to Dom(B)$, where $Dom(B)$ denote the domain of B. It is known that J_r is firmly nonexpansive, and $B^{-1}(0) = F(J_r)$.

Maximal monotone operators have been extensively studied by many authors; see [2-22] and the references therein. One well-known example of such a mapping is ∂f, the subdifferential of a proper closed convex function $f : H \to (-\infty, \infty]$ which is defined by

$$\partial f(x) := \{x^* \in H : f(x) + \langle y-x, x^*\rangle \le f(y), \forall y \in H\}, \quad \forall x \in H.$$

Rockafellar [5] proved that ∂f is a maximal monotone operator. It is easy to verify that $0 \in \partial f(v)$ if and only if $f(v) = \min_{x\in H} f(x)$. Another example is $M + N_C$, M is a single valued maximal monotone mapping that is continuous on C, and N_C is the normal cone mapping

$$N_C(x) := \{x^* \in H : \langle x^*, y-x\rangle \le 0, \forall y \in C\},$$

for $x \in C$ and is empty otherwise. Then, $0 \in Mx + N_C(x)$ iff $x \in C$ satisfies the variational inequalities of $\langle Mx, y - x\rangle \geq 0$ for all $y \in C$.

For approximating zero points of maximal monotone operator T, classical methods for doing this is the proximal point algorithm, proposed by Martinet [19] and generalized by Rockafellar [4-6]. In the case of $T = A + B$, where A and B are monotone operators on H. The following splitting method

$$x_{n+1} = J_{r_n}(I - r_n A)x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ is real number sequence, was proposed by Lions and Mercier [23], by Passty [24] and, in a dual form for convex programming, by Han and Lou [25].

Since many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two monotone nonlinear operators ($T = A + B$), splitting methods recently have been investigated for treating monotone operators; see [26-31] and the references therein. Splitting methods mean an iterative method for which each iteration involves only with the individual operators A and B , but not the sum $A + B$. Indeed, the backward step involves B only, so some portion of T can be put into A to facilitate problem decomposition.

In this paper, we investigate common solutions of fixed point problems and zero points of the sum of two monotone operators based on a viscosity splitting method. Strong convergence theorems for the common solutions of the two problems are established in Hilbert spaces. In order to prove our main results, we also need the following tools.

Lemma 1.1. [31] *Let $A : C \to H$ be a mapping, and $B : H \rightrightarrows H$ a maximal monotone operator. Then $F(J_r(I - rA)) = (A + B)^{-1}(0)$.*

Lemma 1.2 [32] *Let E be a Banach space and let A be an m-accretive operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 1.3 [33] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E, and $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 -$*

$\beta_n)y_n + \beta_n x_n$, $\forall n \geq 1$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.4 [34] *Let* $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the condition* $a_{n+1} \leq (1-t_n)a_n + t_n b_n + e_n$, $\forall n \geq 0$, *where* $\{t_n\}$ *is a number sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ *is a number sequence such that* $\limsup_{n\to\infty} b_n \leq 0$ *and* $\{e_n\}$ *is a number sequence such that* $\sum_{n=0}^{\infty} e_n < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.5 [34] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T : C \to C$ *be a* λ*-strictly pseudocontractive mapping. Define* S_t *by* $S_t = tx + (1-t)Tx$, *where* $t \in [\lambda, 1)$. *Then* S_t *is nonexpansive with* $F(S_t) = F(T)$ *and* $I - T$ *is also demiclosed.*

2. Main results

Now, we are in a position to give our main results.

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping and let* B *be a maximal monotone operator on* H. *Let* $f : C \to C$ *be a* κ*-contractive mapping and let* $T : C \to C$ *be a* λ*-strictly pseudocontractive mapping with fixed points. Assume that* $Dom(B) \subset C$ *and* $F(T) \cap (A+B)^{-1}(0)$ *is not empty. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0,1)$ *and* $\{r_n\}$ *be a positive real number sequence in* $(0, 2\alpha)$. *Let* $\{x_n\}$ *be a sequence generated in the following process:* $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S_\lambda J_{r_n}(y_n - r_n A y_n + e_n), \quad \forall n \geq 1, \end{cases}$$

where $S_\lambda = \lambda x + (1-\lambda)Tx$, $J_{r_n} = (I + r_n B)^{-1}$ *and* $\{e_n\}$ *is a sequence in* H *such that* $\sum_{n=1}^{\infty} \|e_n\| < \infty$. *Assume that the above sequences satisfy the following restrictions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, *where a and b are two real numbers.Then* $\{x_n\}$ *converges strongly to* $q = P_{F(T)\cap(A+B)^{-1}(0)}f(q)$.

Proof. For any $x, y \in C$, we find that

$$\begin{aligned}
&\|(I - r_nA)x - (I - r_nA)y\|^2 \\
&= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + {r_n}^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2.
\end{aligned}$$

Using the restriction (c), we obtain that $I - r_nA$ is nonexpansive. Fixing $p \in (A+B)^{-1}(0) \cap F(S)$, we see that

$$\begin{aligned}
\|y_n - p\| &\leq \alpha_n\|f(x_n) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\leq \big(1 - \alpha_n(1 - \kappa)\big)\|x_n - p\| + \alpha_n(1 - \kappa)\frac{\|f(p) - p\|}{1 - \kappa}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|S_\lambda J_{r_n}(y_n - r_nAy_n + e_n) - p\| \\
&\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|J_{r_n}(y_n - r_nAy_n + e_n) - p\| \\
&\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|(y_n - r_nAy_n + e_n) - p\| \\
&\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|y_n - p\| + \|e_n\| \\
&\leq \beta_n\|x_n - p\| + (1 - \beta_n)\big(1 - \alpha_n(1 - \kappa)\big)\|x_n - p\| \\
&\quad + \alpha_n(1 - \beta_n)(1 - \kappa)\frac{\|f(p) - p\|}{1 - \kappa} + \|e_n\| \\
&\leq \big(1 - \alpha_n(1 - \beta_n)(1 - \kappa)\big)\|x_n - p\| + \alpha_n(1 - \beta_n)(1 - \kappa)\frac{\|f(p) - p\|}{1 - \kappa} + \|e_n\|.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. Note that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + \|f(x_{n-1}) - x_{n-1}\||\alpha_n - \alpha_{n-1}| \\
&\quad + (1 - \alpha_n)\|x_n - x_{n-1}\| \qquad (2.1) \\
&\leq \big(1 - \alpha_n(1 - \kappa)\big)\|x_n - x_{n-1}\| + \|f(x_{n-1}) - x_{n-1}\||\alpha_n - \alpha_{n-1}|
\end{aligned}$$

Set $z_n = y_n - r_n A y_n + e_n$. Using Lemma 1.2, we find that

$$\begin{aligned}
&\|J_{r_n} z_n - J_{r_{n-1}} z_{n-1}\| \\
&\leq \|\frac{r_{n-1}}{r_n}(z_n - z_{n-1}) + (1 - \frac{r_{n-1}}{r_n})(J_{r_n} z_n - z_{n-1})\| \\
&\leq \|z_n - z_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J_{r_n} z_n - z_n\| \\
&\leq \|y_n - y_{n-1}\| + |r_{n-1} - r_n|(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{a}) + \|e_n\| + \|e_{n-1}\|.
\end{aligned} \tag{2.2}$$

Substituting (2.1) into (2.2) yields that

$$\begin{aligned}
\|J_{r_n} z_n - J_{r_{n-1}} z_{n-1}\| \leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + \|f(x_{n-1}) - x_{n-1}\||\alpha_n - \alpha_{n-1}| \\
+ |r_{n-1} - r_n|(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{a}) + \|e_n\| + \|e_{n-1}\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|S_\lambda J_{r_n} z_n - S_\lambda J_{r_{n-1}} z_{n-1}\| \\
&\leq \|J_{r_n} z_n - J_{r_{n-1}} z_{n-1}\| \\
&\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x_{n-1}\| + \|f(x_{n-1}) - x_{n-1}\||\alpha_n - \alpha_{n-1}| \\
&\quad + |r_{n-1} - r_n|(\|Ay_{n-1}\| + \frac{\|J_{r_n} z_n - z_n\|}{a}) + \|e_n\| + \|e_{n-1}\|.
\end{aligned}$$

Using the restrictions (a) and (c), we find that

$$\limsup_{n\to\infty}\big(\|S_\lambda J_{r_n} z_n - S_\lambda J_{r_{n-1}} z_{n-1}\| - \|x_n - x_{n-1}\|\|\|\big) \leq 0.$$

Using Lemma 1.3, we see that $\lim_{n\to\infty}\|S_\lambda J_{r_n} z_n - x_n\| = 0$. Since $x_{n+1} - x_n = (1-\beta_n)(S_\lambda J_{r_n} z_n - x_n)$, we find that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.3}$$

Since $y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n$, we find from the restriction (a) that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.4}$$

Note that

$$\|y_n - p\|^2 \leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2.$$

Hence, we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|S_\lambda J_{r_n}z_n-p\|^2\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|J_{r_n}\big((I-r_nA)y_n+e_n\big)-p\|^2\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|\big((I-r_nA)y_n+e_n\big)-(I-r_nA)p\|^2\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\big(\|(I-r_nA)y_n-(I-r_nA)p\|+\|e_n\|\big)^2\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\big(\|(I-r_nA)y_n-(I-r_nA)p\|^2\\
&\quad +\|e_n\|(\|e_n\|+2\|y_n-p\|)\big)\\
&\le \big(1-\alpha_n(1-\beta_n)\big)\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2\\
&\quad -r_n(2\alpha-r_n)(1-\beta_n)\|Ay_n-Ap\|^2+\|e_n\|(\|e_n\|+2\|y_n-p\|)\\
&\le \|x_n-p\|^2+\alpha_n\|f(x_n)-p\|^2-r_n(2\alpha-r_n)(1-\beta_n)\|Ay_n-Ap\|^2\\
&\quad +\|e_n\|(\|e_n\|+2\|y_n-p\|).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&r_n(1-\beta_n)(2\alpha-r_n)\|Ay_n-Ap\|^2\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\alpha_n\|f(x_n)-p\|^2+\|e_n\|(\|e_n\|+2\|y_n-p\|).
\end{aligned}
$$

Using the restrictions (a), (b) and (c), we find from (2.3) that

$$
\lim_{n\to\infty}\|Ay_n-Ap\|=0. \tag{2.5}
$$

Since J_{r_n} is firmly nonexpansive, we see that

$$
\begin{aligned}
\|J_{r_n}z_n-p\|^2 &\le \langle J_{r_n}z_n-p,(y_n-r_nAy_n)-(p-r_nAp)\rangle\\
&=\frac{1}{2}\Big(\|J_{r_n}z_n-p\|^2+\|(y_n-r_nAy_n)-(p-r_nAp)\|^2\\
&\quad -\|(J_{r_n}z_n-p)-\big((y_n-r_nAy_n)-(p-r_nAp)\big)\|^2\Big)\\
&\le \frac{1}{2}\big(\|J_{r_n}z_n-p\|^2+\|y_n-p\|^2-\|J_{r_n}z_n-y_n\|^2\\
&\quad -\|r_nAy_n-r_nAp\|^2+2r_n\|Ay_n-Ap\|\|J_{r_n}z_n-y_n\|\big).
\end{aligned}
$$

It follows that

$$\begin{aligned}\|J_{r_n}z_n - p\|^2 &\le \|y_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2 \\ &\quad - \|r_nAy_n - r_nAp\|^2 + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\| \\ &\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2 \\ &\quad - \|r_nAy_n - r_nAp\|^2 + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|\end{aligned}$$

Using the convexness of $\|\cdot\|^2$, we find that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|S_\lambda J_{r_n}z_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|J_{r_n}z_n - p\|^2 \\ &\le \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 - (1-\beta_n)\|J_{r_n}z_n - y_n\|^2 \\ &\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\beta_n)\|J_{r_n}z_n - y_n\|^2 &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - p\|^2 \\ &\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\end{aligned}$$

Using the restrictions (a) and (b), we from (2.3) and (2.5) see that

$$\lim_{n\to\infty}\|J_{r_n}z_n - y_n\| = 0. \tag{2.6}$$

Since $P_{F(T)\cap(A+B)^{-1}(0)}f$ is contractive, we see that there exits a unique fixed point. Next, we use q to denote the unique fixed point. Now, we are in a position to show that $\limsup_{n\to\infty}\langle f(q) - q, y_n - q\rangle \le 0$. To show it, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(q) - q, y_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, y_{n_i} - q\rangle.$$

Since $\{y_{n_i}\}$ is bounded, we can choose a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly some point x. We may assume, without loss of generality, that y_{n_i} converges weakly to x.

Now, we are in a position to prove that $x \in F(T)$. Setting $w_n = J_{r_n}z_n$, we find that

$$\|S_\lambda w_n - y_n\| \le \frac{1}{1-\beta_n}\|x_{n+1} - y_n\| + \frac{\beta_n}{1-\beta_n}\|y_n - x_n\|.$$

Using (2.3) and (2.4), we find that This implies that $\lim_{n\to\infty}\|S_\lambda w_n - y_n\| = 0$. Note that

$$\|S_\lambda w_n - w_n\| \le \|S_\lambda w_n - y_n\| + \|y_n - w_n\|.$$

In view of (2.6), we find that $\|S_\lambda w_n - w_n\| \to 0$. In view of demiclosed of the mapping, we find that $x \in F(S_\lambda) = F(T)$.

Now, we are in a position to show that $x \in (A+B)^{-1}(0)$. It follows that

$$y_n - r_n A y_n + e_n \in (I + r_n B) w_n$$

That is, $\frac{y_n - w_n}{r_n} - Ay_n + e_n \in Bw_n$. Since B is monotone, we get, for any $(\mu, \nu) \in B$, that

$$\langle w_n - \mu, \frac{y_n - w_n}{r_n} - Ay_n + e_n - \nu \rangle \ge 0.$$

It follows from (2.6) that

$$\langle x - \mu, -Ax - \nu \rangle \ge 0.$$

This gives that $-Ax \in Bx$, that is, $0 \in (A+B)(x)$. This proves that $x \in (A+B)^{-1}(0)$. This complete the proof that $x \in F(T) \cap (A+B)^{-1}(0)$. Hence

$$\limsup_{n\to\infty} \langle f(q) - q, y_n - q \rangle \le 0.$$

Finally, we show that $x_n \to q$. Notice that

$$\begin{aligned} \|y_n - q\|^2 &\le \alpha_n \langle f(x_n) - q, y_n - q \rangle + (1-\alpha_n)\|x_n - q\|\|y_n - q\| \\ &\le \big(1 - \alpha_n(1-\kappa)\big)\|x_n - q\|\|y_n - q\| + \alpha_n \langle f(q) - q, y_n - q \rangle \end{aligned}$$

This implies that

$$\|y_n - q\|^2 \le \big(1 - \alpha_n(1-\kappa)\big)\|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, y_n - q \rangle$$

It follows that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\le \beta_n \|x_n - q\|^2 + (1-\beta_n)\|S_\lambda J_{r_n}(y - r_n A y_n + e_n) - q\|^2 \\ &\le \beta_n \|x_n - q\|^2 + (1-\beta_n)\|y_n - q\|^2 + \| + \|e_n\|(\|e_n\| + 2\|y_n - q\|) \\ &\le \big(1 - \alpha_n(1-\beta_n)(1-\kappa)\big)\|x_n - q\|^2 + 2\alpha_n(1-\beta_n)\langle f(q) - q, y_n - q \rangle \\ &\quad + \|e_n\|(\|e_n\| + 2\|y_n - q\|). \end{aligned}$$

Using the restrictions (a) and (b), we find from Lemma 1.4 that $x_n \to q$. This completes the proof.

If T is nonexpansive, then we have the following.

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Let $f : C \to C$ be a κ-contractive mapping and let $T : C \to C$ be a nonexpansive mapping with fixed points. Assume that $Dom(B) \subset C$ and $F(T) \cap (A+B)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0,1)$ and $\{r_n\}$ be a positive real number sequence in $(0,2\alpha)$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)TJ_{r_n}(y_n - r_n A y_n + e_n), \quad \forall n \geq 1, \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$ and $\{e_n\}$ is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Assume that the above sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers.Then $\{x_n\}$ converges strongly to $q = P_{F(T)\cap(A+B)^{-1}(0)} f(q)$.

If T is the identity mapping, then we have the following.

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Let $f : C \to C$ be a κ-contractive mapping. Assume that $Dom(B) \subset C$ and $(A+B)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0,1)$ and $\{r_n\}$ be a positive real number sequence in $(0,2\alpha)$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)J_{r_n}(y_n - r_n A y_n + e_n), \quad \forall n \geq 1, \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$ and $\{e_n\}$ is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Assume that the above sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < a \le r_n \le b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers.Then $\{x_n\}$ converges strongly to $q = P_{(A+B)^{-1}(0)} f(q)$.

3. Applications

In this section, we give applications of the main results. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x,y) \ge 0, \quad \forall y \in C. \tag{3.1}$$

In this paper, we use $EP(F)$ to denote the solution set of the equilibrium problem (3.1).

To study the equilibrium problems (3.1), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \le 0$ for all $x,y \in C$;

(A3) for each $x,y,z \in C$,

$$\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Lemma 3.1. [30] *Let C be a nonempty closed convex subset of a real Hilbert space H, F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4) and A_F a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x,y) \ge \langle y - x, z\rangle, \quad \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{3.2}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$ and

$$T_r x = (I + rA_F)^{-1} x, \quad \forall x \in H, r > 0,$$

where T_r is defined as

$$T_r x = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C\}$$

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and Let F_B be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $T : C \to C$ be a λ-strictly pseudocontractive mapping with fixed points. Assume that $F(T) \cap EP(F)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0,1)$ and $\{r_n\}$ be a positive real number sequence in $(0,2\alpha)$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S_\lambda T_{r_n}(y_n - r_n A y_n + e_n), \quad \forall n \geq 1, \end{cases}$$

where $S_\lambda = \lambda x + (1-\lambda)Tx$, $T_{r_n} = (I + rA_F)^{-1}$ and $\{e_n\}$ is a sequence in H such that $\sum_{n=1}^{\infty}\|e_n\| < \infty$. Assume that the above sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$,

where a and b are two real numbers.Then $\{x_n\}$ converges strongly to $q = P_{F(T)\cap EP(F)}f(q)$.

If $T = I$, the identity mapping, we have the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and Let F_B be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Assume that $EP(F)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0,1)$ and $\{r_n\}$ be a positive real number sequence in $(0,2\alpha)$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) T_{r_n}(y_n - r_n A y_n + e_n), \quad \forall n \geq 1, \end{cases}$$

where $T_{r_n} = (I + rA_F)^{-1}$ and $\{e_n\}$ is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Assume that the above sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers.Then $\{x_n\}$ converges strongly to $q = P_{EP(F)}f(q)$.

Recall the classical variational inequality is to find $u \in C$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \forall v \in C.$$

The solution set of the inequality is denoted by $VI(C,A)$ in this section. Let $f : H \to (-\infty, +\infty]$ a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z-x, y \rangle, \quad z \in H\}, \quad \forall x \in H.$$

From Rockafellar [5], we know that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y\in H} f(y)$. Let I_C be the indicator function of C, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{3.3}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator.

Lemma 3.4 [5] *Let C be a nonempty closed convex subset of a real Hilbert space H, $Proj_C$ the metric projection from H onto C, ∂I_C the subdifferential of I_C, where I_C is as defined in (3.2) and $J_\lambda = (I + \lambda \partial I_C)^{-1}$. Then $y = J_\lambda x \Longleftrightarrow y = Proj_C x, \forall x \in H, y \in C$.*

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $T : C \to C$ be a nonexpansive mapping with fixed points. Assume that $F(T) \cap VI(C,A)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0,1)$ and $\{r_n\}$ be a positive real number sequence in $(0,2\alpha)$. Let $\{x_n\}$ be a* *sequence generated in the following process:* $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)TP_C(y_n - r_nAy_n + e_n), \quad \forall n \geq 1, \end{cases}$$

where $\{e_n\}$ *is a sequence in* H *such that* $\sum_{n=1}^{\infty} \|e_n\| < \infty$. *Assume that the above sequences satisfy the following restrictions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers.Then $\{x_n\}$ *converges strongly to* $q = P_{F(T)\cap VI(C,A)} f(q)$.

Proof. Putting $Bx = \partial I_C$, we find from Lemma 3.4 the desired conclusion immediately.

If T is the identity mapping, then we have the following.

Corollary 3.6. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping. Assume that* $VI(C,A)$ *is not empty. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0,1)$ *and* $\{r_n\}$ *be a positive real number sequence in* $(0,2\alpha)$. *Let* $\{x_n\}$ *be a sequence generated in the following process:* $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)P_C(y_n - r_nAy_n + e_n), \quad \forall n \geq 1, \end{cases}$$

where $\{e_n\}$ *is a sequence in* H *such that* $\sum_{n=1}^{\infty} \|e_n\| < \infty$. *Assume that the above sequences satisfy the following restrictions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers.Then $\{x_n\}$ *converges strongly to* $q = P_{VI(C,A)} f(q)$.

Conflict of Interests

The author declares that there is no conflict of interests.

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REFERENCES

[1] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967), 197-228.

[2] R. R. Phelps, Convex Functions, Monotone Operators and Differentiability, Springer-Verlag, New York, 1989.

[3] R. T. Rockafellar, Augmented Lagrangians and applications of the proximal point algorithm in convex programming, Math. Oper. Res. 1 (1976), 97-116.

[4] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1976), 877-898.

[5] R. T. Rockafellar, Characterization of the subdifferentials of convex functions, Pacific J. Math. 17 (1966) 497-510.

[6] R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970), 75-88.

[7] Y. Qing, S.Y. Cho, Proximal point algorithms for zero points of nonlinear operators, Fixed Point Theory Appl. 2014, 2014:42.

[8] Y. Qing, S.Y. Cho, A regularization algorithm for zero points of accretive operators, Fixed Point Theory Appl. 2013, 2013:341.

[9] J. Song, M. Chen, A modified Mann iteration for zero points of accretive operators, Fixed Point Theory Appl. 2013, 2013:347.

[10] M. Zhang, An algorithm for treating asymptotically strict pseudocontractions and monotone operators, Fixed Point Theory Appl. 2014, 2014:52.

[11] C. Wu, L. Sun, A monotone projection algorithm for fixed points of nonlinear operators, Fixed Point Theory Appl. 2013, 2013:318.

[12] C. Wu, A. Liu, Strong convergence of a hybrid projection iterative algorithm for common solutions of operator equations and of inclusion problems, Fixed Point Theory Appl. 2012 (2012), Article ID 90.

[13] C. Wu Mann iteration for zero theorems of accretive operators, J. Fixed Point Theory 2013 (2013), Article ID 3.

[14] C. Wu, S. Lv, Bregman projection methods for zeros of monotone operators, J. Fixed Point Theory 2013 (2013), Article ID 7.

[15] X. Qin, S.Y. Cho, L. Wang, Iterative algorithms with errors for zero points of m-accretive operators, Fixed Point Theory Appl. 2013, 2013:148.

[16] S.Y. Cho, X. Qin, S.M. Kang, Iterative processes for common fixed points of two different families of mappings with applications, J. Glob. Optim. 57 (2013), 1429-1446.

[17] S. Yang, Zero theorems of accretive operators in reflexive Banach spaces. J. Nonlinear Funct. Anal. 2013 (2013), Article ID 2.

[18] M. Zhang, Iterative algorithms for common elements in fixed point sets and zero point sets with applications, Fixed Point Theory Appl. 2012 (2012), Article ID 21.

[19] S.Y. Cho, W. Li, S.M. Kang, Convergence analysis of an iterative algorithm for monotone operators, J. Inequal. Appl. 2013, 2013:199.

[20] B. Martinet, Regularisation d'inequations variationelles par approximations successives, Rev. Francaise Informat. Recherche Operationnelle, 4 (1970), 154-158.

[21] Y. Hecai, On weak convergence of an iterative algorithm for common solutions of inclusion problems and fixed point problems in Hilbert spaces, Fixed Point Theory Appl. 2013 (2013), Article ID 155.

[22] S. Yang, Zero theorems of accretive operators in reflexive Banach spaces. J. Nonlinear Funct. Anal. 2013 (2013), Article ID 2.

[23] X. Qin, Y. Su, Approximation of a zero point of accretive operator in Banach spaces, J. Math. Anal. Appl. 329 (2007), 415-424.

[24] P. L. Lions, B. Mercier, Splitting algorithms for the sum of two nonlinear operators, SIAM J. Numer. Anal. 16 (1979), 964-979.

[25] G. B. Passty, Ergodic convergence to a zero of the sum of monotone operators in Hilbert space, J. Math. Anal. Appl. 72 (1979), 383-390.

[26] S. P. Han, G. Lou, A parallel algorithm for a class of convex programs, SIAM J. Control Optim. 26 (1988), 345-355.

[27] P. Tseng, A modified forward-backward splitting methods for maximal monotone mappings, SIAM J. Control Optim. 38 (2000), 431-446.

[28] P. Tseng, Applications of a splitting algorithm to decomposition in convex programming and variational inequalities, SIAM J. Control Optim. 29 (1991), 119-138.

[29] A. Moudafi, M. Théra, Finding a zero of the sum of two maximal monotone operators, J. Optim. Theory Appl. 94 (1997), 425-448.

[30] H.H. Bauschke, A note on the paper by Eckstein and Svaiter on general projective splitting methods for sums of maximal monotone operators, SIAM J. Control Optim. 48 (2009) 2513-2515.

[31] S. Takahashi, W. Takahashi, M. Toyoda, Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces, J. Optim. Theory Appl. 147 (2010) 27-41.

[32] S.Y. Cho, Strong convergence of an iterative algorithm for sums of two monotone operators, J. Fixed Point Theory 2013 (2013), Article ID 6.

[33] V. Barbu, Nonlinear Semigroups and differential equations in Banach space, Noordhoff, 1976.

[34] H.-G. Chen, R. T. Rockafellar, Convergence rates in forward-backward splitting, SIAM J. Optim. 7 (1997), 421-444.

[35] Z. Xue, H. Zhou, Y.J. Cho, Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces, J. Nonlinear Convex Anal. 1 (2000) 313-320.

[36] H. Zhou, Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 69 (2008), 456-462.