Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 2, 168-191
ISSN: 1927-6303

TRIPLED BEST PROXIMITY POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

JAMNIAN NANTADILOK*, SOMKIAT CHAIPORNJAREANSRI

Department of Mathematics, Lampang Rajabhat University, Lampang 52000, Thailand

Abstract. In this paper, we establish tripled best proximity point theorems for a mixed monotone mapping satisfying the proximally tripled weak (ψ,ϕ) contraction in a partially ordered metric space. Presented theorems extend and improve many existing results in the literature.

Keywords: Partially ordered set; Proximity point; Tripled fixed point; Tripled best proximity point.

2010 AMS Subject Classification: 41A65, 90C30, 47H10.

1. Introduction

Let (X,d) be a metric space and A be a nonempty subset of X. If $T:A\to X$ is a mapping such that the equation $Tx=x$ has at least one solution in A, then x is called a fixed point of T. If the equation $Tx=x$ does not possess a solution, then $d(x,Tx)>0$. In such situation, it is crucial to find an element x such that x is in proximity to Tx. In the setting of a metric space (X,d), if $T:A\to X$, then a best approximation theorem provides sufficient conditions that ascertain the

*Corresponding author

E-mail address: jamnian52@lpru.ac.th

Received November 20, 2014

existence of an element x_0 such that

$$d(x_0, Tx_0) = dist(Tx_0, A),$$

where $dist(A,B) = \inf\{d(x,y) : x \in A \text{ and } y \in B\}$ for any nonempty subsets A and B of X. A point x_0 is known as a *best approximant*. In fact, a well-known best approximation theorem, due to Ky Fan [19], states that if K is a nonempty compact convex subset of a Banach space X and $T : K \to X$ is a single-valued continuous mapping, then there exists an element $x_0 \in K$ such that

$$d(x_0, Tx_0) = dist(Tx_0, K).$$

Later, this result has been generalized by many authors (see [15, 16, 17, 39, 40, 61, 60]). Despite the fact that the existence of an approximate solution is ensured by best approximation theorems, a natural question in this direction is whether it is possible to guarantee the existence of approximate solution that is optimal. In other words, if A and B are nonempty subsets of a normed linear space and $T : A \to B$ is a mapping, then the point to be explored is whether one can find an element $x_0 \in A$ such that

$$d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}.$$

The answer to this poser is provided by best proximity pair theorems. A best proximity pair theorem investigates the conditions under which the optimization problem has a solution. In deed, if T is a multifunction from A to B, then $d(x,Tx) \geq dist(A,B)$, where $d(x,Tx) = \inf\{d(x,y) : y \in Tx\}$. So, the most optimal solution to the problem of minimizing the real-valued function $x \to d(x,Tx)$ over the domain A of the mapping T will be the one for which the value $dist(A,B)$ is attained. In view of this standpoint, best proximity pair theorems are considered to study the conditions that assert the existence of an element x_0 such that

$$d(x_0, Tx_0) = dist(A, B).$$

The pair (x_0, Tx_0) is called a *best proximity pair* of T and the point x_0 is called a *best proximity point* of T. If the mapping under consideration is a self-mapping, it may be observed that a best proximity pair theorem is nothing but a fixed point theorem under certain suitable conditions. For detailed study on fixed point theory, we refer readers to ([20, 28, 58, 59, 62]).

The existence and convergence of best proximity points is an interesting topic of optimization theory which recently attracted the attention of many authors (see [7, 18, 27, 29, 30, 31, 33, 51, 52, 53, 54, 55, 56]). One can also find the existence of best proximity point in the setting of partially ordered metric space in ([4, 38, 49, 50]). Recently, Bhaskar and Lakshmikantham have introduced the concept called mixed monotone mapping and proved coupled fixed point theorems for mappings satisfying the mixed monotone property, which is used to investigate a large class of problems, and they discussed the existence and uniqueness of a solution for a periodic boundary value problem. One can find the existence of coupled fixed points in the setting of partially ordered metric space in ([4, 14, 21, 22, 23, 29, 30, 33, 34, 35, 38, 42, 44, 45, 47, 48, 49, 50, 51, 52]). Moreover, one can also find the existence of tripled fixed points in the setting of various ordered metric spaces in ([1, 2, 3, 6, 8, 9, 10, 11, 13, 24, 32, 37, 41, 46, 57]) and references therein.

2. Preliminaries

Let Φ be the class of all functions $\phi : [0,\infty) \to [0,\infty)$ which satisfy:

(i) ϕ is continuous and nondecreasing,

(ii) $\phi(t) = 0$ if and only if $t = 0$,

(iii) $\phi(t+s) \leq \phi(t) + \phi(s)$, $\forall t,s \in (0,\infty]$.

And let Ψ be the class of all functions $\psi : (0,\infty] \to (0,\infty]$ which satisfy $\lim\limits_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim\limits_{t\to 0^+} \psi(t) = 0$.

In [26], Kumum *et al*. extended the main theoretical result of Luong and Thuan in [36]. The main result of Kumam *et al*. in [26] is the following.

Theorem 2.1. *Let $(X,d,\leq)$ be a partially ordered complete metric space. Let A and B be nonempty closed subsets of the metric space (X,d) such that $A_0 \neq \emptyset$. Let $F : A\times A \to B$ satisfy the following conditions:*

(i) F is a continuous proximally coupled weak (ψ,ϕ) contraction having the proximal mixed monotone property on A such that $F(A_0\times A_0) \subseteq B_0$.

(ii) There exist elements (x_0, y_0) *and* (x_1, y_1) *in* $A_0 \times A_0$ *such that*

$$d(x_1, F(x_0, y_0)) = dist(A, B) \ \ with \ \ x_0 \leq x_1$$

and

$$d(y_1, F(y_0, x_0)) = dist(A, B) \ \ with \ \ y_0 \leq y_1.$$

Then, there exist $(x, y) \in A \times A$ *such that*

$$d(x, F(x, y)) = dist(A, B) \quad and \quad d(y, F(y, x)) = dist(A, B).$$

Motivated and inspired by the above Theorem, we investigate the concept of the proximal mixed monotone property and of a proximally tripled weak (ψ, ϕ) contraction on A. We also prove the existence and uniqueness of tripled best proximity points in the setting of partially ordered metric spaces. Further, we attempt to provide an extension to the Theorem 2.1 above.

Now we recall the definition of a tripled fixed point which recently introduced by Berinde and Borcut in [12]. Let X be a nonempty set and $F : X^3 \to X$ be a given mapping. An element $(x, y, z) \in X^3$ is called a tripled fixed point of the mapping F if

$$x = F(x, y, z) \ \ , \ \ y = F(y, x, y) \ \text{ and } \ z = F(z, y, x).$$

The authors mentioned above also introduced the notion of mixed monotone mapping. If $(X, \leq)$ is a partially ordered set, the mapping F is said to have the mixed monotone property, if

$$x_1, x_2 \in X, \ x_1 \leq x_2 \ \Rightarrow \ F(x_1, y, z) \leq F(x_2, y, z) \ , \ \forall y, z \in X,$$

$$y_1, y_2 \in X, \ y_1 \leq y_2 \ \Rightarrow \ F(x, y_1, z) \geq F(x, y_2, z) \ , \ \forall x, z \in X,$$

$$z_1, z_2 \in X, \ z_1 \leq z_2 \ \Rightarrow \ F(x, y, z_1) \leq F(x, y, z_2) \ , \ \forall x, y \in X.$$

Let A and B be nonempty subsets of a metric space (X, d). We use the following notations in the sequel:

$$dist(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\},$$

$$A_0 = \{x \in A : d(x, y) = dist(A, B), \text{for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = dist(A, B), \text{for some } x \in A\}.$$

In [29], the authors discussed sufficient conditions which guarantee the nonemptiness of A_0 and B_0. Also in [53], the authors proved that $A_0 \subseteq Bd(A)$ and $B_0 \subseteq Bd(B)$ in the setting of normed linear space, where $Bd(K)$ denotes the boundary of K for any $K \subseteq X$.

We now give the following definition.

Definition 2.1. *Let $(X,d,\leq)$ be a partially ordered metric space and A,B are nonempty subsets of X. A mapping $F : A \times A \times A \to B$ is said to have proximal mixed monotone property if $F(x,y,z)$ is proximally nondecreasing in x and z, and is proximally nonincreasing in y, that is, for all $x,y,z \in A$,*

$$\left.\begin{array}{c} x_1 \leq x_2 \leq x_3 \\ d(u_1,F(x_1,y,z)) = dist(A,B) \\ d(u_2,F(x_2,y,z)) = dist(A,B) \\ d(u_3,F(x_3,y,z)) = dist(A,B) \end{array}\right\} \Rightarrow u_1 \leq u_2 \leq u_3;$$

$$\left.\begin{array}{c} y_1 \leq y_2 \leq y_3 \\ d(u_4,F(x,y_1,z)) = dist(A,B) \\ d(u_5,F(x,y_2,z)) = dist(A,B) \\ d(u_6,F(x,y_3,z)) = dist(A,B) \end{array}\right\} \Rightarrow u_4 \geq u_5 \geq u_6;$$

and

$$\left.\begin{array}{c} z_1 \leq z_2 \leq z_3 \\ d(u_7,F(x,y,z_1)) = dist(A,B) \\ d(u_8,F(x,y,z_2)) = dist(A,B) \\ d(u_9,F(x,y,z_3)) = dist(A,B) \end{array}\right\} \Rightarrow u_7 \leq u_8 \leq u_9,$$

where $x_1,x_2,x_3,y_1,y_2,y_3,u_1,u_2,\ldots,u_9 \in A$.

We note that if $A = B$ in the above definition, the notion of the proximal mixed monotone property reduces to that of mixed monotone property. The following lemmas are essential.

Lemma 2.2. *Let $(X,d,\leq)$ be a partially ordered metric space and A,B are nonempty subsets of X. Assume A_0 is nonempty. A mapping $F : A \times A \times A \to B$ has the proximal mixed monotone* *property with* $F(A_0 \times A_0 \times A_0) \subseteq B_0$ *whenever* $x_0, x_1, x_2, x_3, y_0, y_1, y_2, z_0, z_1, z_2$ *in* A *such that*

$$\left.\begin{array}{c} x_0 \le x_1 \le x_2\ ;\ y_0 \ge y_1 \ge y_2\ ;\ z_0 \le z_1 \le z_2 \\ d(x_1, F(x_0, y_0, z_0)) = dist(A,B) \\ d(x_2, F(x_1, y_1, z_1)) = dist(A,B) \\ d(x_3, F(x_2, y_2, z_2)) = dist(A,B) \end{array}\right\} \Rightarrow\ x_1 \le x_2 \le x_3. \tag{1}$$

Proof By hypothesis $F(A_0 \times A_0 \times A_0) \subseteq B_0$, therefore $F(x_2, y_0, z_0) \in B_0$. Hence, there exists $x_1^* \in A$ such that

$$d(x_1^*, F(x_2, y_0, z_0)) = dist(A,B). \tag{2}$$

Using that F is proximal mixed monotone (in particular F is proximally nondecreasing in x) to (1) and (2), we get

$$\left.\begin{array}{c} x_0 \le x_1 \le x_2 \\ d(x_1, F(x_0, y_0, z_0)) = dist(A,B) \\ d(x_2, F(x_1, y_1, z_1)) = dist(A,B) \\ d(x_1^*, F(x_2, y_0, z_0)) = dist(A,B) \end{array}\right\} \Rightarrow\ x_1 \le x_2 \le x_1^*. \tag{3}$$

Analogously, using the fact that F is proximal mixed monotone (in particular, F is proximally nonincreasing in y) to (1) and (2), we get

$$\left.\begin{array}{c} y_0 \ge y_1 \ge y_2 \\ d(x_1^*, F(x_2, y_0, z_0)) = dist(A,B) \\ d(x_3, F(x_2, y_2, z_2)) = dist(A,B) \end{array}\right\} \Rightarrow\ x_1^* \le x_3. \tag{4}$$

From (3) and (4), we conclude that $x_1 \le x_2 \le x_3$. The proof is finished.

Lemma 2.3. *Let* $(X, d, \le)$ *be a partially ordered metric space and* A, B *are nonempty subsets of* X. *Assume* A_0 *is nonempty. A mapping* $F : A \times A \times A \to B$ *has the proximal mixed monotone property with* $F(A_0 \times A_0 \times A_0) \subseteq B_0$ *whenever* $x_0, x_1, x_2, y_0, y_1, y_2, y_3, z_0, z_1, z_2$ *in* A *such that*

$$\left.\begin{array}{c} x_0 \le x_1 \le x_2\ ;\ y_0 \ge y_1 \ge y_2\ ;\ z_0 \le z_1 \le z_2 \\ d(y_1, F(y_0, x_0, z_0)) = dist(A,B) \\ d(y_2, F(y_1, x_1, z_1)) = dist(A,B) \\ d(y_3, F(y_2, x_2, z_2)) = dist(A,B) \end{array}\right\} \Rightarrow\ y_1 \ge y_2 \ge y_3. \tag{5}$$

Proof By hypothesis $F(A_0 \times A_0 \times A_0) \subseteq B_0$, therefore $F(y_2, x_1, z_1) \in B_0$. Hence, there exists $y_1^* \in A$ such that

$$d(y_1^*, F(y_2, x_1, z_1)) = dist(A, B). \tag{6}$$

Using that F is proximal mixed monotone (in particular F is proximally nondecreasing in x) to (5) and (6), we get

$$\left.\begin{array}{c} x_0 \leq x_1 \leq x_2 \ ; \ y_0 \geq y_1 \geq y_2 \\ d(y_1, F(y_0, x_0, z_0)) = dist(A, B) \\ d(y_2, F(y_1, x_1, z_1)) = dist(A, B) \\ d(y_1^*, F(y_2, x_1, z_1)) = dist(A, B) \end{array}\right\} \Rightarrow \ y_1 \geq y_2 \geq y_1^*. \tag{7}$$

Analogously, using the fact that F is proximal mixed monotone (in particular, F is proximally nonincreasing in y) to (5) and (6), we get

$$\left.\begin{array}{c} x_0 \leq x_1 \leq x_2; z_0 \leq z_1 \leq z_2 \\ d(y_1^*, F(y_2, x_1, z_1)) = dist(A, B) \\ d(y_3, F(y_2, x_2, z_2)) = dist(A, B) \end{array}\right\} \Rightarrow \ y_1^* \geq y_3. \tag{8}$$

From (7) and (8), we conclude that $y_1 \geq y_2 \geq y_3$. The proof is finished.

Lemma 2.4. *Let $(X, d, \leq)$ be a partially ordered metric space and A, B are nonempty subsets of X. Assume A_0 is nonempty. A mapping $F : A \times A \times A \to B$ has the proximal mixed monotone property with $F(A_0 \times A_0 \times A_0) \subseteq B_0$ whenever $x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, z_3$ in A such that*

$$\left.\begin{array}{c} x_0 \leq x_1 \leq x_2 \ ; \ y_0 \geq y_1 \geq y_2 \ ; \ z_0 \leq z_1 \leq z_2 \\ d(z_1, F(z_0, y_0, x_0)) = dist(A, B) \\ d(z_2, F(z_1, y_1, x_1)) = dist(A, B) \\ d(z_3, F(z_2, y_2, x_2)) = dist(A, B) \end{array}\right\} \Rightarrow \ z_1 \leq z_2 \leq z_3. \tag{9}$$

Proof The proof is similar to that of Lemma 2.2 and Lemma 2.3.

As in [26], we similarly give the following definition.

Definition 2.2. *Let $(X, d, \leq)$ be a partially ordered metric space and A, B are nonempty subsets of X. Assume A_0 is nonempty. A mapping $F : A \times A \times A \to B$ is said to be proximally tripled* *weak (ψ,ϕ) contraction on A, whenever*

$$\left.\begin{array}{c} x_1 \leq x_2 \ ; \ y_1 \geq y_2 \ ; \ z_1 \leq z_2 \\ d(u,F(x_1,y_1,z_1)) = dist(A,B) \\ d(v,F(x_2,y_2,z_2)) = dist(A,B) \end{array}\right\} \Rightarrow \ \phi(d(u,v)) \leq \frac{1}{3}\phi\left(d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)\right)$$
$$-\psi\left(\frac{d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)}{3}\right),$$

where $x_1,x_2,y_1,y_2,z_1,z_2,u,v \in A$.

One can see that, if $A = B$ in the above definition, the notion of proximally tripled weak (ψ,ϕ) contraction on A reduces to that of a tripled weak (ψ,ϕ) contraction.

3. Main results

Let $(X,d,\leq)$ be a partially complete metric space endowed with the product space $X\times X\times X$ with the following partial order:

for $(x,y,z),(u,v,w) \in X\times X\times X$,

$$(u,v,w) \preceq (x,y,z) \ \Leftrightarrow \ x \geq u, y \leq v, z \geq w.$$

Theorem 3.1. *Let $(X,d,\leq)$ be a partially ordered complete metric space. Let A,B are nonempty closed subsets of the metric space (X,d) such that $A_0 \neq \emptyset$. Let $F: A\times A\times A \to B$ satisfy the following conditions:*

(i) F is continuous proximally tripled weak (ψ,ϕ) contraction on A having the proximal mixed monotone property on A such that $F(A_0\times A_0\times A_0) \subseteq B_0$.

(ii) There exist elements (x_0,y_0,z_0) and (x_1,y_1,z_1) in $A_0\times A_0\times A_0$ such that

$$\begin{aligned} d(x_1,F(x_0,y_0,z_0)) &= dist(A,B) \ \ with \ \ x_0 \leq x_1, \\ d(y_1,F(y_0,x_0,z_0)) &= dist(A,B) \ \ with \ \ y_0 \geq y_1, and \\ d(z_1,F(z_0,y_0,x_0)) &= dist(A,B) \ \ with \ \ z_0 \leq z_1. \end{aligned}$$

Then, there exists (x,y,z) in $A\times A\times A$ such that $d(x,F(x,y,z)) = dist(A,B)$, $d(y,F(y,x,z)) = dist(A,B)$ and $d(z,F(z,y,x)) = dist(A,B)$

Proof. By hypothesis, there exist elements (x_0,y_0,z_0) and (x_1,y_1,z_1) in $A_0\times A_0\times A_0$ such that

$$\begin{aligned} d(x_1,F(x_0,y_0,z_0)) &= dist(A,B) \text{ with } x_0\leq x_1,\\ d(y_1,F(y_0,x_0,z_0)) &= dist(A,B) \text{ with } y_0\geq y_1, \text{and}\\ d(z_1,F(z_0,y_0,x_0)) &= dist(A,B) \text{ with } z_0\leq z_1. \end{aligned}$$

Since $F(A_0\times A_0\times A_0)\subseteq B_0$, there exists element (x_2,y_2,z_2) in $A_0\times A_0\times A_0$ such that

$$\begin{aligned} d(x_2,F(x_1,y_1,z_1)) &= dist(A,B),\\ d(y_2,F(y_1,x_1,z_1)) &= dist(A,B), \text{and}\\ d(z_2,F(z_1,y_1,x_1)) &= dist(A,B). \end{aligned}$$

And also there exists element (x_3,y_3,z_3) in $A_0\times A_0\times A_0$ such that

$$\begin{aligned} d(x_3,F(x_2,y_2,z_2)) &= dist(A,B),\\ d(y_3,F(y_2,x_2,z_2)) &= dist(A,B), \text{and}\\ d(z_3,F(z_2,y_2,x_2)) &= dist(A,B). \end{aligned}$$

Hence from Lemma 2.2, Lemma 2.3 and Lemma 2.4, we obtain that $x_1\leq x_2\leq x_3, y_1\geq y_2\geq y_3$ and $z_1\leq z_2\leq z_3$. Continuing this process, we can construct the sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ in A_0 such that

$$d(x_{n+1},F(x_n,y_n,z_n))=dist(A,B)\ ,\ \forall n\in\mathbb{N} \text{ with } x_0\leq x_1\leq\cdots\leq x_n\leq x_{n+1}\leq\cdots, \tag{10}$$

$$d(y_{n+1},F(y_n,x_n,z_n))=dist(A,B)\ ,\ \forall n\in\mathbb{N} \text{ with } y_0\geq y_1\geq\cdots\geq y_n\geq y_{n+1}\geq\cdots, \tag{11}$$

and

$$d(z_{n+1},F(z_n,y_n,x_n))=dist(A,B)\ ,\ \forall n\in\mathbb{N} \text{ with } z_0\leq z_1\leq\cdots\leq z_n\leq z_{n+1}\leq\cdots. \tag{12}$$

Then $d(x_n,F(x_{n-1},y_{n-1},z_{n-1}))=d(A,B), d(x_{n+1},F(x_n,y_n,z_n))=dist(A,B)$ and we also have $x_{n-1}\leq x_n, y_{n-1}\geq y_n$ and $z_{n-1}\leq z_n, \forall n\in\mathbb{N}$. Now using the fact that F is proximally tripled weak (ψ,ϕ) contraction on A, we have

$$\begin{aligned}\phi(d(x_n,x_{n+1})) \leq\ & \frac{1}{3}\phi\left(d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)\right) \\ & -\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)}{3}\right),\end{aligned} \tag{13}$$

similarly,

$$\begin{aligned}\phi(d(y_n,y_{n+1})) \leq\ & \frac{1}{3}\phi\left(d(y_{n-1},y_n)+d(x_{n-1},x_n)+d(z_{n-1},z_n)\right) \\ & -\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)}{3}\right),\end{aligned} \tag{14}$$

and

$$\begin{aligned}\phi(d(z_n,z_{n+1})) \leq\ & \frac{1}{3}\phi\left(d(z_{n-1},z_n)+d(y_{n-1},y_n)+d(x_{n-1},x_n)\right) \\ & -\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)}{3}\right).\end{aligned} \tag{15}$$

Adding (13), (14) and (15), we get

$$\begin{aligned}& \phi(d(x_n,x_{n+1}))+\phi(d(y_n,y_{n+1}))+\phi(d(z_n,z_{n+1})) \\ \leq\ & \phi\left(d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)\right) \\ & -3\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)}{3}\right).\end{aligned} \tag{16}$$

By property (iii) of ϕ, we have

$$\begin{aligned}& \phi\left(d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1})\right) \\ \leq\ & \phi(d(x_n,x_{n+1}))+\phi(d(y_n,y_{n+1}))+\phi(d(z_n,z_{n+1}))\end{aligned} \tag{17}$$

From (16) and (17), we get

$$\begin{aligned}& \phi\left(d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1})\right) \\ \leq\ & \phi\left(d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)\right) \\ & -3\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n)}{3}\right).\end{aligned} \tag{18}$$

From (18) and using the fact that ϕ is nondecreasing, we get

$$d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1}) \leq d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(z_{n-1},z_n). \tag{19}$$

Putting $\delta_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1})$, then the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1})] = \delta. \tag{20}$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. By taking the limit as $n \to \infty$ on both sides of (18) and ϕ is continuous, we have

$$\begin{aligned}\phi(\delta) &\leq \lim_{n\to\infty} \phi(\delta_{n-1}) - 3 \lim_{n\to\infty} \psi\left(\frac{\delta_{n-1}}{3}\right) \\ &= \phi(\delta) - 3 \lim_{n\to\infty} \psi\left(\frac{\delta_{n-1}}{3}\right) \\ &< \phi(\delta)\ , \text{ since } \lim_{t\to r} \psi(t) > 0, \text{ for all } r > 0.\end{aligned}$$

This is a contradiction. Therefore $\delta = 0$, that is

$$\begin{aligned}\lim_{n\to\infty} (\delta_n) &= \lim_{n\to\infty} [d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1})] \\ &= 0.\end{aligned} \tag{21}$$

Now, we prove that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences. On the contrary, assume that at least one of $\{x_n\}, \{y_n\}$ and $\{z_n\}$ is not a Cauchy sequence. This means that

$$\lim_{n,m\to\infty} d(x_n, x_m) \not\to 0 \quad \text{or} \quad \lim_{n,m\to\infty} d(y_n, y_m) \not\to 0 \quad \text{or} \quad \lim_{n,m\to\infty} d(z_n, z_m) \not\to 0$$

and consequently,

$$\lim_{n,m\to\infty} [d(x_n, x_m) + d(y_n, y_m) + d(z_n, z_m)] \not\to 0.$$

Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}, \{x_{m_k}\}$ of $\{x_n\}$; $\{y_{n_k}\}, \{y_{m_k}\}$ of $\{y_n\}$ and $\{z_{n_k}\}, \{z_{m_k}\}$ of $\{z_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$,

$$d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) \geq \varepsilon. \tag{22}$$

This means that

$$d(x_{n_k-1}, x_{m_k}) + d(y_{n_k-1}, y_{m_k}) + d(z_{n_k-1}, z_{m_k}) < \varepsilon. \tag{23}$$

Using (22) and (23) and the triangle inequality, we get

$$\begin{aligned}\varepsilon &\le d(x_{n_k},x_{m_k})+d(y_{n_k},y_{m_k})+d(z_{n_k},z_{m_k})\\ &\le d(x_{n_k},x_{n_k-1})+d(x_{n_k-1},x_{m_k})+d(y_{n_k},y_{n_k-1})+d(y_{n_k-1},y_{m_k})\\ &\qquad +d(z_{n_k},z_{n_k-1})+d(z_{n_k-1},z_{m_k})\\ &\le d(x_{n_k},x_{n_k-1})+d(y_{n_k},x_{y_k-1})+d(z_{n_k},z_{n_k-1})+\varepsilon.\end{aligned}$$

Letting $k\to\infty$ and using (21), we obtain

$$\lim_{n\to\infty}[d(x_{n_k},x_{m_k})+d(y_{n_k},y_{m_k})+d(z_{n_k},z_{m_k})]=\varepsilon. \tag{24}$$

By the triangle inequality, we have

$$\begin{aligned}d(x_{n_k},x_{m_k})&+d(y_{n_k},y_{m_k})+d(z_{n_k},z_{m_k})\\ &\le d(x_{n_k},x_{n_k+1})+d(x_{n_k+1},x_{m_k+1})+d(x_{m_k+1},x_{m_k})\\ &\quad +d(y_{n_k},y_{n_k+1})+d(y_{n_k+1},y_{m_k+1})+d(y_{m_k+1},y_{m_k})\\ &\qquad +d(z_{n_k},z_{n_k+1})+d(z_{n_k+1},z_{m_k+1})+d(z_{m_k+1},z_{m_k})\\ &=\left[d(x_{n_k},x_{n_k+1})+d(y_{n_k},y_{n_k+1})+d(z_{n_k},z_{n_k+1})\right]\\ &\quad +\left[d(x_{m_k},x_{m_k+1})+d(y_{m_k},y_{m_k+1})+d(z_{m_k},z_{m_k+1})\right]\\ &\qquad +\left[d(x_{n_k+1},x_{m_k+1})+d(y_{n_k+1},y_{m_k+1})+d(z_{n_k+1},z_{m_k+1})\right].\end{aligned} \tag{25}$$

Using the property of ϕ, we obtain

(26)

$$\phi(\gamma_k)\le\phi(\delta_{n_k})+\phi(\delta_{m_k})+\phi(d(x_{n_k+1},x_{m_k+1}))+\phi(d(y_{n_k+1},y_{m_k+1}))+\phi(d(z_{n_k+1},z_{m_k+1})),$$

where $\gamma_k=d(x_{n_k},x_{m_k})+d(y_{n_k},y_{m_k})+d(z_{n_k},z_{m_k})$,

$\delta_{n_k}=d(x_{n_k},x_{n_k+1})+d(y_{n_k},y_{n_k+1})+d(z_{n_k},z_{n_k+1})$ and

$\delta_{m_k}=d(x_{m_k},x_{m_k+1})+d(y_{n_k},y_{n_k+1})+d(z_{n_k},z_{n_k+1})$.

Since $x_{n_k} \geq x_{m_k}, y_{n_k} \leq y_{m_k}$ and $z_{n_k} \geq z_{m_k}$, using the fact that F is a proximally tripled weak (ψ, ϕ) contraction on A, we get

$$\begin{aligned}
\phi(d(x_{n_k+1}, x_{m_k+1})) &\leq \frac{1}{3}\phi\left(d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k})\right) \\
&\quad - \psi\left(\frac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k})}{3}\right) \\
&= \frac{1}{3}\phi(\gamma_k) - \psi\left(\frac{\gamma_k}{3}\right), \qquad (27)
\end{aligned}$$

and

$$\begin{aligned}
\phi(d(y_{m_k+1}, y_{n_k+1})) &\leq \frac{1}{3}\phi\left(d(y_{m_k}, y_{n_k}) + d(x_{m_k}, x_{n_k}) + d(z_{m_k}, z_{n_k})\right) \\
&\quad - \psi\left(\frac{d(x_{m_k}, x_{n_k}) + d(y_{m_k}, y_{n_k}) + d(z_{m_k}, z_{n_k})}{3}\right) \\
&= \frac{1}{3}\phi(\gamma_k) - \psi\left(\frac{\gamma_k}{3}\right), \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
\phi(d(z_{n_k+1}, z_{m_k+1})) &\leq \frac{1}{3}\phi\left(d(z_{n_k}, z_{m_k}) + d(y_{n_k}, y_{m_k}) + d(x_{n_k}, x_{m_k})\right) \\
&\quad - \psi\left(\frac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k})}{3}\right) \\
&= \frac{1}{3}\phi(\gamma_k) - \psi\left(\frac{\gamma_k}{3}\right). \qquad (29)
\end{aligned}$$

From (26) - (29), we get

$$\phi(\gamma_k) \leq \phi(\delta_{n_k}) + \phi(\delta_{m_k}) + \phi(\gamma_k) - 3\psi(\frac{\gamma_k}{3}). \qquad (30)$$

Letting $k \to \infty$ and using (21), (24) and (30), we have

$$\phi(\varepsilon) \leq \phi(0) + \phi(0) + \phi(\varepsilon) - 3\psi(\frac{\varepsilon}{3}) < \phi(\varepsilon).$$

This is a contradiction. Therefore, we can conclude that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences. Since A is a closed subset of a complete metric space X, these sequences have limits. Hence, there exist $x, y, z \in A$ such that $x_n \to x, y_n \to y$ and $z_n \to z$. Then $(x_n, y_n, z_n) \to (x, y, z)$ in $A \times A \times A$. Since F is continuous, we have that $F(x_n, y_n, z_n) \to F(x, y, z), F(y_n, x_n, z_n) \to (y, x, z)$ and $F(z_n, y_n, x_n) \to (z, y, x)$. But from (10), (11) and (12), we know that the sequences $\{d(x_{n+1}, F(x_n, y_n, z_n))\}, \{d(y_{n+1}, F(y_n, x_n, z_n))\}$ and $\{d(z_{n+1}, F(z_n, y_n, x_n))\}$ are constant sequences with the value $dist(A,B)$. Therefore $d(x,F(x,y,z)) = dist(A,B)$, $d(y,F(y,x,z)) = dist(A,B)$ and $d(z,F(z,y,x)) = dist(A,B)$. This completes our proof.

Corollary 3.2. *Let $(X,d,\leq)$ be a partially ordered complete metric space. Let A be a nonempty closed subset of the metric space (X,d). Let $F : A\times A\times A \to A$ satisfy the following conditions:*

(i) F is continuous having the proximal mixed monotone property and proximally tripled weak (ψ,ϕ) contraction on A.

(ii) There exist (x_0,y_0,z_0) and (x_1,y_1,z_1) in $A\times A\times A$ such that $x_1 = F(x_0,y_0,z_0)$ with $x_0 \leq x_1$, $y_1 = F(y_0,x_0,z_0)$ with $y_0 \geq y_1$ and $z_1 = F(z_0,y_0,x_0)$ with $z_0 \leq z_1$.

Then, there exists $(x,y,z) \in A\times A\times A$ such that $d(x,F(x,y,z)) = 0, d(y,F(y,x,z)) = 0$ and $d(z,F(z,y,x)) = 0$.

We also note that Theorem 3.1 is still valid for F not necessarily continuous, if A has the following property that

$$\{x_n\} \text{ is a nondecreasing sequence in } A \text{ such that } x_n \to x, \text{ then } x_n \leq x, \tag{31}$$

$$\{y_n\} \text{ is a nonincreasing sequence in } A \text{ such that } y_n \to y, \text{ then } y_n \geq y, \tag{32}$$

$$\{z_n\} \text{ is a nondecreasing sequence in } A \text{ such that } z_n \to z, \text{ then } z_n \leq z. \tag{33}$$

Theorem 3.3. *Assume the conditions* (31)*,* (32) *and* (33) *and A_0 is closed in X instead of continuity of F in Theorem 3.1, then the conclusion of Theorem 3.1 holds.*

Proof Similar to the proof of Theorem 3.1, there exist sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ in A satisfying the following conditions:

$$d(x_{n+1},F(x_n,y_n,z_n)) = d(A,B) \text{ with } x_n \leq x_{n+1} \text{ , } \forall n \in \mathbb{N}, \tag{34}$$

$$d(y_{n+1},F(y_n,x_n,z_n)) = d(A,B) \text{ with } y_n \geq y_{n+1} \text{ , } \forall n \in \mathbb{N}, \tag{35}$$

$$d(z_{n+1},F(z_n,y_n,x_n)) = d(A,B) \text{ with } z_n \leq z_{n+1} \text{ , } \forall n \in \mathbb{N}. \tag{36}$$

Moreover, $x_n \to x, y_n \to y$ and $z_n \to z$. From (31), (32) and (33), we get $x_n \leq x, y_n \geq y$ and $z_n \leq z$, respectively. Note that the sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ are in A_0 and A_0 is closed. Therefore,

$(x,y,z)\in A_0\times A_0\times A_0$. Since $F(A_0\times A_0\times A_0)\subseteq B_0$, there exist $F(x,y,z),F(y,x,z)$ and $F(z,y,x)$ in B_0. Therefore, there exist $(x^*,y^*,z^*)\in A_0\times A_0\times A_0$ such that

$$d(x^*,F(x,y,z))=d(A,B),d(y^*,F(y,x,z))=d(A,B) \text{ and } d(z^*,F(z,y,x))=d(A,B). \tag{37}$$

Since $x_n\leq x, y_n\geq y$ and $z_n\leq z$. By using the fact that F is a proximally tripled weak (ψ,ϕ) contraction on A for (34),(35),(36) and (37), we get

$$\begin{aligned}
\phi(d(x_{n+1},x^*)) &\leq \frac{1}{3}\phi\left(d(x_n,x)+d(y_n,y)+d(z_n,z)\right)\\
&\quad -\psi\left(\frac{d(x_n,x)+d(y_n,y)+d(z_n,z)}{3}\right),\ \text{for all } n,\\
\phi(d(y^*,y_{n+1})) &\leq \frac{1}{3}\phi\left(d(y,y_n)+d(x,x_n)+d(z,z_n)\right)\\
&\quad -\psi\left(\frac{d(y,y_n)+d(x,x_n)+d(z,z_n)}{3}\right),\ \text{for all } n \quad \text{and}\\
\phi(d(z_{n+1},z^*)) &\leq \frac{1}{3}\phi\left(d(z_n,z)+d(y_n,y)+d(x_n,x)\right)\\
&\quad -\psi\left(\frac{d(z_n,z)+d(y_n,y)+d(x_n,x)}{3}\right),\ \text{for all } n.
\end{aligned}$$

Since $x_n\to x, y_n\to y$ and $z_n\to z$, by taking the limit on the above inequalities, we get $x=x^*, y=y^*$ and $z=z^*$. Hence from (37), we get that

$$d(x,F(x,y,z))=dist(A,B),d(y,F(y,x,z))=dist(A,B) \text{ and } d(z,F(z,y,x))=dist(A,B).$$

Our proof is finished.

We note that the hypothesis in Theorem 3.1 and Theorem 3.3 do not guarantee the uniqueness of the tripled best proximity point. We give the following example.

Example 3.1. *Let $X=\{(0,0,1),(1,0,0),(-1,0,0),(0,0,-1)\}\subset\mathbb{R}^3$ and consider the usual order*

$$(x,y,z)\preceq(u,v,w)\ \Leftrightarrow\ x\leq u, y\leq v \text{ and } z\leq w.$$

Thus, $(X,\preceq)$ is a partially ordered set. We note that (X,d_3) is a complete metric space, where d_3 is the Euclidean metric. Let $A=\{(0,0,1),(1,0,0)\}$ and $B=\{(0,0,-1),(-1,0,0)\}$ be closed subsets of X. Then $dist(A,B)=\sqrt{2}$, $A=A_0$ and $B=B_0$. Let $F:A\times A\times A\to B$ be defined as

$$F\left((x_1,x_2,x_3),(y_1,y_2,y_3),(z_1,z_2,z_3)\right)=(-x_3,-x_2,-x_1).$$

Then, one can see that F is continuous and $F(A_0 \times A_0 \times A_0) \subseteq B_0$. The only comparable pairs of points in A are $x \preceq x$ for $x \in A$, hence the proximal mixed monotone property and the proximally tripled weak (ψ, ϕ) contraction on A are obviously satisfied.

One can show that the other hypotheses of the theorem are also satisfied. However, F has many tripled best proximity points, such as $((0,0,1),(0,0,1),(0,0,1))$; $((1,0,0),(1,0,0),(1,0,0))$; $((1,0,0),(1,0,0),(0,0,1))$, etc. Hence, not unique.

However, we can prove that the tripled best proximity point is in fact unique, provided that the product space $A \times A \times A$ endowed with the partial order mentioned above has the following property :

(38) Every pair of elements has either a lower bound or an upper bound.

This condition is equivalent to the following statement.

(39) For every pair of $(x,y,z),(x^*,y^*,z^*) \in A \times A \times A$. There exists $(z_1,z_2,z_3) \in A \times A \times A$ which is comparable to (x,y,z) and (x^*,y^*,z^*).

Theorem 3.4. *In addition to the hypothesis of Theorem 3.1 (resp. Theorem 3.3), suppose that for any two elements (x,y,z) and (x^*,y^*,z^*) in $A_0 \times A_0 \times A_0$,*

(40) *there exists $(z_1,z_2,z_3) \in A_0 \times A_0 \times A_0$ such that (z_1,z_2,z_3) is comparable to (x,y,z) and (x^*,y^*,z^*),*

then F has a unique tripled best proximity point.

Proof From Theorem 3.1 (resp. Theorem 3.3), the set of tripled best proximity points of F is nonempty. Suppose that there exist (x,y,z) and (x^*,y^*,z^*) in $A \times A \times A$ which are tripled best proximity points. That is

$$d(x,F(x,y,z)) = dist(A,B),$$

$$d(y,F(y,x,z)) = dist(A,B),$$

$$d(z,F(z,y,x)) = dist(A,B),$$

and

$$d(x^*, F(x^*, y^*, z^*)) = dist(A, B),$$

$$d(y^*, F(y^*, x^*, z^*)) = dist(A, B),$$

$$d(z^*, F(z^*, y^*, x^*)) = dist(A, B).$$

We consider two cases :

Case I : Suppose (x, y, z) is comparable. Let (x, y, z) is comparable to (x^*, y^*, z^*) with respect to the ordering in $A \times A \times A$. Using the fact that F is a proximally tripled weak (ψ, ϕ) contraction on A to $d(x, F(x, y, z)) = dist(A, B)$ and $d(x^*, F(x^*, y^*, z^*)) = dist(A, B)$, we get

$$\begin{aligned} \phi(d(x,x^*)) &\leq \frac{1}{3}\phi\left(d(x,x^*) + d(y,y^*) + d(z,z^*)\right) \\ &\quad - \psi\left(\frac{d(x,x^*) + d(y,y^*) + d(z,z^*)}{3}\right). \end{aligned} \tag{41}$$

Similarly, we get

$$\begin{aligned} \phi(d(y,y^*)) &\leq \frac{1}{3}\phi\left(d(y,y^*) + d(x,x^*) + d(z,z^*)\right) \\ &\quad - \psi\left(\frac{d(y,y^*) + d(x,x^*) + d(z,z^*)}{3}\right) \end{aligned} \tag{42}$$

and

$$\begin{aligned} \phi(d(z,z^*)) &\leq \frac{1}{3}\phi\left(d(z,z^*) + d(y,y^*) + d(x,x^*)\right) \\ &\quad - \psi\left(\frac{d(z,z^*) + d(y,y^*) + d(x,x^*)}{3}\right). \end{aligned} \tag{43}$$

Adding (41), (42) and (43), we get

$$\begin{aligned} \phi(d(x,x^*)) + \phi(d(y,y^*)) + \phi(d(z,z^*)) &\leq \phi\left(d(x,x^*) + d(y,y^*) + d(z,z^*)\right) \\ &\quad - 3\psi\left(\frac{d(x,x^*) + d(y,y^*) + d(z,z^*)}{3}\right). \end{aligned} \tag{44}$$

By the property (iii) of ϕ, we obtain

$$\phi(d(x,x^*) + d(y,y^*) + d(z,z^*)) \leq \phi(d(x,x^*)) + \phi(d(y,y^*)) + \phi(d(z,z^*)). \tag{45}$$

From (44) and (45), we get

$$\begin{aligned}\phi(d(x,x^*)+d(y,y^*)+d(z,z^*)) &\leq \phi\left(d(x,x^*)+d(y,y^*)+d(z,z^*)\right)\\ &\quad -3\psi\left(\frac{d(x,x^*)+d(y,y^*)+d(z,z^*)}{3}\right).\end{aligned} \tag{46}$$

This implies that $3\psi\left(\frac{d(x,x^*)+d(y,y^*)+d(z,z^*)}{3}\right)\leq 0$. Using the property of ψ, we get $d(x,x^*)+d(y,y^*)+d(z,z^*)=0$. Hence $d(x,x^*)=d(y,y^*)=d(z,z^*)=0$. So $x=x^*, y=y^*$ and $z=z^*$.

Case II : Suppose (x,y,z) is not comparable. Let (x,y,z) be not comparable to (x^*,y^*,z^*), then there exists $(u_1,v_1,w_1)\in A_0\times A_0\times A_0$ which is comparable to (x,y,z) and (x^*,y^*,z^*). Since $F(A_0\times A_0\times A_0)\subseteq B_0$, there exists $(u_2,v_2,w_2)\in A_0\times A_0\times A_0$ such that

$$d(u_2,F(u_1,v_1,w_1))=dist(A,B),$$
$$d(v_2,F(v_1,u_1,w_1))=dist(A,B), \text{ and}$$
$$d(w_2,F(w_1,v_1,u_1))=dist(A,B).$$

Without loss of generality, assume that $(u_1,v_1,w_1)\leq(x,y,z)$ (i.e., $x\geq u_1, y\leq v_1$ and $z\geq w_1$). Note that $(u_1,v_1,u_1)\leq(x,y,x)$ implies that $(y,x,y)\leq(v_1,u_1,v_1)$.

From Lemma 2.2, Lemma 2.3 and Lemma 2.4, we get

$$\left.\begin{array}{c} u_1\leq x, v_1\geq y \text{ and } w_1\leq z\\ d(u_2,F(u_1,v_1,w_1))=dist(A,B)\\ d(x,F(x,y,z))=dist(A,B)\end{array}\right\}\Rightarrow\quad u_2\leq x;$$

$$\left.\begin{array}{c} u_1\leq x \text{ and } v_1\geq y\\ d(v_2,F(v_1,u_1,w_1))=dist(A,B)\\ d(y,F(y,x,z))=dist(A,B)\end{array}\right\}\Rightarrow\quad v_2\geq y \qquad \text{and;}$$

$$\left.\begin{array}{c} v_1\geq y \text{ and } w_1\leq z\\ d(w_2,F(w_1,v_1,u_1))=dist(A,B)\\ d(z,F(z,y,x))=dist(A,B)\end{array}\right\}\Rightarrow\quad w_2\leq z.$$

From the above three inequalities, we obtain $(u_2, v_2, w_2) \leq (x, y, z)$. Continuing this process, we get sequences $\{u_n\}, \{v_n\}$ and $\{w_n\}$ such that

$$d(u_{n+1}, F(u_n, v_n, w_n)) = dist(A, B),$$

$$d(v_{n+1}, F(v_n, u_n, w_n)) = dist(A, B), \text{ and}$$

$$d(w_{n+1}, F(w_n, v_n, u_n)) = dist(A, B),$$

with $(u_n, v_n, w_n) \leq (x, y, z)$, $\forall n \in \mathbb{N}$. By using the fact that F is a proximally tripled weak (ψ, ϕ) contraction on A, we get

$$\left.\begin{array}{c} u_n \leq x, v_n \geq y \text{ and } w_n \leq z \\ d(u_{n+1}, F(u_n, v_n, w_n)) = dist(A, B) \\ d(x, F(x, y, z)) = dist(A, B) \end{array}\right\} \tag{47}$$
$$\Rightarrow \phi(d(u_{n+1}, x)) \leq \frac{1}{3}\phi\left(d(u_n, x) + d(v_n, y) + d(w_n, z)\right) - \psi\left(\frac{d(u_n, x) + d(v_n, y) + d(w_n, z)}{3}\right).$$

Similarly, we can have that

$$\left.\begin{array}{c} y \leq v_n, x \geq u_n \text{ and } w_n \leq z \\ d(y, F(y, x, z)) = dist(A, B) \\ d(v_{n+1}, F(v_n, u_n w_n)) = dist(A, B) \end{array}\right\} \tag{48}$$
$$\Rightarrow \phi(d(y, v_{n+1})) \leq \frac{1}{3}\phi\left(d(y, v_n) + d(x, u_n) + d(z, w_n)\right) - \psi\left(\frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3}\right),$$

and

$$\left.\begin{array}{c} w_n \leq z, v_n \geq y \text{ and } u_n \leq x \\ d(w_{n+1}, F(w_n, v_n, u_n)) = dist(A, B) \\ d(z, F(z, y, x)) = dist(A, B) \end{array}\right\} \tag{49}$$
$$\Rightarrow \phi(d(z, w_{n+1})) \leq \frac{1}{3}\phi\left(d(z, w_n) + d(y, v_n) + d(x, u_n)\right) - \psi\left(\frac{d(z, w_n) + d(y, v_n) + d(x, u_n)}{3}\right).$$

Adding (47), (48) and (49), we obtain

$$\begin{aligned}
&\phi(d(u_{n+1},x))+\phi(d(v_{n+1},y))+\phi(d(w_{n+1},z)) \\
&\qquad\le \phi\left(d(u_n,x)+d(v_n,y)+d(w_n,z)\right) \\
&\qquad\quad -3\psi\left(\frac{d(u_n,x)+d(v_n,y)+d(w_n,z)}{3}\right).
\end{aligned} \tag{50}$$

By the property (iii) of ϕ, we get

$$\begin{aligned}
&\phi\left(d(u_{n+1},x)+d(v_{n+1},y)+d(w_{n+1},z)\right) \\
&\le \quad \phi(d(u_{n+1},x))+\phi(d(v_{n+1},y))+\phi(d(w_{n+1},z)).
\end{aligned} \tag{51}$$

From (50) and (51), we obtain

$$\begin{aligned}
&\phi\left(d(u_{n+1},x)+d(v_{n+1},y)+d(w_{n+1},z)\right) \\
&\qquad \le \phi\left(d(u_n,x)+d(v_n,y)+d(w_n,z)\right)-3\psi\left(\frac{d(u_n,x)+d(v_n,y)+d(w_n,z)}{3}\right).
\end{aligned} \tag{52}$$

This implies that

$$\phi\left(d(u_{n+1},x)+d(v_{n+1},y)+d(w_{n+1},z)\right) \le \phi\left(d(u_n,x)+d(v_n,y)+d(w_n,z)\right).$$

Using the fact that ϕ is nondecreasing, we get

$$d(u_{n+1},x)+d(v_{n+1},y)+d(w_{n+1},z) \le d(u_n,x)+d(v_n,y)+d(w_n,z). \tag{53}$$

This means that the sequence $\{d(u_n,x)+d(v_n,y)+d(w_n,z)\}$ is decreasing. Therefore, there exists $\alpha \ge 0$ such that

$$\lim_{n\to\infty}[d(u_n,x)+d(v_n,y)+d(w_n,z)] \;=\; \alpha. \tag{54}$$

We show that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Taking the limit as $n \to \infty$ in (52), we have that

$$\begin{aligned}
\phi(\alpha) &\le \phi(\alpha) - 3\lim_{n\to\infty}\psi\left(\frac{d(u_n,x)+d(v_n,y)+d(w_n,z)}{3}\right) \\
&< \phi(\alpha).
\end{aligned}$$

This is a contradiction. Thus $\alpha = 0$, that is

$$\lim_{n\to\infty}[d(u_n,x)+d(v_n,y)+d(w_n,z)] \;=\; 0,$$

so we have that $u_n \to x, v_n \to y$ and $w_n \to z$. Analogously, we can prove that $u_n \to x^*, v_n \to y^*$ and $w_n \to z^*$. Therefore, $x = x^*, y = y^*$ and $z = z^*$. Our proof is finished.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] H. Aydi, M. Abbas, Tripled coincidence and fixed point results in partially metric spaces, Appl. General Topology, 2(2012), 193–206.

[2] H. Aydi, E. Karapinar, Tripled fixed points in ordered metric spaces, Bull. Math. Anal. Appl. 4 (2012), 197–207.

[3] H. Aydi, M. Abbas, W. Sintunavarat, P. Kumam, Tripled fixed point of W- compatible mappings in abstract metric spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 134.

[4] A. Abkar, M. Gabeleh, Generalized cyclic contractions in partially ordered metric spaces, Optim. Lett, (2011), doi:10.1007/s11590-011-0379-y.

[5] A. Abkar, M. Gabeleh, Best proximity points for cyclic mappings in ordered metric spaces, J. Optim. Theory Appl. 150 (2011), 188–193.

[6] H. Aydi, E. Karapinar, S. Radenović, Tripled coincidence fixed point results for Boyd-Wong and Matkowski type contractions, RACSAM, 107 (2013), 339–353.

[7] H. Aydi, E. Karapinar, P. Salimi, I. M. Erhan, Best proximity points of generalized almost -Geraghty contractive non-self mappings, Fixed Point Theory Appl. 2014 (2014), Article ID 34.

[8] H. Aydi, E. Karapinar, W. Shatanawi, Tripled coincidence point results for generalized contractions in ordered generalized metric spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 101.

[9] H. Aydi, E. Karapinar, Z. Mustafa, Some tripled coincidence point theorems for almost generalized contractions in ordered metric spaces, Tamkang J. Math. 44 (2013), 233–251.

[10] H. Aydi and E. Karapinar, Tripled coincidence point theorems for a class of contractions in ordered metric spaces, Acta Universitatis Apulensis, 23 (2012), 195–208.

[11] H. Alsulami, A. Roldan, E. Karapinar and S. Radenović, Some inevitable remarks on ”Tripled fixed point theorems for mixed monotone Kannan type contractive mappings”, J. Appl. Math. in press.

[12] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011), 4889–4897.

[13] V. Ćojbasić Rajić, S. Radenović, A note on tripled fixed point of w-compatible mappings in tvs-cone metric spaces, Thai J. Math. (2013), in press.

[14] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006), 1379–1393.

[15] S. S. Basha, P. Veeramani, Best approximations and best proximity pairs, Acta Sci. Math.(Szeged), 63 (1997), 289–300.

[16] G. Beer, D. V. Pai, Proximal maps, prox maps and coincidence points, Number. Funct. Anal. Optim. 11 (1990), 429–448.

[17] G. Beer, D. V. Pai, The prox map, J. Math. Anal. Appl. 156 (1991), 428–443.

[18] A. A. Eldred and P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2006), 1001–1006.

[19] K. Fan, Extensions of two fixed point theorems of F. E. Browder, Math. Z. 122 (1969), 234–240.

[20] K. Goebel, W. A. Kirk, Topics in metric fixed point theory, Cambridge University Press, Cambridge, 1990.

[21] N. Hussain, M. Abbas, A. Azam, J. Ahmad, Coupled coincidence point results for a generalized compatible pair with applications, Fixed Point Theory Appl. 2014 (2014), Article ID 62.

[22] M. Jleli, B. Samet, Remarks on the paper: Best proximity point theorems: an exploration of a common solution to approximation and optimization problems, Appl. Math. Comput. 228 (2014), 366–370.

[23] M. A. Kutbi, A. Azam, J. Ahmad, C. Di Bari, Some common coupled fixed point results for generalized contraction in complex-valued metric spaces, J. Appl. Math. 2013 (2013), Article ID 352927.

[24] E. Karapinar, Tripled fixed point theorems on partially ordered metric spaces, Stud. Univ. Babes-Bolyai Math. 58 (2013), 75–85.

[25] E. Karapinar, A. Roldan, C. Rodan, J. Martinez-Moreno, A note on N-fixed point theorems for nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl. 2013 (2013), Article ID 310.

[26] P. Kumam, V. Pragadeeswarar and M. Marudai and K. Sitthithakerngkiet, Coupled best proximity points in ordered metric spaces, Fixed Point Theory Appl. 2014 (2014), Article ID 107.

[27] P. Kumam, H. Aydi, E. Karapinar, W. Sintunavarat, Best proximity points and extension of Mizoguchi-Takahashis fixed point theorems, Fixed Point Theory Appl. 2013 (2013), Article ID 242.

[28] M. A. Khamsi, W. A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, Wiley-Interscience, New York, (2001).

[29] W. A. Kirk, S. Reich, P. Veeramani, Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24 (2003), 851–862.

[30] W. K. Kim, K. H. Lee, Existence of best proximity pairs and equilibrium pairs, J. Math. Anal. Appl. 316 (2006), 433–446.

[31] Z. Kadelburg, S. Radenović, A note on some recent best proximity point results for non-self mappings, Gulf J. Math. 1 (2013), 36–41.

[32] Z. Kadelburg, S. Radenović, A note on tripled fixed points and fixed points of 3-order in partially ordered metric spaces, Math. Sci. Appl. E-Notes, 1 (2013), 230–237.

[33] E. Karapinar, W. Sintunavarat, The existence of an optimal approximate solution theorems for generalized α-proximal contraction nonself mappings and applications, Fixed Point Theory Appl. 2013 (2013), Article ID 323.

[34] B. V. Luong, N. X. Thuan, Coupled fixed point theorems for mixed monotone mappings and an application to integral equations, Comput. Math. Appl. 62 (2011), 4238–4248.

[35] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341–4349.

[36] B. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011), 983–992.

[37] V. Parvaneh, J. R. Roshan, S. Radenović, Existence of tripled coincidence point in ordered b-metric spaces and applications to a system of integral equations, Fixed Point Theory Appl. 2013 (2013), Article ID 130.

[38] V. Pragadeeswarar, M. Marudai, Best proximity points: approximation and optimization in partially ordered metric spaces, Optim. Lett. (2012), doi:10.1007/s11590-012-0529-x.

[39] J. B. Prolla, Fixed point theorems for set valued problems and existence of best approximations, Number. Funct. Anal. Optim. 5 (1982/1983), 449–455.

[40] S. Reich, Approximate selections, best approximations, fixed points and invariant sets, J. Math. Anal. 62 (1978), 104–113.

[41] S. Radenović, A note on tripled coincidence and tripled common fixed point theorems in partially ordered metric spaces, Apl. Math. Comput. 236 (2014), 367–372.

[42] S. Radenović, Remarks on some coupled coincidence point results in partially ordered metric spaces, Arab J Math. Sci. 20 (2014), 29–39.

[43] A. Roldan, J. Martinez-Moreno, C. Rojdan, E. Karapinar, Some remarks on multidimensional fixed point theorems, Fixed Point Theory Appl. 15 (2014), 545–558.

[44] S. Radenović, Remarks on some coupled fixed point results in partial metric spaces, Nonlinear Funct. Anal. Appl. 18 (2013), 39–50.

[45] S. Radenović, Remarks on some recent coupled coincidence point results in symmetric G-metric spaces, J. Oper. 2013 (2013), Article ID 290525.

[46] S. Radenović, S. Pantelić, P. Salimi, J. Vu-jaković, A note on some tripled coincidence point results in Gmetric spaces, Inter. J. Math. Sci. Eng. Appl. 6 (2012), 23–38.

[47] B. Samet, E. Karapinar and H. Aydi and V. Rajic, Discussion on some coupled fixed point theorems, Fixed Point Theory Appl. 2013 (2013), Article ID 50.

[48] W. Shatanawi, Partially ordered metric spaces and coupled fixed point results, Comput. Math. Appl. 60 (2010), 2508–2515.

[49] W. Sintunavarat, P. Kumam, Coupled best proximity point theorem in metric spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 93.

[50] S. Sadiq Basha, Discrete optimization in partially ordered sets, J. Glob. Optim. (2011), doi:10.1007/s10898-011-9774-2.

[51] W. Sintunavarat, P. Kumam, The existence theorems of an optimal approximate solution for generalized proximal contraction mappings, Abstr. Appl. Anal. (2013), Article ID 375604.

[52] V. Sankar Raj, A best proximity point theorem for weakly contractive non-self-mappings, Nonlinear Anal. 74 (2011), 4804–4808.

[53] S. Sadiq Basha, P. Veeramani, Best proximity pair theorems for multifunctions with open fibers, J. Approx. Theory 103 (2000), 119–129.

[54] M. A. Al-Thagafi, N. Shahzad, Convergence and existence results for best proximity points, Nonlinear Anal. 70 (2009), 3665–3671.

[55] P. S. Srinivasan, P. Veeramani, On existence of equilibrium pair for constrained generalized games, Fixed Point Theory Appl. 1 (2004), 21–29.

[56] M. De la Sen, R. P. Agarwal, Some fixed point-type results for a class of extended cyclic self-mappings with a more general contractive condition, Fixed Point Theory Appl. 2011 (2011), Article ID 59.

[57] K. P. R. Rao, G. N. V. Kishore, A unique common tripled fixed point theorem in partially ordered cone metric spaces, Bull. Math. Anal. Appl. 4 (2011), 213–222.

[58] S. P. Singh, B. Watson, P. Srivastava, Fixed point Theory and Best Approximation: The KKM-Map Principle, Kluwer Academic Publishers, New York, (1997).

[59] W. Takahashi, Nonlinear Functional Analysis: Fixed Point Theory and Its Applications, Yokohama Publishers Inc, Yokohama, (2000).

[60] V. Vetrivel, P. Veeramani, P. Bhattacharyya, Some extensions of Fan's best approximation theorem, Numer.Fucnt. Anal. Optim. 13 (1992), 397–402.

[61] S. M. Sehgal, S. P. Singh, A generalization to multifunctions of Fan's best approximation theorem, Proc. Am. Math. Soc. 102 (1988), 534–537.

[62] E. Zeidler, Nonlinear Functional Analysis and Its Applications - I: Fixed Point Theorems, Springer, New York, (1986).