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SOME FIXED POINT THEOREMS IN COMPLETE PARTIAL B-METRIC SPACES

HOSSEIN PIRI*, HOJJAT AFSHARI

Department of Mathematics, Faculty of Basic Sciences, University of Bonab, Bonab 5551-761167, Iran

Abstract. In this paper, we prove some fixed point theorems in the framework of complete partial b-metric spaces. The article also includes an example which shows the validity of our results.

Keywords: fixed point, partial b-metric space, contraction.

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1. Introduction and preliminaries

Fixed point theory is one of the most important tools in development of mathematics because it plays a essential role in applications of many branches of mathematics. For this reason several researchers studied the fixed point of contractive maps (see for example [10] and references therein). In 1992, Polish mathematician Banach [4] proved a very important result regarding a contraction mapping, known as Banach contraction principle. It is one of the fundamental result in fixed point theory. Due to its importance, several authors have obtained many interesting extensions and generalizations of metric spaces and Banach contraction principle; see

*Corresponding author

E-mail address: h.piri@bonabu.ac.ir

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[14, 25, 26, 27, 29, 30, 31, 32, 33, 34] and references therein. In this sequel, in 1989, Bakhtin [5] introduced the concept of b-metric spaces and presented the contraction mapping in b-metric spaces that is generalization of the Banach contraction principle in metric spaces (see also Czerwik [9]). After that, fixed point results in b-metric spaces were studied by several researchers; see [3, 6, 20, 19, 13, 21, 11, 7, 22] and references therein.

On the other hand, Matthews [15, 16] introduced the notion of a partial metric space which is a generalization of usual metric space. Also he generalized the Banach contraction principle in the context of complete partial metric spaces. Recently, many researchers have focused on partial metric spaces and obtained many useful fixed point results in these spaces (see [28, 18, 24, 3, 1, 12, 13] and references therein).

Very recently, Shukla [23] introduced partial b-metric spaces as a generalization of both b-metric spaces and partial metric spaces. Moreover, he proved Banach contraction principle as well as the Kannan type fixed point theorem in partial b-metric spaces.

In this paper, motivated and inspired by ideas of some recent papers such as Mustafa et al. [17], Suzuki [25], Wong [35] and Shukla [23], we obtain some fixed point theorems in partial b-metric spaces. Our result is the generalization of the result announced by Suzuki [25], Wong [35] and Shukla [23] and some others.

Throughout this paper, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of real numbers, the set of nonnegative real numbers and the set of positive integers, respectively.

Definition 1.1. [5] Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to \mathbb{R}^+$ is said to be a b-metric if for all $x,y,z \in X$ the following conditions are satisfied:

(bM_1) $d(x,y) = 0$ if and only if $x = y$;

(bM_2) $d(x,y) = d(y,x)$;

(bM_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space (with constant s).

Remark 1.1. The class of b-metric spaces is larger than the class of metric spaces since any metric space is a b-metric space with constant $s = 1$. Therefore, it is obvious that b-metric spaces generalize metric spaces. We can present an example, which show that introducing a

b-metric space instead of a metric space is meaningful since there exist b-metric spaces which are not metric spaces.

Example 1.1. Let $X = [0,\infty)$ and $d\colon X\times X \to \mathbb{R}^+$ defined by $d(x,y) = \mid x-y \mid^p$, where p is a real number such that $p>1$. Let $x,y,z \in X$. By taking $u = x-z$ and $v = z-y$, we have

$$\begin{aligned} \mid x-y \mid^p = \mid u+v \mid^p &\leq (\mid u \mid + \mid v \mid)^p \\ &\leq (2\max\{\mid u \mid, \mid v \mid\})^p \\ &\leq 2^p(\mid x-z \mid^p + \mid z-y \mid^p). \end{aligned}$$

Which implies that $d(x,y) \leq 2^p[d(x,z)+d(z,y)]$. Therefore (X,d) is a $b-$metric space with coefficient 2^p. On the other hand, for $x>z>y$, we have

$$\mid x-y \mid^p = \mid u+v \mid^p = (u+v)^p > u^p+v^p = (x-z)^p+(z-y)^p = \mid x-z \mid^p + \mid z-y \mid^p,$$

which implies that $d(x,y) > d(x,z)+d(z,y)$. Therefore (X,d) is not a metric space.

Definition 1.2. [15] Let X be a nonempty set. A mapping $p\colon X\times X \to \mathbb{R}^+$ is said to be a partial metric on X if for all $x,y,z \in X$ the following conditions are satisfied:

(p_1) $x=y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
(p_2) $p(x,x) \leq p(x,y)$;
(p_3) $p(x,y) = p(y,x)$;
(p_4) $p(x,z) \leq p(x,y)+p(y,z)-p(y,y)$.

In this case, the pair (X,d) is called a partial metric space.

The most basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$. For more examples of partial metric spaces, we refer to [28, 18, 24, 3, 1, 12, 13].

Definition 1.3. [23] Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $p_b\colon X\times X \to \mathbb{R}^+$ is said to be a partial b-metric on X if for all $x,y,z \in X$ the following conditions are satisfied:

(p_{b_1}) $x=y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$;
(p_{b_2}) $p_b(x,x) \leq p_b(x,y)$;

(p_{b_3}) $p_b(x,y) = p_b(y,x)$;

(p_{b_4}) $p_b(x,z) \leq s[p_b(x,y) + p_b(y,z)] - p_b(y,y)$.

In this case, the pair (X, p_b) is called a partial b-metric space (with constant s).

In a partial b-metric space (X, p_b), if $x, y \in X$ and $p_b(x,y) = 0$, then $x = y$, but the converse may not be true. Obviously every partial metric space is a partial b-metric space with the coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. The following example, shows that the converse of these facts need not hold.

Example 1.2. Let $X = \{0, 1, 2\}$ and define $p_b : X \times X \to \mathbb{R}^+$ as follows:

$$p_b(0,0) = 0, \quad p_b(1,1) = 4, \quad p_b(0,1) = p_b(1,0) = 8,$$
$$p_b(2,2) = 2, \quad p_b(0,2) = p_b(2,0) = 5, \quad p_b(1,2) = p_b(2,1) = 4.$$

Because $p_b(2,2) = 2 \neq 0$. Thus, condition (bM_1) is not established, therefore (X, p_b) is not b-metric space. Since

$$8 = p_b(o,1) > p_b(o,2) + p_b(2,1) - p_b(2,2) = 5 + 4 - 2 = 7,$$

thus, condition (p_4) is not established, therefore, (X, p_b) is not partial metric space. But, we know (X, p_b) is a partial b-metric spaces with constant $s \geq 2$.

For more example, we refer to Shukla [23].

Definition 1.4. [23] Let (X, p_b) be a partial b-metric space with constant s, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge to $x \in X$, if $\lim_{n\to\infty} p_b(x_n, x) = p_b(x,x)$;

(ii) The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be Cauchy in (X, p_b), if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists and is finite;

(iii) (X, p_b) is said to be complete if for every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X there exists $x \in X$ such that

$$\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x,x).$$

2. Main results

Theorem 2.1. *Let (X, p_b) be a complete partial b-metric space with constant $s \geq 1$ and let T be a self mapping on X. Suppose that $L \in [0, \infty)$ and there exist functions $\alpha_i, i = 1,2,3,4,5,6$ of $(0, \infty)$ into $[0, \infty)$ such that*

(A) each α_i is upper semicontinuous from the right;

(B) $\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t) + \alpha_5(t) + L\alpha_6(t) < \frac{1}{2s^2}t$, for all $t > 0$,

(C) for any distinct $x, y \in X$,

$$\frac{1}{2s} p_b(x, Tx) < p_b(x,y) \Rightarrow \tag{1}$$

$$\begin{aligned} p_b(x,y)p_b(Tx,Ty) \leq\ & \alpha_1(p_b(x,y))p_b(x,y) + \alpha_2(p_b(x,y))p_b(x,Ty) \\ & + \alpha_3(p_b(x,y))p_b(Tx,y) + \alpha_4(p_b(x,y))p_b(x,Tx) \\ & + \alpha_5(p_b(x,y))d(y,Ty) \\ & + L\alpha_6(p_b(x,y))\min\{p_b(x,y), p_b(x,Ty), p_b(Tx,y)\}. \end{aligned} \tag{2}$$

Then T has a unique fixed point.

Proof. Let us first show that if fixed points of T exists, then it is unique. Let $x, y \in X$ be two distinct fixed points of T, that is, $Tx = x \neq y = Ty$. Therefore $p_b(x,y) > 0$. If $p_b(x,x) = 0$, we have $0 = \frac{1}{2s}p_b(x,x) = \frac{1}{2s}p_b(x,Tx) < p_b(x,y)$. If $p_b(x,x) > 0$. From p_{b_2}, we get $\frac{1}{2s}p_b(x,Tx) = \frac{1}{2s}p_b(x,x) < p_b(x,x) \leq p_b(x,y)$. So from p_{b_2}, (B) and (C), we obtain

$$\begin{aligned} & p_b(x,y)^2 = p_b(x,y)p_b(Tx,Ty) \\ & \leq \alpha_1(p_b(x,y))d(x,y) + \alpha_2(p_b(x,y))p_b(x,Ty) \\ & \quad + \alpha_3(p_b(x,y))d(Tx,y) + \alpha_4(p_b(x,y))p_b(x,Tx) \\ & \quad + \alpha_5(p_b(x,y))p_b(y,Ty) + \alpha_6(p_b(x,y))\min\{p_b(x,y), p_b(x,Ty), p_b(Tx,y)\} \\ & \leq \alpha_1(p_b(x,y))p_b(x,y) + \alpha_2(p_b(x,y))p_b(x,Ty) \\ & \quad + \alpha_3(p_b(x,y))p_b(Tx,y) + \alpha_4(p_b(x,y))p_b(x,Tx) \\ & \quad + \alpha_5(p_b(x,y))p_b(y,Ty) + L\alpha_6(p_b(x,y))p_b(x,y) \end{aligned}$$

$$
\begin{aligned}
&= \alpha_1(p_b(x,y))p_b(x,y) + \alpha_2(p_b(x,y))p_b(x,y) \\
&\quad + \alpha_3(p_b(x,y))p_b(x,y) + \alpha_4(p_b(x,y))p_b(x,x) \\
&\quad + \alpha_5(p_b(x,y))p_b(y,y) + L\alpha_6(p_b(x,y))p_b(x,y) \\
&\leq \alpha_1(p_b(x,y))p_b(x,y) + \alpha_2(p_b(x,y))p_b(x,y) \\
&\quad + \alpha_3(p_b(x,y))p_b(x,y) + \alpha_4(p_b(x,y))p_b(x,y) \\
&\quad + \alpha_5(p_b(x,y))p_b(x,y) + L\alpha_6(p_b(x,y))p_b(x,y) \\
&= [\alpha_1(p_b(x,y)) + \alpha_2(p_b(x,y)) + \alpha_3(p_b(x,y)) \\
&\quad + \alpha_4(p_b(x,y)) + \alpha_5(p_b(x,y)) + L\alpha_6(p_b(x,y))]p_b(x,y) \\
&< \frac{1}{2s^2}p_b(x,y) < p_b(x,y).
\end{aligned}
$$

This is a contradiction. So $x = y$. Choose $x_1 \in X$. Set

$$x_2 = Tx_1,\ \cdots, x_{n+1} = Tx_n = T^{n+1}x_1, \cdots, p_n = p_b(x_n, Tx_n). \tag{3}$$

If there exists $n \in \mathbb{N}$ such that $p_n = 0$ the proof is complete. So we assume that for every $n \in \mathbb{N}$, $0 < p_n = p_b(x_n, Tx_n) = p_b(x_n, x_{n+1})$. Therefore, we have

$$\frac{1}{2s}p_b(x_n, Tx_n) < p_b(x_n, Tx_n), \quad \forall n \in \mathbb{N}. \tag{4}$$

From assumption of theorem, we have,

$$
\begin{aligned}
&p_b(x_n, Tx_n)p_b(Tx_n, T^2x_n) \\
&\quad \leq \alpha_1(p_b(x_n, Tx_n))p_b(x_n, Tx_n) + \alpha_2(p_b(x_n, Tx_n))p_b(x_n, T^2x_n) \\
&\quad\quad + \alpha_3(p_b(x_n, Tx_n))p_b(Tx_n, Tx_n) + \alpha_4(p_b(x_n, Tx_n))p_b(x_n, Tx_n) \\
&\quad\quad + \alpha_5(p_b(x_n, Tx_n))p_b(Tx_n, T^2x_n) \\
&\quad\quad + L\alpha_6(p_b(x_n, Tx_n))\min\{p_b(x_n, Tx_n), p_b(x_n, T^2x_n), p_b(Tx_n, Tx_n)\} \\
&\quad \leq \alpha_1(p_b(x_n, Tx_n))p_b(x_n, Tx_n) + s\alpha_2(p_b(x_n, Tx_n))p_b(x_n, Tx_n) \\
&\quad\quad + s\alpha_2(p_b(x_n, Tx_n))p_b(Tx_n, T^2x_n) - \alpha_2(p_b(x_n, Tx_n))p_b(Tx_n, Tx_n) \\
&\quad\quad + s\alpha_3(p_b(x_n, Tx_n))p_b(Tx_n, x_n) + s\alpha_3(p_b(x_n, Tx_n))p_b(x_n, Tx_n)
\end{aligned}
$$

$$
\begin{aligned}
&-\alpha_3(p_b(x_n,Tx_n))p_b(x_n,x_n)+\alpha_4(p_b(x_n,Tx_n))p_b(x_n,Tx_n)\\
&+\alpha_5(p_b(x_n,Tx_n))p_b(Tx_n,T^2x_n)+L\alpha_6(p_b(x_n,Tx_n))p_b(x_n,Tx_n)\\
\leq\ &\alpha_1(p_b(x_n,Tx_n))p_b(x_n,Tx_n)+s\alpha_2(p_b(x_n,Tx_n))p_b(x_n,Tx_n)\\
&+s\alpha_2(p_b(x_n,Tx_n))p_b(Tx_n,T^2x_n)+s\alpha_3(p_b(x_n,Tx_n))p_b(Tx_n,x_n)\\
&+s\alpha_3(p_b(x_n,Tx_n))p_b(x_n,Tx_n)+\alpha_4(p_b(x_n,Tx_n))p_b(x_n,Tx_n)\\
&+\alpha_5(p_b(x_n,Tx_n))p_b(Tx_n,T^2x_n)+L\alpha_6(p_b(x_n,Tx_n))p_b(x_n,Tx_n),
\end{aligned}
$$

and equivalently,

$$
\begin{aligned}
p_b(x_n,Tx_n)p_b(Tx_n,T^2x_n)\leq\ &[\alpha_1(p_b(x_n,Tx_n))+s\alpha_2(p_b(x_n,Tx_n))+2s\alpha_3(p_b(x_n,Tx_n))\\
&+\alpha_4(p_b(x_n,Tx_n))+L\alpha_6(p_b(x_n,Tx_n))]p_b(x_n,Tx_n)\\
&+[s\alpha_2(p_b(x_n,Tx_n))+\alpha_5(p_b(x_n,Tx_n))]p_b(Tx_n,T^2x_n).
\end{aligned}
$$

So from (3), we get

$$
p_np_{n+1}\leq[\alpha_1(p_n)+s\alpha_2(p_n)+2s\alpha_3(p_n)+\alpha_4(p_n)+L\alpha_6(p_n)]p_n+[s\alpha_2(p_n)+\alpha_5(p_n)]\,p_{n+1}
$$

and the equivalently

$$
p_{n+1}\leq\frac{\alpha_1(p_n)+s\alpha_2(p_n)+2s\alpha_3(p_n)+\alpha_4(p_n)+L\alpha_6(p_n)}{p_n-s\alpha_2(p_n)-\alpha_5(p_n)}p_n. \tag{5}
$$

On the other hand, we get from (B) that

$$
\begin{aligned}
\alpha_1(t)+2s\alpha_2(t)+2s\alpha_3(t)+\alpha_4(t)+\alpha_5(t)+L\alpha_6(t)\leq\ &2s[\alpha_1(t)+\alpha_2(t)+\alpha_3(t)\\
&+\alpha_4(t)+\alpha_5(t)+L\alpha_6(t)]\\
&<2s\frac{1}{2s^2}t=\frac{1}{s}t\leq t,\quad\forall t>0.
\end{aligned}
$$

Therefore, we have

$$\frac{\alpha_1(t)+s\alpha_2(t)+2s\alpha_3(t)+\alpha_4(t)+L\alpha_6(t)}{t-s\alpha_2(t)-\alpha_5(t)}<1, \quad \forall t>0. \tag{6}$$

It follows from (5) and (6) that

$$p_b(x_{n+1},Tx_{n+1})=p_{n+1}<p_n=p_b(x_n,Tx_n), \quad \forall n\in\mathbb{N}. \tag{7}$$

Therefore, $\{p_n\}_{n=1}^{\infty}$ is a decreasing sequence of real numbers which is bounded from below. So $\{p_n\}_{n=1}^{\infty}$ converges to some point $p\in[0,\infty)$. If $p>0$. From (6) and (A), we get

$$\begin{aligned} p=\lim_{n\to\infty}p_n=\limsup_{n\to\infty}p_n &\le \limsup_{n\to\infty}\frac{\alpha_1(p_n)+s\alpha_2(p_n)+2s\alpha_3(p_n)+\alpha_4(p_n)+\alpha_6(p_n)}{p_n-s\alpha_2(p_n)-\alpha_5(p_n)}p_n \\ &= \frac{\alpha_1(p)+s\alpha_2(p)+2s\alpha_3(p)+\alpha_4(p)+\alpha_6(p)}{p-s\alpha_2(p)-\alpha_5(p)}p<p. \end{aligned}$$

This is a contradiction. Hence

$$\lim_{n\to\infty}p_n=\lim_{n\to\infty}p_b(x_n,Tx_n)=0. \tag{8}$$

Now, we claim that

$$\lim_{n,m\to\infty}p_b(x_n,x_m)=0 \tag{9}$$

Arguing by contradiction, we assume that there exist $\varepsilon>0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n)>q(n)>n, \;\; p_b(x_{p(n)},x_{q(n)})\ge\varepsilon, \;\; p_b(x_{p(n)-1},x_{q(n)})<\varepsilon, \quad \forall n\in\mathbb{N}. \tag{10}$$

Therefore, we have

$$\begin{aligned} \varepsilon\le p_b(x_{p(n)},x_{q(n)}) &\le s[p_b(x_{p(n)},x_{p(n)-1})+p_b(x_{p(n)-1},x_{q(n)})]-p_b(x_{p(n)-1},x_{p(n)-1}) \\ &\le sp_b(x_{p(n)},x_{p(n)-1})+sp_b(x_{p(n)-1},x_{q(n)}) \\ &\le sp_b(x_{p(n)},x_{p(n)-1})+s\varepsilon \\ &= sp_b(x_{p(n)-1},Tx_{p(n)-1})+s\varepsilon. \end{aligned}$$

Using (8), we get

$$\varepsilon \leq \liminf_{n\to\infty} p_b(x_{p(n)},x_{q(n)}) \leq \limsup_{n\to\infty} p_b(x_{p(n)},x_{q(n)}) \leq s\varepsilon. \tag{11}$$

On the other hand, we find from (8) that $\lim_{n\to\infty} p_b(x_{p(n)},Tx_{p(n)}) = 0$. So there exists $N \in \mathbb{N}$ such that $\frac{1}{2s}p_b(x_{p(n)},Tx_{p(n)}) < \varepsilon$, $\forall n > N$, and from (10), we have

$$\frac{1}{2s}p_b(x_{p(n)},Tx_{p(n)}) < p_b(x_{p(n)+1},x_{q(n)+1}) = p_b(Tx_{p(n)},Tx_{q(n)}), \quad \forall n > N.$$

Therefore, from (C) for every $n > N$, we obtain

$$\begin{aligned}
&p_b(x_{p(n)},x_{q(n)})p_b(Tx_{p(n)},Tx_{q(n)})\\
&\quad\leq \alpha_1(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)}) + \alpha_2(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},Tx_{p(n)})\\
&\qquad+\alpha_3(p_b(x_{p(n)},x_{q(n)}))p_b(x_{q(n)},Tx_{q(n)}) + \alpha_4(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},Tx_{q(n)})\\
&\qquad+\alpha_5(p_b(x_{p(n)},x_{q(n)}))p_b(Tx_{p(n)},x_{q(n)})\\
&\qquad+L\alpha_6(p_b(x_{p(n)},x_{q(n)}))\min\{p_b(x_{p(n)},x_{q(n)}),p_b(x_{p(n)},Tx_{q(n)}),p_b(Tx_{p(n)},x_{q(n)})\}\\
&\leq \alpha_1(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)}) + \alpha_2(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},Tx_{p(n)})\\
&\qquad+\alpha_3(p_b(x_{p(n)},x_{q(n)}))p_b(x_{q(n)},Tx_{q(n)}) + \alpha_4(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},Tx_{q(n)})\\
&\qquad+\alpha_5(p_b(x_{p(n)},x_{q(n)}))p_b(Tx_{p(n)},x_{q(n)}) + L\alpha_6(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)}),
\end{aligned}$$

and this is equivalent to

$$\begin{aligned}
&p_b(x_{p(n)},x_{q(n)})p_b(x_{p(n)+1},x_{q(n)+1})\\
&\quad\leq \alpha_1(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)}) + \alpha_2(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},Tx_{p(n)})\\
&\qquad+\alpha_3(p_b(x_{p(n)},x_{q(n)}))p_b(x_{q(n)},Tx_{q(n)}) + s\alpha_4(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)})\\
&\qquad+s\alpha_4(p_b(x_{p(n)},x_{q(n)}))p_b(x_{q(n)},Tx_{q(n)}) - \alpha_4(p_b(x_{p(n)},x_{q(n)}))p_b(x_{q(n)},x_{q(n)})\\
&\qquad+s\alpha_5(p_b(x_{p(n)},x_{q(n)}))p_b(Tx_{p(n)},x_{p(n)}) + s\alpha_5(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)})\\
&\qquad-\alpha_5(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{p(n)}) + L\alpha_6(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)})\\
&\quad\leq \alpha_1(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)}) + \alpha_2(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},Tx_{p(n)})\\
&\qquad+\alpha_3(p_b(x_{p(n)},x_{q(n)}))p_b(x_{q(n)},Tx_{q(n)}) + \alpha_4(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)})
\end{aligned}$$

$$+\alpha_4(p_b(x_{p(n)},x_{q(n)}))p_b(x_{q(n)},Tx_{q(n)})+\alpha_5(p_b(x_{p(n)},x_{q(n)}))p_b(Tx_{p(n)},x_{p(n)})$$
$$+\alpha_5(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)})+L\alpha_6(p_b(x_{p(n)},x_{q(n)}))p_b(x_{p(n)},x_{q(n)}).$$

It follows that

$$\begin{aligned}
&p_b(x_{p(n)},x_{q(n)})p_b(x_{p(n)+1},x_{q(n)+1})\\
&\le [\alpha_1(p_b(x_{p(n)},x_{q(n)}))+s\alpha_4(p_b(x_{p(n)},x_{q(n)}))\\
&\quad +s\alpha_5(p_b(x_{p(n)},x_{q(n)}))+L\alpha_6(p_b(x_{p(n)},x_{q(n)}))]p_b(x_{p(n)},x_{q(n)})\\
&\quad +[\alpha_2(p_b(x_{p(n)},x_{q(n)}))+s\alpha_5(p_b(x_{p(n)},x_{q(n)}))]p_b(Tx_{p(n)},x_{p(n)})\\
&\quad +[\alpha_3(p_b(x_{p(n)},x_{q(n)}))+s\alpha_4(p_b(x_{p(n)},x_{q(n)}))]p_b(x_{q(n)},Tx_{q(n)}).
\end{aligned}$$

Hence from (B), we obtain

$$\begin{aligned}
p_b(x_{p(n)},x_{q(n)})p_b(x_{p(n)+1},x_{q(n)+1}) &\le \frac{1}{2s^3}[p_b(x_{p(n)},x_{q(n)})]^2\\
&\quad+\frac{1}{2s^2}p_b(x_{p(n)},x_{q(n)})p_b(Tx_{p(n)},x_{p(n)})\\
&\quad+\frac{1}{2s^2}p_b(x_{p(n)},x_{q(n)})p_b(x_{q(n)},Tx_{q(n)}).
\end{aligned} \tag{12}$$

From (8), (10), (11) and (12), we conclude that

$$\begin{aligned}
\varepsilon^2 &\le \limsup_{n\to\infty}[p_b(x_{p(n)},x_{q(n)})p_b(x_{p(n)+1},x_{q(n)+1})]\\
&\le \limsup_{n\to\infty}[\frac{1}{2s^3}(p_b(x_{p(n)},x_{q(n)})^2\\
&\quad+\frac{1}{2s^3}p_b(x_{p(n)},x_{q(n)})p_b(Tx_{p(n)},x_{p(n)})\\
&\quad+\frac{1}{2s^3}p_b(x_{p(n)},x_{q(n)})p_b(x_{q(n)},Tx_{q(n)})]\\
&\le \frac{1}{2s^2}(s\varepsilon)^2=\frac{1}{2}\varepsilon^2<\varepsilon^2.
\end{aligned}$$

This is a contradiction. Hence $\lim_{m,n\to\infty} d(x_n,x_m)=0$. By completeness of (X,p_b) there exists $x\in X$ such that

$$p_b(x,x)=\lim_{n\to\infty}p_b(x_n,x)=\lim_{n,m\to\infty}p_b(x_n,x_m)=0. \tag{13}$$

We shall prove that for every $n \in \mathbb{N}$,

$$(I)\ \frac{1}{2s} p_b(x_n, Tx_n) < p_b(x_n, x),$$
$$or \tag{14}$$
$$(II)\ \frac{1}{2s} p_b(Tx_n, T^2x_n) < p_b(Tx_n, x).$$

Arguing by contradiction, we assume that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2s} p_b(x_m, Tx_m) \geq p_b(x_m, x) \text{ and } \frac{1}{2s} p_b(Tx_m, T^2x_m) \geq p_b(Tx_m, x). \tag{15}$$

Therefore, from (5), (7) and (15), we obtain

$$\begin{aligned} p_b(x_m, Tx_m) &\leq s[p_b(x_m, x) + p_b(x, Tx_m)] - p_b(x, x) \\ &\leq s p_b(x_m, x) + s p_b(x, Tx_m) \\ &\leq s \frac{1}{2s} p_b(x_m, Tx_m) + s \frac{1}{2s} p_b(Tx_m, T^2x_m) \\ &< \frac{1}{2} p_b(x_m, Tx_m) + \frac{1}{2} p_b(x_m, Tx_m) = p_b(x_m, Tx_m). \end{aligned}$$

This is a contradiction. Hence (14) holds. From part (I) of (14), we get

$$\begin{aligned} p_b(x_n, x) p_b(Tx_n, Tx) &\leq \alpha_1(p_b(x_n, x)) p_b(x_n, x) + \alpha_2(d(x_n, x)) p_b(x_n, Tx) \\ &\quad + \alpha_3(p_b(x_n, x)) p_b(Tx_n, x) + \alpha_4(p_b(x_n, x)) p_b(x_n, Tx_n) \\ &\quad + \alpha_5(p_b(x_n, x)) d(x, Tx) \\ &\quad + L\alpha_6(p_b(x_n, x)) \min\{p_b(x_n, x), p_b(x_n, Tx), p_b(Tx_n, x)\} \\ &\leq \alpha_1(p_b(x_n, x)) p_b(x_n, x) + s\alpha_2(p_b(x_n, x)) p_b(x_n, Tx_n) \\ &\quad + s\alpha_2(p_b(x_n, x)) p_b(Tx_n, Tx) + \alpha_3(p_b(x_n, x)) p_b(Tx_n, x) \\ &\quad + \alpha_4(p_b(x_n, x)) p_b(x_n, Tx_n) + s\alpha_5(p_b(x_n, x)) p_b(x, Tx_n) \\ &\quad + s\alpha_5(p_b(x_n, x)) p_b(Tx_n, Tx)) + L\alpha_6(p_b(x_n, x)) p_b(x_n, x), \end{aligned}$$

which is equivalent to

$$
\begin{aligned}
p_b(x_n,x)p_b(Tx_n,Tx) \le {} & [\alpha_1(p_b(x_n,x)) + L\alpha_6(p_b(x_n,x))]p_b(x_n,x) \\
& + [s\alpha_2(p_b(x_n,x)) + \alpha_4(p_b(x_n,x))]p_b(x_n,Tx_n) \\
& + [s\alpha_2(p_b(x_n,x)) + s\alpha_5(p_b(x_n,x))]p_b(Tx_n,Tx) \\
& + [\alpha_3(p_b(x_n,x))p_b(Tx_n,x) + s\alpha_5(p_b(x_n,x))]p_b(x,Tx_n).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
p_b(Tx_n,Tx) \le {} & \frac{\alpha_1(p_b(x_n,x)) + L\alpha_6(p_b(x_n,x))}{p_b(x_n,x) - s\alpha_2(p_b(x_n,x)) - s\alpha_5(p_b(x_n,x)}p_b(x_n,x) \\
& + \frac{s\alpha_2(p_b(x_n,x)) + \alpha_4(p_b(x_n,x))}{p_b(x_n,x) - s\alpha_2(p_b(x_n,x)) - s\alpha_5(p_b(x_n,x))}p_b(x_n,Tx_n) \\
& + \frac{\alpha_3(p_b(x_n,x)) + \alpha_5(p_b(x_n,x))}{p_b(x_n,x) - s\alpha_2(p_b(x_n,x)) - s\alpha_5(p_b(x_n,x))}p_b(x,Tx_n). \qquad (16)
\end{aligned}
$$

From (6) and (16), we have

$$
\begin{aligned}
p_b(Tx_n,Tx) &\le p_b(x_n,x) + p_b(x_n,Tx_n) + p_b(x,Tx_n) \\
&= p_b(x_n,x) + p_n + p_b(x,x_{n+1}).
\end{aligned}
$$

Using (8) and (13), we obtain

$$
\lim_{n\to\infty} p_b(Tx_n,Tx) = 0. \qquad (17)
$$

Since

$$
\begin{aligned}
p_b(x,Tx) &\le s[p_b(x,Tx_n) + p_b(Tx_n,Tx)] - p_b(Tx_n,Tx_n) \\
&\le s[p_b(x,Tx_n) + p_b(Tx_n,Tx)] \\
&= p_b(x,x_{n+1}) + p_b(Tx_n,Tx),
\end{aligned}
$$

we find from (13) and (17) that $x = Tx$. From part (II) of (14), we get

$$
\begin{aligned}
& p_b(x_n,x)p_b(T^2x_n,Tx) \\
& \leq \alpha_1(p_b(Tx_n,x))p_b(Tx_n,x) + \alpha_2(p_b(Tx_n,x))p_b(Tx_n,Tx) \\
& \quad + \alpha_3(p_b(Tx_n,x))p_b(T^2x_n,x) + \alpha_4(p_b(Tx_n,x))p_b(Tx_n,T^2x_n) \\
& \quad + \alpha_5(p_b(Tx_n,x))p_b(x,Tx) \\
& \quad + L\alpha_6(p_b(Tx_n,x))\min\{p_b(Tx_n,x),p_b(Tx_n,Tx),p_b(T^2x_n,x)\} \\
& \leq \alpha_1(p_b(Tx_n,x))p_b(Tx_n,x) + s\alpha_2(p_b(Tx_n,x))p_b(Tx_n,T^2x_n) \\
& \quad + s\alpha_2(p_b(Tx_n,x))p_b(T^2x_n,Tx) - \alpha_2(p_b(Tx_n,x))p_b(T^2x_n,T^2x_n) \\
& \quad + \alpha_3(p_b(Tx_n,x))p_b(T^2x_n,x) + \alpha_4(p_b(Tx_n,x))p_b(Tx_n,T^2x_n) \\
& \quad + s\alpha_5(p_b(Tx_n,x))p_b(x,T^2x_n) + s\alpha_5(p_b(Tx_n,x))p_b(T^2x_n,Tx) \\
& \quad - \alpha_5(p_b(Tx_n,x))p_b(T^2x_n,T^2x_n) + L\alpha_6(p_b(Tx_n,x))p_b(Tx_n,x) \\
& \leq \alpha_1(p_b(Tx_n,x))p_b(Tx_n,x) + s\alpha_2(p_b(Tx_n,x))p_b(Tx_n,T^2x_n) \\
& \quad + s\alpha_2(p_b(Tx_n,x))p_b(T^2x_n,Tx) + \alpha_3(p_b(Tx_n,x))p_b(T^2x_n,x) \\
& \quad + \alpha_4(p_b(Tx_n,x))p_b(Tx_n,T^2x_n) + s\alpha_5(p_b(Tx_n,x))p_b(x,T^2x_n) \\
& \quad + s\alpha_5(p_b(Tx_n,x))p_b(T^2x_n,Tx) + L\alpha_6(p_b(Tx_n,x))d(Tx_n,x),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
p_b(Tx_n,x)p_b(T^2x_n,Tx) \leq {} & [\alpha_1(p_b(Tx_n,x)) + L\alpha_6(p_b(Tx_n,x))]p_b(Tx_n,x) \\
& + [s\alpha_2(p_b(Tx_n,x)) + \alpha_4(p_b(Tx_n,x))]p_b(Tx_n,T^2x_n) \\
& + [s\alpha_2(p_b(Tx_n,x)) + s\alpha_5(p_b(Tx_n,x))]p_b(T^2x_n,Tx) \\
& + [\alpha_3(p_b(Tx_n,x)) + s\alpha_5(p_b(Tx_n,x))]p_b(T^2x_n,x).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
p_b(T^2x_n,Tx) < {} & \frac{\alpha_1(p_b(Tx_n,x))+L\alpha_6(p_b(Tx_n,x))}{p_b(Tx_n,x)-s\alpha_2(p_b(Tx_n,x))-s\alpha_5(p_b(Tx_n,x)}p_b(Tx_n,x) \\
& + \frac{s\alpha_2(p_b(Tx_n,x))+\alpha_4(p_b(Tx_n,x))}{p_b(Tx_n,x)-s\alpha_2(p_b(Tx_n,x))-s\alpha_5(p_b(Tx_n,x))}p_b(Tx_n,T^2x_n) \\
& + \frac{\alpha_3(p_b(Tx_n,x))+s\alpha_5(p_b(Tx_n,x))}{p_b(Tx_n,x)-s\alpha_2(p_b(Tx_n,x))-s\alpha_5(p_b(Tx_n,x))}p_b(x,T^2x_n). \qquad (18)
\end{aligned}
$$

From (6) and (18), we conclude that

$$
\begin{aligned}
0 \le p_b(T^2x_n,Tx) &\le p_b(Tx_n,x)+p_b(Tx_n,T^2x_n)+p_b(x,T^2x_n) \\
&= p_b(x_{n+1},x)+p_{n+1}+p_b(x,x_{n+2}).
\end{aligned}
$$

Hence from (8) and (13), we obtain

$$\lim_{n\to\infty} p_b(T^2x_n,Tx)=0. \qquad (19)$$

Since

$$
\begin{aligned}
p_b(x,Tx) &\le s[p_b(x,T^2x_n)+p_b(T^2x_n,Tx)]-p_b(T^2x_n,T^2x_n) \\
&\le s[p_b(x,T^2x_n)+p_b(T^2x_n,Tx)] \\
&= sp_b(x,x_{n+2})+p_b(T^2x_n,Tx),
\end{aligned}
$$

we obtain from (13) and (19) that $x = Tx$.

Theorem 2.2. *Let (X, b) be a complete partial b-metric space with coefficient $s>1$ and $T: X \to X$ be a mapping satisfying the following condition:*

$$b(Tx,Ty) \le \lambda b(x,y), \forall x,y \in X,$$

where $\lambda \in [0,1)$. Then T has a unique fixed point $u \in X$ and $b(u,u)=0$.

Proof. It is sufficient to take $\alpha_1(t)=\frac{1}{2s^3}t$, $\lambda=\frac{1}{2s^3}$ and $\alpha_2(t)=\alpha_3(t)=\alpha_4(t)=\alpha_5(t)=\alpha_6(t)=0$ in Theorem .

Theorem 2.3. *Let (X, b) be a complete partial b-metric space with coefficient $s>1$ and $T: X \to X$ be a mapping satisfying the following condition:*

$$b(Tx,Ty) \le \lambda[b(x,Tx)+b(y,Ty)], \forall x,y \in X,$$

where $\lambda \in [0, \frac{1}{2})$ *and* $\lambda \neq \frac{1}{s}$.

Proof. It is sufficient to take $\alpha_4(t) = \alpha_5(t) = \frac{1}{2s^3}t$, $\lambda = \frac{1}{2s^3}$ and $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \alpha_6(t) = 0$ in Theorem .

Example 2.1. *Put* $X = \mathbb{N} \cup \{m + \frac{1}{n+2} : m, n \in \mathbb{N}\}$. *Let* P *be a real number such that* $p > 1$ *and* $d \colon X \times X \to \mathbb{R}^+$ *be defined by*

$$p_b(x,y) = \max\{x,y\}^p + |\, x - y \,|^p \ .$$

Define a mapping T *on* X *by*

$$T(x) = \begin{cases} 1 & x \in \mathbb{N}, \\ 7m + \frac{1}{n+2} & x = m + \frac{1}{n}. \end{cases}$$

Then T *satisfies in the assumption of Theorem 2.1.*

Proof. It is obviously that (X, p_b) is a complete partial b-metric space with coefficient $s = 2^p$, but it is neither a b-metric nor a partial metric space [23] and 1 is a unique fixed point of T. if $n < m$, we have

$$\begin{aligned} &\frac{1}{2}d(m + \frac{1}{n+2}, T(m + \frac{1}{n+2})) \\ &= \frac{1}{2}d(m + \frac{1}{n+2}, 7m + \frac{1}{n+2}) \\ &= \frac{1}{2}\left(\max\{m + \frac{1}{n+2}, 7m + \frac{1}{n+2}\} + |\, m + \frac{1}{n+2} - 7m - \frac{1}{n+2} \,|\right) \\ &= \frac{1}{2}(7m + \frac{1}{n+2} + 6m) \\ &= \frac{1}{2}(13m + \frac{1}{n+2}) \end{aligned}$$

and

$$\begin{aligned} d(m + \frac{1}{n+2}, n + \frac{1}{n+2}) &= \max\{m + \frac{1}{n+2}, n + \frac{1}{n+2}\} + |\, m + \frac{1}{n+2} - n - \frac{1}{n+2} \,| \\ &= m + \frac{1}{n+2} + m - n \\ &< 2m + \frac{1}{n+2}. \end{aligned}$$

So for $n < m$, $\frac{1}{2}d(m + \frac{1}{n+2}, T(m + \frac{1}{n+2})) \not< d(m + \frac{1}{n+2}, n + \frac{1}{n+2})$. Therefore T satisfies all the conditions of the Theorem 2.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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