Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 4, 462-478
ISSN: 1927-6303

ON LOCAL NONUNIQUE RANDOM FIXED POINT THEOREMS FOR DHAGE TYPE CONTRACTIVE MAPPINGS IN POLISH SPACES

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Abstract. The authors present some local nonumique random fixed point theorems for random mappings in a separable complete metric space satisfying certain Dhage type contractive condition. A case of a metric space with three metrics is also considered for discussion. The established local nnonunique fixed point theorems also have extended to partially ordered polish space and fixed point theorems with PPF dependence.

Keywords: Polish space; contractive mapping; random fixed point theorem.

2010 AMS Subject Classification: 47H10, 54G25.

1. Introduction

Throughout the rest of the paper, let X denote a polish space, *i.e.*, a complete, separable metric space with a metric d. Let $(\Omega, \mathscr{A})$ denote a measurable space with σ-algebra $\mathscr{A}$. A function $x : \Omega \to X$ is said to be a random variable if it is measurable. A mapping $T : \Omega \times X \to X$ is called random mapping if $T(.,x)$ is measurable for each $x \in X$. A random mapping on a metric space X is denoted by $T(\omega,x)$ or simply $T(\omega)x$ for $\omega \in \Omega$ and $x \in X$. A random mapping $T(\omega)$ is said to be continuous on X into itself if the mapping $T(\omega,\cdot)$ is continuous on X for each

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Received June 30, 2014

$\omega \in \Omega$. A measurable function $x : \Omega \to X$ is called a random fixed point of the random mapping $T(\omega)$ if $T(\omega)x(\omega) = x(\omega)$ for all $\omega \in \Omega$. Random fixed point theory for random operators in separable Banach spaces is initiated by Hans [8] and Spacek [13] and further developed by several authors in the literature. A brief survey of such random fixed point theorems appear in Joshi and Bose [9]. The details of random operators and random solutions of random equations appears in Bharucha-Reid [3], Hans [8] and Spacek [13].

The following is the key result for the development of random fixed point theory and applications to random equations.

Theorem 1.1. *Let X be a Polish space, that is, a complete and separable metric space. Then,*

(a) *If $\{x_n(\omega)\}$ is a sequence of random variables converging to $x(\omega)$ for all $\omega \in \Omega$, then $x(\omega)$ is also a random variable.*

(b) *If $T(\omega,\cdot)$ is continuous for each $\omega \in \Omega$ and $x : \Omega \to X$ is a random variable, then $T(\omega)x$ is also a random variable.*

Given a metric space (X,d), we define a closed ball $\mathscr{B}_d[x_0,r]$ in X centered at x_0 of radius r for some $x_0 \in X$ and for some real number $r > 0$ by

$$\mathscr{B}_d[x_0,r] = \{x \in X \mid d(x_0,x) \leq r\}.$$

The deterministic or classical local nonunique fixed pint theorems for the contractive maps in a closed ball $B[x_0,r]$ have been proved in Achari [1] and Dhage [5]. However, the random local nonunique fixed point theorems seem to have not been proved so ar in the literature. The aim of the present study is to establish a couple of local random fixed point theorems for the random mappings on Polish spaces. In he following section we prove the main results of this paper.

2. Main results

Our first local nonunique random fixed point theorem (in short RFPT) is as follows.

Theorem 2.1 *Let $(\Omega,\mathscr{A})$ be a measurable space and let (X,d) be a Polish space. Let $\mathscr{B}_d[x_0,r]$ be a closed ball centered at x_0 of radius r w.r.t. the metric d for some $x_0 \in X$ and for some real number $r > 0$. Let $T(\omega)$ be a continuous random selfmapping of a complete and separable* *metric space X into itself satisfying for each $\omega \in \Omega$,*

$$\begin{aligned} 0 \le \min\Big\{ & d(T(\omega)x, T(\omega)y), d(x, T(\omega)x), d(y, T(\omega)y), \\ & \frac{d(x, T(\omega)x)[1+d(y, T(\omega)y)]}{1+d(x,y)}, \frac{d(y, T(\omega)y)[1+d(x, T(\omega)x)]}{1+d(x,y)} \Big\} \\ & + b(\omega) \min\big\{ d(x, T(\omega)y), d(y, T(\omega)x) \big\} \\ & \le q(\omega) \max\big\{ d(x,y), [\min\{d(x, T(\omega)x), d(y, T(\omega)y)\}] \big\} \end{aligned} \tag{2.1}$$

for all $x, y \in X$, where $b : \Omega \to \mathbb{R}$ and $q : \Omega \to \mathbb{R}_+$ are measurable functions satisfying $0 \le q(\omega) < 1$. Furthermore if

$$d(x_0, T(\omega)x_0) \le [1 - q(\omega)]\, r \tag{2.2}$$

for all $\omega \in \Omega$, then $T(\omega)$ has a random fixed point in $B_d[x_0, r]$ which is unique if $b(\omega) > q(\omega)$ for all $\omega \in \Omega$.

Proof. Let $x : \Omega \to X$ be arbitrary measurable function and consider the sequence of successive iterates of $T(\omega)$ at x defined by

$$x_0 = x, x_1 = T(\omega)x, ..., x_n = T(\omega)x_{n-1} \tag{2.3}$$

for each $n \in \mathbb{N}$. Clearly, $\{x_n\}$ is a sequence of measurable functions on Ω into X. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then the result follows immediately. Now we assume that $x_n \ne x_{n+1}$ for each $n \in \mathbb{N}$. We shall show that $\{x_n\}$ is Cauchy sequence of measurable functions on Ω into X. Taking $x = x_0$ and $y = x_1$ in (2.2), we obtain

$$\begin{aligned} 0 \le \min\Big\{ & d(T(\omega)x_0, T(\omega)x_1), d(x_0, T(\omega)x_0), d(x_1, T(\omega)x_1), \\ & \frac{d(x_0, T(\omega)x_0)[1+d(x_1, T(\omega)x_1)]}{1+d(x_0,x_1)}, \frac{d(x_1, T(\omega)x_1)[1+d(x_0, T(\omega)x_0)]}{1+d(x_0,x_1)} \Big\} \\ & + b(\omega) \min\big\{ d(x_0, T(\omega)x_1), d(x_1, T(\omega)x_0) \big\} \\ & \le q(\omega) \max\big\{ d(x_0,x_1), [\min\{d(x_0, T(\omega)x_0), d(x_1, T(\omega)x_1)\}] \big\}, \end{aligned}$$

which further gives

$$0 \le \min\Big\{ d(x_1,x_2), d(x_0,x_1), d(x_1,x_2),$$

$$\frac{d(x_0,x_1)[1+d(x_1,x_2)]}{1+d(x_0,x_1)}, \frac{d(x_1,x_2)[1+d(x_0,x_1)]}{1+d(x_0,x_1)}\Big\}$$

$$+b(\omega)\min\big\{d(x_0,x_2),d(x_1,x_1)\big\}$$

$$\le q(\omega)\max\big\{d(x_0,x_1),[\min\{d(x_0,x_1),d(x_1,x_2)\}]\big\},$$

or,

$$0 \le \min\left\{d(x_1,x_2),d(x_0,x_1),\frac{d(x_0,x_1)[1+d(x_1,x_2)]}{1+d(x_0,x_1)}\right\}$$

$$+b(\omega)\min\big\{d(x_0,x_2),0\big\}$$

$$\le q(\omega)\max\big\{d(x_0,x_1),[\min\{d(x_0,x_1),d(x_1,x_2)\}]\big\}.$$

This further gives

$$\begin{aligned}&\min\left\{d(x_1,x_2),d(x_0,x_1),\frac{d(x_0,x_1)[1+d(x_1,x_2)]}{1+d(x_0,x_1)}\right\}\\&\qquad\le q(\omega)\max\big\{d(x_0,x_1),[\min\{d(x_0,x_1),d(x_1,x_2)\}]\big\}.\end{aligned}\tag{2.4}$$

If

$$\min\{d(x_1,x_2),d(x_0,x_1)\}=d(x_0,x_1),$$

then

$$d(x_0,x_1)\le\frac{d(x_0,x_1)[1+d(x_1,x_2)]}{1+d(x_0,x_1)}.$$

Hence, from (2.3) it follows that

$$d(x_0,x_1)\le q(\omega)d(x_0,x_1),$$

which is a contraction since $q=q(\omega)<1$ for all $\omega\in\Omega$. So

$$\min\{d(x_1,x_2),d(x_0,x_1)\}=d(x_1,x_2).$$

Now there are two cases. In the first case we have

$$d(x_1,x_2)\le q\,d(x_0,x_1).$$

In the second case we have

$$\frac{d(x_0,x_1)[1+d(x_1,x_2)]}{1+d(x_0,x_1)}\le q\,d(x_0,x_1),$$

which further gives

$$d(x_1,x_2) \leq q\,d(x_0,x_1).$$

Proceeding in this way, by induction, it follows that

$$d(x_n,x_{n+1}) \leq qd(x_{n-1},x_n)$$

for each $n \in \mathbb{N}$. From (2.3) it follows that

$$\begin{aligned} d(x_n,x_{n+1}) &\leq qd(x_{n-1},x_n) \\ &\leq q^2 d(x_{n-2},x_{n-1}) \\ &\vdots \\ &\leq q^n d(x_0,x_1). \end{aligned} \tag{2.5}$$

Now for any positive integer p, we obtain by triangle inequality,

$$\begin{aligned} d(x_n,x_{n+p}) &\leq d(x_n,x_{n+1}) + \ldots + d(x_{n+p-1},x_{n+p}) \\ &\leq q^n d(x_0,x_1) + \ldots + q^{n+p-1} d(x_0,x_1) \\ &\leq \left[q^n + q^{n+1} + \ldots + q^{n+p-1}\right] d(x_0,x_1) \\ &\leq \frac{q^n(1-q^{p-1})}{1-q} \\ &\leq \frac{q^n}{1-q} \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.6}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X.

Next we show that $\{x_n\} \subset B_d[x_0,r]$. Now,

$$d(x_0,x_1) = d(x_0,T(\omega)x_0) \leq \left[1-q\right] r \leq r.$$

Again,

$$\begin{aligned} d(x_0,x_2) &\leq d(x_0,x_1) + d(x_1,x_2) \\ &\leq (1-q)r + qd(x_0,x_1) \\ &\leq (1-q)r + q(1-q)r \end{aligned}$$

$$\leq \left[1-q^2\right]r$$
$$\leq r. \qquad [q<1]$$

In general for any $n \in \mathbb{N}$, one has

$$\begin{aligned} d(x_0,x_n) &\leq d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{n-1},x_n) \\ &\leq \left[1+q+q^2+\cdots+q^n\right]d(x_0,x_1) \\ &\leq \left[1+q+q^2+\cdots+q^n\right](1-q)r \\ &\leq \frac{\left[1-q^n\right]}{1-q}(1-q)r \\ &\leq r. \qquad [q<1] \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in the closed ball $\mathscr{B}_d[x_0,r]$. The metric space X being $T(\omega)$-orbitally complete, $B_d[x_0,r]$ is also $T(\omega)$-orbitally complete. Hence, there is a measurable function $x^*:\Omega\to\mathscr{B}_d[x_0,r]$ such that $\lim_{n\to\infty}x_n=x^*$. Again as $T(\omega)$ is continuous, we have

$$T(\omega)x^*(\omega)=\lim_{n\to\infty}T(\omega)x_n(\omega)=\lim_{n\to\infty}x_{n+1}(\omega)=x^*(\omega)$$

for each $\omega\in\Omega$. Thus x^* is a random fixed point of the random mapping $T(\omega)$ in $\mathscr{B}_d[x_0,r]$ and the sequence $\{T^nx_0\}$ of successive iterations converges to $x^*(\omega)$. Next, we assume that $b>q$ on Ω. If $y^*(\neq x^*)$ is another random fixed point of the random mapping $T(\omega)$ in $\mathscr{B}_d[x_0,r]$, then from (2.2) we get a contradiction. This completes the proof.

Corollary 2.1. *Let (X,d) be a complete metric space. Let $\mathscr{B}_d[x_0,r]$ be a closed ball centered at x_0 of radius r w.r.t the metric d for some $x_0\in X$ and for some real number $r>0$. Let T be a continuous selfmapping of X into itself satisfying*

$$\begin{aligned} 0 \leq \min\Big\{&d(Tx,Ty),d(x,Tx),d(y,Ty), \\ &\frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)},\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\Big\} \\ &+b\min\left\{d(x,Ty),d(y,Tx)\right\} \\ &\leq q\max\left\{d(x,y),[\min\{d(x,Tx),d(y,Ty)\}]\right\} \end{aligned} \tag{2.7}$$

for all $x, y \in X$, where $b \in \mathbb{R}$ and $q \in \mathbb{R}_+$ are two numbers satisfying $0 \le q < 1$. Furthermore if

$$d(x_0, Tx_0) \le [1-q]\, r. \tag{2.8}$$

Then T has a fixed point in $\mathscr{B}_d[x_0, r]$ which is unique if $b > q$.

Remark 2.1. We remark that Corollary 2.1 includes the nonunique local fixed point theorems of Achari [1] and Dhage [5] as special cases.

Next we prove a local nonunique random fixed point theorem in a metric space X with three metrics on it. It is possible that a metric space may be complete w.r.t. a metric, but may not be complete w.r.t. another metric on X. The study of fixed point theorems along these lines is initiated by Maia [10]. In these circumstance we prove the following local nonunique random fixed point result in the area of random fixed point theory.

Theorem 2.2. *Let $(\Omega, \mathscr{A})$ be a measurable space and let X be a Polish space with three metrics d, d_1 and d_2. Let $\mathscr{B}_{d_2}[x_0, r]$, $\mathscr{B}_{d_1}[x_0, r]$ and $\mathscr{B}_d[x_0, r]$ denote the closed balls centered at x_0 of radius r w.r.t. the metrics d_2, d_1 and d respectively. Suppose that $T : \Omega \times X \to X$ be a random mapping satisfying (2.1). Assume that the following conditions hold in X.*

(a) *$d_2(x, y) \le d_1(x, y) \le d(x, y)$ for all $x, y \in X$.*
(b) *X is complete w.r.t. d_1.*
(b) *$T(\omega)$ is continuous w.r.t. d_2.*

Furthermore, if the condition (2.2) holds, then $T(\omega)$ has a random fixed point in $\mathscr{B}_{d_2}[x_0, r]$ and which is unique if $b(\omega) > q(\omega)$ for each $\omega \in \Omega$.

Proof. Let $x : \Omega \to X$ be arbitrary measurable function and consider the sequence of successive iterates of $T(\omega)$ at x defined by

$$x_0 = x,\, x_1 = T(\omega)x, \ldots., x_n = T(\omega)x_{n-1} \tag{2.9}$$

for each $n \in \mathbb{N}$. Clearly, $\{x_n\}$ is a sequence of measurable functions on Ω into X. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then the result follows immediately. Now we assume that $x_n \ne x_{n+1}$ for each $n \in \mathbb{N}$. It can be shown as in the proof of Theorem 2.2 that $\{x_n\} \subset \mathscr{B}_d[x_0, r]$ satisfying

$$d(x_n, x_{n+p}) \le \frac{q^n}{1-q} d(x_0, x_1)$$

for some positive integer p. By hypothesis (a),

$$d_1(x_n, x_{n+p}) \leq \frac{q^n}{1-q} d(x_0, x_1) \to 0 \text{ as } n \to \infty.$$

This shows that $\{x_n\}$ is Cauchy sequence of measurable functions in X w.r.t. the metric d_1. From hypothesis (a) it follows that

$$\mathscr{B}_d[x_0, r] \subseteq \mathscr{B}_{d_1}[x_0, r] \subseteq \mathscr{B}_{d_2}[x_0, r].$$

Hence,

$$\{x_n\} \subseteq \mathscr{B}_{d_1}[x_0, r] \subseteq \mathscr{B}_{d_2}[x_0, r].$$

Since the metric space (X, d_1) is complete, the closed ball $\mathscr{B}_{d_1}[x_0, r]$ is also complete. Hence, there is a point $x^* \in \mathscr{B}_{d_1}[x_0, r]$ such that $\lim_{n\to\infty} x_n(\omega) = x^*(\omega)$ for each $\omega \in \Omega$. Again, by hypotheses (a), $x^* \in \mathscr{B}_{d_2}[x_0, r]$ and

$$d_2(x_n, x^*) \leq d_1(x_n, x^*) \to 0 \quad \text{as} \quad n \to \infty.$$

Now the random operator $T(\omega)$ is orbitally continuous w.r.t. d_2, so we have

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for each $\omega \in \Omega$. Thus x^* is a random fixed point of the random mapping $T(\omega)$ in $\mathscr{B}_{d_2}[x_0, r]$. Next assume that $b > q$ on Ω. If $y^* (\neq x^*)$ is another random fixed point of the random mapping $T(\omega)$ in $\mathscr{B}_{d_2}[x_0, r]$, then from (2.1) we get a contradiction. This completes the proof.

Corollary 2.2. *Let X be a metric space with three metrics d, d_1 and d_2. Let $\mathscr{B}_{d_2}[x_0, r]$, $\mathscr{B}_{d_1}[x_0, r]$ and $\mathscr{B}_d[x_0, r]$ denote the closed balls centered at x_0 of radius r w.r.t. the metrics d_2, d_1 and d respectively. Suppose that $T : X \to X$ be a mapping satisfying (2.7). Assume that the following conditions hold in X.*

(a) *$d_2(x, y) \leq d_1(x, y) \leq d(x, y)$ for all $x, y \in X$.*

(b) *X is complete w.r.t. d_1.*

(b) *T is continuous w.r.t. d_2.*

Furthermore, if the condition (2.8) holds, then T has a random fixed point $\xi(\omega)$ in $\mathscr{B}_{d_2}[x_0, r]$ and the sequence $\{T^n(\omega)x_0\}$ converges to $\xi(\omega)$. Furthermore, ξ is unique if $b > q$.

3. Local RFPTs in partially ordered metric spaces

We equip the metric space X with an order relation $\leq$ which is a reflexive, antisymmetric and transitive relation in X. The metric space X together with the order relation $\leq$ is called a partially ordered metric space denoted by $(X,\leq,d)$. A random mapping $T:\Omega\times X\to X$ is called nondecreasing if for any $x,y\in X$ with $x\leq y$ we have that $T(\omega)x\leq T(\omega)y$ for all $\omega\in\Omega$. Similarly, the random mapping $T:\Omega\times X\to X$ is called nonincreasing if for any $x,y\in X$, $x\leq y$ implies $T(\omega)x\geq T(\omega)y$ for all $\omega\in\Omega$. A monotone random mapping which is either nondecreasing or nonincreasing on X.

The investigation of the existence of fixed points in partially ordered sets was first considered in Ram and Reuriungs [12]. This study is continued in Nieto and Rodriguer-Lopez [11], Dhage *et al.* [7] and many others by assuming the existence of only lower solution instead of usual approach where both the lower and upper solutions are assumed to exist. The fixed point theorems of this type are applicable to obtain existence and uniqueness results for nonlinear ordinary differential equations (cf. Nieto and Lopez [11]). Below we prove some local nonunique random fixed point theorems for monotone random mappings in separable and complete metric spaces.

Theorem 3.1. *Let $(\Omega,\mathscr{A})$ be a measurable space and let (X,d) be a partially ordered Polish space. Let $\mathscr{B}_d[x_0,r]$ be a closed ball centered at x_0 of radius r w.r.t. the metric d for some $x_0\in X$ and for some real number $r>0$. Let $T:\Omega\times X\to X$ be a monotone nondecreasing random mapping satisfying for each $\omega\in\Omega$,*

$$\begin{aligned}0\leq \min\Big\{&d(T(\omega)x,T(\omega)y),d(x,T(\omega)x),d(y,T(\omega)y),\\&\frac{d(x,T(\omega)x)[1+d(y,T(\omega)y)]}{1+d(x,y)},\frac{d(y,T(\omega)y)[1+d(x,T(\omega)x)]}{1+d(x,y)}\Big\}\\&+b(\omega)\min\big\{d(x,T(\omega)y),d(y,T(\omega)x)\big\}\\&\leq q(\omega)\max\big\{d(x,y),[\min\{d(x,T(\omega)x),d(y,T(\omega)y)\}]\big\}\end{aligned}\tag{3.1}$$

for all comparable elements $x,y\in X$, where $b:\Omega\to\mathbb{R}$ and $q:\Omega\to\mathbb{R}_+$ are measurable functions satisfying $0\leq q(\omega)<1$. Further if $T(\omega)$ is continuous and there exists a measurable function $x_0:\Omega\to X$ such that $x_0\leq T(\omega)x_0$ satisfying (2.2) for all $\omega\in\Omega$, then the random mapping $T(\omega)$ has a random fixed point $\xi(\omega)$ in $\mathscr{B}_{d_0}[x_0,r]$ and the random sequence of successive iterations

$\{T^n(\omega)(x_n)\}$ converges monotonically to ξ. Furthermore, the random fixed point ξ is unique if every pair of elements of X has a lower and an upper bound in X and $b > q$ on Ω.

Proof. Define a sequence $\{x_n\}$ of successive approximations of $T(\omega)$ by

$$x_{n+1} = T(\omega)x_n, n = 0, 1, 2, \ldots.$$

Clearly $\{x_n\}$ is a sequence of measurable functions from Ω into X such that

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq \cdots.$$

Now, it can be shown as in the proof of Theorem 2.1 that $\{x_n\}$ is a Cauchy sequence of measurable functions in $\mathscr{B}_d[x_0, r]$. The rest of the proof is similar to Theorem 2.1 and we omit the details.

Corollary 3.1. *Let $(X, \leq, d)$ be a partially ordered complete metric space. Let $\mathscr{B}_d[x_0, r]$ be a closed ball centered at x_0 of radius r w.r.t. the metric d for some $x_0 \in X$ and for some real number $r > 0$. Let $T : X \to X$ be a monotone nondecreasing mapping satisfying*

$$\begin{aligned} 0 \leq \min\Big\{ & d(Tx,Ty), d(x,Tx), d(y,Ty), \\ & \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}, \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} \Big\} \\ & + b\min\big\{d(x,Ty), d(y,Tx)\big\} \\ & \leq q\max\big\{d(x,y), [\min\{d(x,Tx), d(y,Ty)\}]\big\} \end{aligned} \tag{3.2}$$

for all comparable elements $x, y \in X$, where $b \in \mathbb{R}$ and $q \in \mathbb{R}_+$ are two numbers satisfying $0 \leq q < 1$. Further if T is continuous and there exists a measurable function $x_0 \in X$ such that $x_0 \leq Tx_0$ satisfying the condition (2.8), then the mapping T has a fixed point $x^* \in \mathscr{B}_d[x_0, r]$ and the sequence $\{T^n x_n\}$ of successive iterations converges monotonically to x^*. Furthermore, the fixed point x^* is unique if every pair of elements of X has a lower and an upper bound in X and $b > q$.

4. Nonunique PPF dependant random fixed point theory

A fixed point theory of nonlinear operators which is PPF dependent, theory is depending on past, present and future data was developed in Bernfield *et al.* [2]. The domain space of the nonlinear operator was taken as $C(I,E)$, $I=[a,b]\subset\mathbb{R}$ and the range space as E, a Banach space. An important example of such a nonlinear operator is a delay differential equation. The PPF fixed point theorems are applied to ordinary nonlinear functional differential equations for proving the existence of solutions.

In the present section we obtain a successful fusion of above two ideas and prove some nonunique PPF dependent random fixed point theorems for random mappings in separable metric spaces. In the PPF dependent classical fixed point theory, the Razumikkin or minimal class of functions plays a significant role both in proving existence as well as uniqueness of PPF dependent fixed points of the mappings under consideration. Let E be a metric space and let I be a given closed and bounced interval in $\mathbb{R}$, the set of real numbers. Let $E_0=C(I,E)$ denote the class of continuous mappings from I to E. We equip the class $C(J,E)$ with metric d_0 defined by

$$d_0(x,y)=\sup_{t\in J} d(x(t),y(t)).$$

The following result is obvious.

Lemma 4.1. *If (E,d) is complete then the metric space (E_0,d_0) is also complete.*

When E is a Banach space and let $E_0=C(J,E)$ be a space of continuous E -valued function defined on J Then minimal class of functions related to a fixed $c\in J$ is defined as

$$\mathscr{M}_c=\{\phi\in E_0 \mid \|\phi\|_{E_0}=\|\phi(c)\|_E\}.$$

Now we are in a position to state our fixed point results concerning the existence of fixed points with PPF dependence. In a metric space X, we define the minimal class $\mathscr{M}_c$ as

$$\mathscr{M}_c=\{\phi,\psi\in E_0 \mid d_0(\phi,\psi)=d(\phi(c),\psi(c))\}.$$

Now we are in a position to state our main result of this section.

Theorem 4.1. *Let $(\Omega,\mathscr{A})$ be a measurable space and E, a separable complete metric space and let $\mathscr{B}_{d_0}[\phi_0,r]$ be a closed ball centered at ϕ_0 of radius r w.r.t. the metric d_0 for some $\phi_0\in E_0$ and for some real number $r>0$. Let $T:\Omega\times E_0\to E$ be a continuous random mapping satisfying* *for each* $\omega \in \Omega$,

$$0 \le \min\Big\{ d(T(\omega)\phi, T(\omega)\psi), d(\phi(c,\omega), T(\omega)\phi), d(\psi(c,\omega), T(\omega)\psi), \frac{d(\phi(c,\omega), T(\omega)\phi)[1 + d(\psi(c,\omega), T(\omega)\psi)]}{1 + d_0(\phi,\psi)}, \frac{d(\psi(c,\omega), T(\omega)\psi)[1 + d(\phi(c,\omega), T(\omega)\phi)]}{1 + d_0(\phi,\psi)} \Big\} \tag{4.1}$$

$$+ b(\omega) \min\{ d(\phi(c,\omega), T(\omega)\psi), d(\psi(c,\omega), T(\omega)\phi) \}$$

$$\le q(\omega) \max\{ d_0(\phi,\psi), [\min\{ d(\phi(c,\omega), T(\omega)\phi), d(\psi(c,\omega), T(\omega)\psi) \}] \}$$

for all $\phi, \psi \in E_0$, *where* $b : \Omega \to \mathbb{R}$ *and* $q : \Omega \to \mathbb{R}_+$ *are measurable functions satisfying* $0 \le q(\omega) < 1$ *for all* $\omega \in \Omega$ *and* $c \in I$. *Furthermore, if*

$$d(\phi_0(c,\omega), T(\omega)\phi_0) \le [1 - q(\omega)]\, r \tag{4.2}$$

for all $\omega \in \Omega$, *then* $T(\omega)$ *has a random fixed point* $\xi(\omega)$ *with PPF dependence in* $\mathscr{B}_{d_0}[\phi_0, r]$ *and the sequence* $\{\phi_n\}$ *of successive iterations in* $\mathscr{M}_c$ *converges to* ξ. *The PPF dependent fixed point* $\xi(\omega)$ *is unique if* $\mathscr{M}_c$ *is closed and* $b > q$ *on* Ω.

Proof. Let $\phi_0 : \Omega \to E_0$ be an arbitrary measurable function and define a sequence $\{x_n\}$ in E_0 as follows. Suppose that $T(\omega)\phi_0 = x_1$ for some $x_1 \in E$ Then choose $\phi_1 \in E_0$ such that $\phi_1(c,\omega) = x_1$ for some fixed $c \in I$ and

$$d_0(\phi_0, \phi_1) = d(\phi_0(c,\omega), \phi_1(c,\omega))$$

for all $\omega \in \Omega$. Again let $T(\omega)\phi_1 = x_2$ for some $x_2 \in E$. Then choose $\phi_2(c,\omega) = x_2$ for each fixed $c \in I$ and

$$d_0(\phi_1, \phi_2) = d(\phi_1(c,\omega), \phi_2(c,\omega))$$

for all $\omega \in \Omega$. Proceeding in this way, we obtain

$$T(\omega)\phi_{n-1} = x_n = \phi_n(c,\omega)$$

with

$$d_0(\phi_{n-1}, \phi_n) = d(\phi_{n-1}(c,\omega), \phi_n(c,\omega),\ n \in \mathbb{N}, \tag{4.3}$$

for all $\omega \in \Omega$. Clearly, $\{\phi_n\}$ is a sequence of measurable functions from Ω into E_0. Consequently $\{\phi_n(c)\}$ is a sequence of measurable functions from Ω into E. We show that $\phi_n(c,\omega)$ is a Cauchy sequence in E. Taking $\phi = \phi_0$ and $\psi = \phi_1$ in the inequality (2.2) we obtain

$$\begin{aligned} 0 \leq \min\Big\{ & d(T(\omega)\phi_0, T(\omega)\phi_1), d(\phi_0(c,\omega), T(\omega)\phi_0), d(\phi_1(c,\omega), T(\omega)\phi_1), \\ & \frac{d(\phi_0(c,\omega), T(\omega)\phi_0)[1 + d(\phi_1(c,\omega), T(\omega)\phi_1)]}{1 + d_0(\phi_0, \phi_1)}, \\ & \frac{d(\phi_1(c,\omega), T(\omega)\phi_1)[1 + d(\phi_0(c,\omega), T(\omega)\phi_0)]}{1 + d_0(\phi_0, \phi_1)}\Big\} \\ & + b(\omega)\min\{d(\phi_0(c,\omega), T(\omega)\phi_1), d(\phi_1(c,\omega), T(\omega)\phi_0)\} \\ & \leq q(\omega)\max\{d_0(\phi_0, \psi_1), [\min\{d(\phi_0(c,\omega), T(\omega)\phi_0), d(\phi_1(c,\omega), T(\omega)\phi_1)\}]\}, \end{aligned} \tag{4.4}$$

which further gives

$$\begin{aligned} 0 \leq \min\Big\{ & d(\phi_1(c,\omega), \phi_2(c,\omega)), d(\phi_0(c,\omega), \phi_1(c,\omega)), d(\phi_1(c,\omega), \phi_2(c,\omega)), \\ & \frac{d(\phi_0(c,\omega), \phi_1(c,\omega))[1 + d(\phi_1(c,\omega), \phi_2(c,\omega))]}{1 + d_0(\phi_0, \phi_1)}, \\ & \frac{d(\phi_1(c,\omega), \phi_2(c,\omega))[1 + d(\phi_0(c,\omega), \phi_1(c,\omega))]}{1 + d_0(\phi_0, \phi_1)}\Big\} \\ & + b(\omega)\min\{d(\phi_0(c,\omega), \phi_2(c,\omega)), d(\phi_1(c,\omega), \phi_1(c,\omega))\} \\ & \leq q(\omega)\max\{d_0(\phi_0, \phi_1), [\min\{d(\phi_0(c,\omega), \phi_1(c,\omega)), d(\phi_1(c,\omega), \phi_2(c,\omega))\}]\}. \end{aligned} \tag{4.5}$$

From the expression (4.5) it follows that

$$\begin{aligned} 0 \leq \min\Big\{ & d_0(\phi_1, \phi_2), d_0(\phi_0, \phi_1), d_0(\phi_1, \phi_2), \\ & \frac{d_0(\phi_0, \phi_1)[1 + d_0(\phi_1, \phi_2)]}{1 + d_0(\phi_0, \phi_1)}, \frac{d_0(\phi_1, \phi_2)[1 + d_0(\phi_0, \phi_1)]}{1 + d_0(\phi_0, \phi_1)}\Big\} \\ & + b(\omega)\min\{d_0(\phi_0, \phi_2), d_0(\phi_1, \phi_1)\} \\ & \leq q(\omega)\max\{d_0(\phi_0, \phi_1), [\min\{d_0(\phi_0, \phi_1), d_0(\phi_1, \phi_2)\}]\}. \end{aligned} \tag{4.6}$$

Now proceeding as in the proof of Theorem 2.1, it can be proved that

$$d_0(\phi_1, \phi_2) \leq q d_0(\phi_0, \phi_1).$$

Proceeding in this way, by induction,

$$d_0(\phi_n, \phi_{n+1}) \leq q d_0(\phi_{n-1}, \phi_n) \tag{4.7}$$

for each n, $n = 1, 2, \ldots$. By a repeated application of the inequality (4.6), we obtain

$$\begin{aligned} d_0(\phi_n, \phi_{n+1}) &\leq q d_0(\phi_{n-1}, \phi_n) \\ &\vdots \\ &\leq q^n d_0(\phi_0, \phi_1). \end{aligned} \tag{4.8}$$

Now for any positive integer p, by triangle inequality,

$$\begin{aligned} d_0(\phi_n, \phi_{n+p}) &\leq d_0(\phi_n, \phi_{n+1}) + \cdots + d_0(\phi_{n+p-1}, \phi_{n+p}) \\ &\leq q^n \left(1 + q + \cdots + q^{p-1}\right) d_0(\phi_0, \phi_1) \\ &\leq \frac{q^n}{(1-q)} d_0(\phi_0, \phi_1) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.9}$$

Since

$$d(\phi_n(c, \omega), \phi_{n+p}(c, \omega)) = d_0(\phi_n, \phi_{n+1})$$

for all $\omega \in \Omega$, we have that $\left\{T(\omega)\phi_n\right\}$ is also Cauchy sequence in E.

Next we show that $\{\phi_n\} \subset B_d[x_0, r]$. Now,

$$d(\phi_0, \phi_1) = d(\phi_0(c), T(\omega)\phi_0) \leq \left[1 - q\right] r \leq r.$$

Again,

$$\begin{aligned} d(\phi_0, \phi_2) &\leq d(\phi_0, \phi_1) + d(\phi_1, \phi_2) \\ &\leq (1-q)r + q d(\phi_0, \phi_1) \\ &\leq (1-q)r + q(1-q)r \\ &\leq \left[1 - q^2\right] r \\ &\leq r. \qquad \left[q < 1\right] \end{aligned}$$

In general for any $n \in \mathbb{N}$, one has

$$\begin{aligned} d(\phi_0, \phi_n) &\leq d(\phi_0, \phi_1) + d(\phi_1, \phi_2) + \cdots + d(\phi_{n-1}, \phi_n) \\ &\leq \left[1 + q + q^2 + \cdots + q^n\right] d(\phi_0, \phi_1) \\ &\leq \left[1 + q + q^2 + \cdots + q^n\right] (1 - q) r \\ &\leq \frac{\left[1 - q^n\right]}{1 - q} (1 - q) r \\ &\leq r. \qquad [q < 1] \end{aligned}$$

Hence $\{\phi_n\}$ is a Cauchy sequence in the closed ball $\mathscr{B}_{d_0}[\phi_0, r]$. As E_0 is a complete metric space, there exists a measurable function $\phi^* : \Omega \to \mathscr{B}_{d_0}[\phi_0, r]$ such that $\phi_n \to \phi^*$ and

$$T(\omega)\phi_n = \phi_{n+1}(c, \omega) \to \phi^*(c, \omega)$$

as $n \to \infty$. To prove that ϕ^* is a PPF dependent random fixed point of $T(\omega)$, we first observe that since $T(\omega)$ is continuous on E_0, $T(\omega)$ is a continuous at ϕ^*. Hence for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_0(\phi_{n+1}, \phi^*) < \delta \implies d(T\phi_{n+1}, T\phi^*) < \frac{\varepsilon}{2}.$$

Also since $T(\omega)\phi_n \to \phi^*(c, \omega)$, for $\gamma = \min\left\{\frac{\varepsilon}{2}, \delta\right\}$ there exists $n_0 \in \mathbb{N}$ such that

$$d(T(\omega)\phi_n, \phi^*(c, \omega)) < \gamma$$

for $n \geq n_o$. Thus,

$$\begin{aligned} d(T(\omega)\phi^*, \phi^*(c, \omega)) &\leq d(T(\omega)\phi^*, T(\omega)\phi_n) + d(T(\omega)\phi_n, \phi^*(c, \omega)) \\ &< \frac{\varepsilon}{2} + \gamma < \varepsilon. \end{aligned} \tag{4.10}$$

Since ε is arbitrary, we have

$$T(\omega)\phi^*(\omega) = \phi^*(c, \omega)$$

for all $\omega \in \Omega$. To prove the uniqueness, assume that $\mathscr{M}_c$ is closed in E_0 and $b > q$ on Ω. Then $\phi^* \in \mathscr{M}_c$. Let ψ^* be another PPF dependent fixed point of $T(\omega)$ in $\mathscr{M}_c$. Now by virtue of $\mathscr{M}_c$, we obtain

$$d_0(\phi^*(\omega), \psi^*(\omega)) = d(T(\omega)\phi^*(\omega), T(\omega)\psi^*(\omega)) = d(\phi^*(c, \omega), \psi^*(c, \omega))$$

for all $\omega \in \Omega$. If we substitute $x = \phi^*$ and $y = \psi^*$ in (4.1), then we get a contradiction. Hence, $\phi^*(\omega) = \psi^*(\omega)$ for all $\omega \in \Omega$. This completes the proof.

Corollary 4.1. *Let E be a complete metric space and let $\mathscr{B}_{d_0}[\phi_0, r]$ be a closed ball in E_0 centered at ϕ_0 of radius r w.r.t. the metric d_0 for some $\phi_0 \in E_0$. Let $T : E_0 \to E$ be a continuous mapping satisfying*

$$\begin{aligned} 0 \leq \min\Big\{ & d(T\phi, T\psi), d(\phi(c), T(\phi), d(\psi(c), T\psi), \\ & \frac{d(\phi(c), T\phi)[1 + d(\psi(c), T(\psi)]}{1 + d_0(\phi, \psi)}, \\ & \frac{d(\psi(c), T\psi)[1 + d(\phi(c), T\phi)]}{1 + d_0(\phi, \psi)}\Big\} \\ & + b \min\left\{d(\phi(c), T\psi), d(\psi(c), T\phi)\right\} \\ & \leq q \max\left\{d_0(\phi, \psi), [\min\{d(\phi(c), T\phi), d(\psi(c), T\psi)\}]\right\} \end{aligned} \tag{4.11}$$

for all $\phi, \psi \in E_0$, where $b \in \mathbb{R}$ and $q \in \mathbb{R}_+$ are two numbers satisfying $0 \leq q < 1$. Furthermore, if

$$d(\phi_0(c), T\phi_0) \leq [1 - q]\, r, \tag{4.12}$$

then T has a fixed point ξ with PPF dependence in $\mathscr{B}_{d_0}[\phi_0, r]$ and the sequence $\{\phi_n\}$ of successive iterations in $\mathscr{M}_c$ converges to ξ. Again it is clear that the PPF dependent fixed point $\xi(\omega)$ is unique if $\mathscr{M}_c$ is closed and $b > q$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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