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ON THE GENERAL PRINCIPLE OF MULTI-STEP FIXED POINT ITERATIVE SCHEMES

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Abstract. In this study, we establish a general principle for multi-step fixed point iterative schemes in the sense of Rhoades and Soltuz [B.E. Rhoades, S.M. Soltuz, The equivalence between Mann-Ishikawa iterations and multi-step iteration, Nonlinear Analysis 58 (2004), 219-228] to approximate fixed point of the maps satisfying the generic type contractive definition in Banach spaces. Our result is an extension and generalization of the results by B. E. Rhoades [2-4] and many other results in the literature (see for instance [10-12] and their references).

Keywords: Fixed point; Ishikawa iterative schemes; Mann iterative schemes; Noor iterative schemes; Multi-step fixed point iterative schemes.

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1. Introduction

The first theorem on fixed point theory was established by Polish Mathematician Stefan Banach [5] in 1922. This theorem is also known as Banach fixed point theorem or Contraction mapping theorem. Banach fixed point theorem has been applied to many different areas. The Mann iterative scheme known as one-step iterative scheme [6], invented in 1953 and was used

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to prove the convergence of the sequence to a fixed point of many valued mapping where the Banach fixed point theorem is failed. Later, in 1974 S. Ishikawa in [7] derived a new iteration scheme known as two-step iterative scheme in order to establish the convergence of Lipschitzian pseudocontractive map when Mann iteration scheme failed to converge. M. A. Noor in [8-9] introduced and analyzed three-step iterative scheme to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. In 2004, Rhoades and Soltuz [1] generalized the above discussed iterative schemes by introducing the multi-step iterative scheme, from which it can be obtainable the other iterative schemes. There are a number of papers in literature on the general principle of different fixed point iterative schemes under different types of contractive definitions, see for instance [2-4, 10-12]. However, there is no general principle for general fixed point iterative scheme under general contractive definition which generates the rest. In order to solve this problem, in this paper we introduce a general principle for multi-step fixed point iterative schemes (general fixed point iterative scheme) (2.1) by using a general contractive definition (3.1).

The main objective of the present paper is to establish a general principle for multi-step fixed point iterative schemes under a general contractive definition.

2. Preliminaries

In this section, we discuss some basic definitions and results which are used as the auxiliary tools in our main work.

Throughout this paper, $\mathbf{N}$ will denote the set of all positive integers. Let X be a complete Banach space, E be a non-empty bounded closed convex subset of X and T be a self-map on E. The point $x \in X$ is called a fixed point of the map T iff $T(x) = x$. The set of fixed points of the map T is denoted by $F(T) = \{p \in X : T(p) = p\}$.

The multi-step fixed point iterative schemes (General fixed point iterative Schemes) of T in [1] are defined by,

$$\left\{\begin{array}{l} x_0 \in E; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^{(1)}; \\ y_n^{(i)} = (1-\beta_n^{(i)})x_n + \beta_n^{(i)} T y_n^{(i+1)}; \\ y_n^{(q-1)} = (1-\beta_n^{(q-1)})x_n + \beta_n^{(q-1)} T x_n; i = 1,2,\ldots,q-2, q \geq 2, n \in \mathbf{N} \end{array}\right\} \tag{2.1}$$

where,$\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^{(i)}\}_{n=0}^{\infty}$ are real sequences on $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The Mann iterative schemes (one-step iterative schemes) of T in [6] are defined by,

$$\left\{\begin{array}{l} x_0 \in E; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n; n \in \mathbf{N} \end{array}\right\} \tag{2.2}$$

where, $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of non-negative numbers satisfying (i) $\alpha_0 = 1$, (ii) $0 \leq \alpha_n < 1$, and (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The Ishikawa iterative schemes (two-step iterative schemes) of T in [7] are defined by,

$$\left\{\begin{array}{l} x_0 \in E; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \\ y_n = (1-\beta_n)x_n + \beta_n T x_n; n \in \mathbf{N} \end{array}\right\} \tag{2.3}$$

where,$\{\alpha_n\}_{n=0}^{\infty}$, and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of non-negative numbers such that (i) $0 \leq \alpha_n < 1$, and $0 \leq \beta_n < 1$, (ii)$\lim_{n\to\infty} sup(\beta_n) < 1$.

Similarly, Noor iterative schemes (three-step iterative schemes) of T in [8-9] are defined by

$$\left\{\begin{array}{l} x_0 \in E; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \\ y_n = (1-\beta_n)x_n + \beta_n T z_n; \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n; n \in \mathbf{N} \end{array}\right\} \tag{2.4}$$

where, the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ are convergent, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Using the multi-step fixed point iterative schemes defined by (2.1), we can easily obtained the Mann (one-step), Ishikawa (two-step) and Noor (three-step) iterative schemes by putting $\beta_n^{(1)} = \beta_n^{(2)} = \cdots = \beta_n^{(q-2)} = \beta_n^{(q-1)} = 0$, $\beta_n^{(2)} = \beta_n^{(3)} = \cdots = \beta_n^{(q-2)} = \beta_n^{(q-1)} = 0$, and $\beta_n^{(3)} = \beta_n^{(4)} = \cdots = \beta_n^{(q-2)} = \beta_n^{(q-1)} = 0$ respectively. In the similar way, for different values of $\{\alpha_n\}_{n=0}^{\infty}$, and $\{\beta_n^{(i)}\}_{n=0}^{\infty}$, we can easily obtained all of the rest fixed point iterative schemes form the multi-step fixed point iterative schemes defined by (2.1). Thus, it is clear that multi-step fixed point iterative schemes defined by (2.1) is a general fixed point iterative scheme comparing with other fixed point iterative schemes.

In 1986, M.S. Khan et al. [13] introduced the general Mann iterative scheme as follows:

If X is a complete metric space, T is a self-map of X and A is a lower triangular matrix with non-negative entries, zero column limits, and row sum 1, then the General Mann iterative schemes $\{x_n\}$ of T is defined by

$$x_0 \in X; x_{n+1} := Tv_n, v_n = \sum_{k=0}^{n} \alpha_{nk} x_k \tag{2.5}$$

where $a_{ij} \in A; i, j = 0, 1, 2, \cdots, n; n \in \mathbf{N}$.

Note that Mann iterative schemes defined by (2.2) is a special case of the General Mann iterative schemes defined by (2.5).

A matrix A is called a weighted mean matrix if A is a lower triangular matrix with non-zero entries $a_{nk} = \dfrac{p_k}{P_n}$, where $\{p_k\}$ is a non-negative sequence with p_0 positive and $P_n = \sum_{k=0}^{n} p_k \to \infty$.

If X is a metric space, then for all $x, y \in X$, a $F-$norm r on X is defined by

(i) $r(x) \geq 0$, and $r(x) = 0$ iff $x = 0$;

(ii) $r(x+y) \leq r(x) + r(y)$;

(iii) $r(ax) \leq r(x)$ for all scalars a with $|\, a \,| \leq 1$.

In [2] B.E. Rhoades established the following general principle for General Mann iterative scheme:

Theorem 2.1. (see [2]) *Let T be a self-map of a closed convex subset E of a complete metric space (X,d). Let $\{x_n\}$ be a general Mann iteration of T with A equivalent to convergence. Suppose that $\{x_n\}$ converges to a point $p \in E$. If there exist constants $\alpha, \beta, \gamma, \delta \geq 0, \delta < 1$, such that it is possible to write*

$$d(Tx_n, Tp) \leq \alpha d(x_n, p) + \beta d(x_n, Tx_n) + \gamma d(p, Tx_n) + \delta max\{d(p, Tp), d(x_n, Tp)\}.$$

Then p is a fixed point of T. ♦

Later, in [3] B.E. Rhoades also established the following theorem as a general principle for Ishikawa iterative scheme:

Theorem 2.2. (see [3]) *Let E be a closed convex subset of a Banach space X, T be a self-map on E , and $\{x_n\}$ an Ishikawa iterative scheme satisfying $\lim_{n\to\infty}\alpha_n > 0$, and such that $x_n \to p$. Suppose that there exist constants $\alpha,\beta,\gamma,\delta \geq 0, \beta < 1$ such that, for all n sufficiently large, it is possible to write*

$$\|Tx_n - Ty_n\| \leq \alpha\|x_n - Ty_n\| + \beta\|x_n - Tx_n\|$$

and

$$\|Tp - Tx_n\| \leq \alpha\|x_n - p\| + \gamma\|x_n - Tx_n\| + \delta\|p - Tx_n\| + \beta max\{\|p - Tp\|, \|x_n - Tp\|\}$$

Then p is a fixed point of T. ♦

In this present paper we extended the above results for multi-step fixed point iterative schemes defined by (2.1). The main purpose of this paper is to establish a general principle for multi-step fixed point iterative schemes which will generate the general principle of other fixed point iterative schemes.

3. Main results and Discussion

In this section we state and prove our main result with some related corollaries.

Theorem 3.1. *Let X be a complete Banach space and E be a non-empty bounded closed convex subset of X. Let T be a self-map on E satisfying the following general contractive definition:*

$$\|Tp - Tx_n\| \leq a\|x_n - p\| + b\|x_n - Tx_n\| + c\|p - Tx_n\| + d max\{\|p - Tp\|, \|x_n - Tp\|\} \quad (3.1)$$

for all $x, y \in E$ and $a,b,c,d \geq 0, d < 1$. Let $\{x_n\}$ be a multi-step iterative scheme defined by (2.1) and satisfying $\lim_{n\to\infty}\alpha_n > 0$, $\lim_{n\to\infty}\beta_n^{(i)} > 0$, such that $x_n \to p$. Suppose that there exist some constants $\gamma_1^{(i)} \geq 0; 0 \leq \gamma_2^{(i)} < 1; i = 1,2,\cdots,q-2; q \geq 2; n \in \mathbf{N}$ such that for all sufficiently large n, it is possible to write

$$\left.\begin{cases} \|Tx_n - Ty_n^{(1)}\| \leq \gamma_1^{(1)}\|x_n - Ty_n^{(1)}\| + \gamma_2^{(1)}\|x_n - Tx_n\| \\ \|Ty_n^{(1)} - Ty_n^{(2)}\| \leq \gamma_1^{(2)}\|x_n - Ty_n^{(2)}\| + \gamma_2^{(2)}\|x_n - Ty_n^{(1)}\| \\ \|Ty_n^{(2)} - Ty_n^{(3)}\| \leq \gamma_1^{(3)}\|x_n - Ty_n^{(3)}\| + \gamma_2^{(3)}\|x_n - Ty_n^{(2)}\| \\ \vdots \\ \|Ty_n^{(q-2)} - Ty_n^{(q-1)}\| \leq \gamma_1^{(q-1)}\|x_n - Ty_n^{(q-1)}\| + \gamma_2^{(q-1)}\|x_n - Ty_n^{(q-2)}\| \end{cases}\right\} \quad (3.2)$$

Then p is a fixed point of T.

Proof. From the first equation of (2.1) we have $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^{(1)}$, and it can be written as in the form $x_1 = Tz_1$, where $x_1 = \{x_n\}, z_1 = \left\{Ty_n^{(1)}\right\}$ and W_1 is the weighted mean matrix generated by

$$p_0^{(1)} > 0, p_k^{(1)} = \frac{\alpha_k}{\prod_{j=1}^{k}(1-\alpha_j)}, k > 0.$$

Now, since $\lim_{n\to\infty}\alpha_n > 0$, then W_1 is equivalent to convergence, and hence $\lim_{n\to\infty} x_n = p$ implies that $\lim_{n\to\infty} Ty_n^{(1)} = p$.

In the similar way if we construct the weighted mean matrices $W_i; i = 1,2,\cdots,q-1$ generated by

$$p_0^{(i)} > 0, p_k^{(i)} = \frac{\beta_k^{(i)}}{\prod_{j=1}^{k}(1-\beta_j^{(i)})}; k > 0,$$

then with respect to the Second and Third equations of (2.1), we have

$$y_n^{(i)} = (1-\beta_n^{(i)})x_n + \beta_n^{(i)} Ty_n^{(i+1)};$$
$$y_n^{(q-1)} = (1-\beta_n^{(q-1)})x_n + \beta_n^{(q-1)} Tx_n; i = 1,2,\cdots,q-2; q \geq 2; n \in \mathbf{N}$$

and from these we can easily shown that $\lim_{n\to\infty} x_n = p$ implies $\lim_{n\to\infty} Ty_n^{(i)} = p$ and $\lim_{n\to\infty} Ty_n^{(q-1)} = p; i = 1,2,\cdots,q-2; q \geq 2; n \in \mathbf{N}$.

Thus, we have

$$\begin{aligned} \|Tx_n - p\| \leq \|Tx_n - Ty_n^{(1)}\| + \|Ty_n^1 - Ty_n^2\| + \|Ty_n^2 - Ty_n^{(3)}\| + \cdots \\ + \|Ty_n^{(q-2)} - Ty_n^{(q-1)}\| + \|Ty_n^{(q-1)} - p\| \end{aligned} \tag{3.3}$$

Now, from the inequalities of (3.2) and (3.3) we can write,

$$\begin{aligned} \|Tx_n - p\| \leq \gamma_1^{(1)}\|x_n - Ty_n^{(1)}\| + \gamma_2^{(1)}\|x_n - Tx_n\| + \gamma_1^{(2)}\|x_n - Ty_n^{(2)}\| + \gamma_2^{(2)}\|x_n - Ty_n^{(1)}\| \\ + \gamma_1^{(3)}\|x_n - Ty_n^{(3)}\| + \gamma_2^{(3)}\|x_n - Ty_n^{(2)}\| + \cdots + \gamma_1^{(q-1)}\|x_n - Ty_n^{(q-1)}\| \\ + \gamma_2^{(q-1)}\|x_n - Ty_n^{(q-2)}\| + \|Ty_n^{(q-1)} - p\| \end{aligned} \tag{3.4}$$

Taking limit as $n \to \infty$ on both sides of (3.4) we get,

$$\lim_{n\to\infty}\|Tx_n - p\| \leq \gamma_2^{(1)} \lim_{n\to\infty}\|Tx_n - p\| \tag{3.5}$$

According to our assumption, we have $0 \leq \gamma_2^{(1)} < 1$, hence from (3.5) we obtained

$$\lim_{n\to\infty} Tx_n = p.$$

Further, taking limit as $n \to \infty$ on both sides of (3.1) and using $\lim_{n\to\infty} Tx_n = p$, we yields

$$\|Tp - p\| \le d\|Tp - Tp\| \tag{3.6}$$

Since, $0 \le d < 1$, hence from (3.6) we can write, $Tp = p$.

This completes our proof. ⧫

Next, in [12] B.E. Rhoades used the following contractive definition:

Let X be a complete Banach space and E be a non-empty bounded closed convex subset of X. Let T be a self-map on E. For some constants $c_1, k \ge 0, k < 1$ and for each pair of points $x, y \in X$, we can write

$$\|Tx - Ty\| \le kmax\{c_1\|x-y\|, \|x-Tx\| + \|y-Ty\|, \|x-Ty\| + \|y-Tx\|\} \tag{3.7}$$

Now, we state and prove our Corollary 3.1 to extend the above context for multi-step iterative schemes defined by (2.1).

Corollary 3.1. *Let X be a complete Banach space and E be a non-empty bounded closed convex subset of X. Let T be a self-map on E satisfying the contractive definition (3.7). If the multi-step iterative schemes defined by (2.1) and satisfying the conditions $0 \le \alpha_n \le 1$; $0 \le \beta_n^{(i)} \le 1$; $\lim_{n\to\infty} \le \beta_n^{(i)} = 0$; $i = 1, 2, \cdots, q-2$; $q \ge$; $n \in N$ converges to a point p, then p is a fixed point of T.*

Proof. By the context of our Theorem 3.1 we prove this corollary and for this it is sufficient to prove that the contractive definitions (3.1) and (3.2) are obtained from the contractive definition (3.7).

First, we replace x and y by p and x_n respectively in (3.7), we get

$$\begin{aligned}\|Tp - Tx_n\| &\le kmax\{c_1\|p-x_n\|, \|p-Tp\| + \|x_n-Tx_n\|, \|p-Tx_n\| + \|x_n-Tp\|\} \\ &\le kc_1\|p-x_n\| + k\|x_n-Tx_n\| + k\|p-Tx_n\| + kmax\{\|p-Tp\|, \|x_n-Tp\|\}\end{aligned} \tag{3.8}$$

Now, since $c_1, k \ge 0, k < 1$ and a, b, c, $d \ge 0$, $d < 1$, hence from (3.8) we can write

$$\|Tp - Tx_n\| \le a\|p-x_n\| + b\|x_n-Tx_n\| + c\|p-Tx_n\| + dmax\{\|p-Tp\|, \|x_n-Tp\|\}.$$

Therefore,(3.1) is satisfied.

Replacing x and y by x_n and $y_n^{(1)}$ respectively in (3.7), we get

$$\begin{aligned}\|Tx_n - Ty_n^{(1)}\| \leq kmax\{c_1\|x_n - y_n^{(1)}\|, \|x_n - Tx_n\| + \|y_n^{(1)} - Ty_n^{(1)}\|, \|x_n - Ty_n^{(1)}\| \\ + \|y_n^{(1)} - Tx_n\|\}\end{aligned} \tag{3.9}$$

If we take $i = 1$, then from (2.1), we obtain the following:

$$\left\{\begin{array}{l}\|x_n - y_n^{(1)}\| = \beta_n^{(1)}\|x_n - Tx_n\| \\ \|y_n^{(1)} - Ty_n^{(1)}\| = \left(1 - \beta_n^{(1)}\right)\|x_n - Ty_n^{(1)}\|\beta_n^{(1)}\|Tx_n - Ty_n^{(1)}\| \\ and \\ \|y_n^{(1)} - Tx_n\| = \left(1 - \beta_n^{(1)}\right)\|x_n - Tx_n\|\end{array}\right\} \tag{3.10}$$

Now, combining (3.9) and (3.10), we get

$$\begin{aligned}\|Tx_n - Ty_n^{(1)}\| &\leq kmax\{c_1\beta_n^{(1)}\|x_n - Tx_n\|, \|x_n - Tx_n\| + \left(1 - \beta_n^{(1)}\right)\|x_n - Ty_n^{(1)}\| \\ &\quad + \beta_n^{(1)}\|Tx_n - Ty_n^{(1)}\|, \|x_n - Ty_n^{(1)}\| + \left(1 - \beta_n^{(1)}\right)\|x_n - Tx_n\|\} \\ &\leq max\{c_1k\beta_n^{(1)}\|x_n - Tx_n\|, k\|x_n - Tx_n\| + k\left(1 - \beta_n^{(1)}\right)\|x_n - Ty_n^{(1)}\| \\ &\quad + k\beta_n^{(1)}\|Tx_n - Ty_n^{(1)}\|, k\|x_n - Ty_n^{(1)}\| + k\left(1 - \beta_n^{(1)}\right)\|x_n - Tx_n\|\}\end{aligned}$$

That is,

$$\begin{aligned}\|Tx_n - Ty_n^{(1)}\| \leq max\{c_1k\beta_n^{(1)}\|x_n - Tx_n\|, \frac{k}{1 - k\beta_n^{(1)}}\|x_n - Tx_n\| + \frac{k\left(1 - \beta_n^{(1)}\right)}{1 - k\beta_n^{(1)}}\|x_n - Ty_n^{(1)}\| \\ + k\beta_n^{(1)}\|Tx_n - Ty_n^{(1)}\|, k\|x_n - Ty_n^{(1)}\| + k\left(1 - \beta_n^{(1)}\right)\|x_n - Tx_n\|\}\end{aligned} \tag{3.11}$$

For any fixed positive integer M, if we suppose that $n > M$ implies $c_1\beta_n^{(1)} < 1$ and $\beta_n^{(1)} < \frac{(1-k)}{k}$, then for $n > M$ and $\gamma_2^{(1)} = max_{n>M}\left\{c_1\beta_n^{(1)}, \frac{k}{\left(1 - k\beta_n^{(1)}\right)}, k\left(1 - \beta_n^{(1)}\right)\right\}$ we can write,

$$\|Tx_n - Ty_n^{(1)}\| \leq k\|x_n - Ty_n^{(1)}\| + \gamma_2^{(1)}\|x_n - Tx_n\|.$$

This satisfies the first inequality of (3.2).

Now, replacing x and y by $y_n^{(1)}$ and $y_n^{(2)}$ respectively in (3.7), we get

$$\begin{aligned}\|Ty_n^{(1)}-Ty_n^{(2)}\| \leq kmax\{c_1\|y_n^{(1)}-y_n^{(2)}\|, \|y_n^{(1)}-Ty_n^{(1)}\|+\|y_n^{(2)}-Ty_n^{(2)}\|, \|y_n^{(1)}-Ty_n^{(2)}\| \\ +\|y_n^{(2)}-Ty_n^{(1)}\|\}\end{aligned} \tag{3.12}$$

If we take $i=2$, then from (2.1), we obtained the following:

$$\left\{\begin{array}{l}\|y_n^{(1)}-y_n^{(2)}\| \leq \beta_n^{(1)}\|x_n-Ty_n^{(2)}\|+\beta_n^{(2)}\|x_n-Tx_n\| \\ \|y_n^{(1)}-Ty_n^{(1)}\| \leq \|x_n-Ty_n^{(1)}\|+\beta_n^{(1)}\|x_n-Ty_n^{(2)}\| \\ \|y_n^{(2)}-Ty_n^{(2)}\| \leq \|x_n-Ty_n^{(2)}\|+\beta_n^{(2)}\|x_n-Tx_n\| \\ and \\ \|y_n^{(1)}-Ty_n^{(2)}\| \leq \left(1-\beta_n^{(1)}\right)\|x_n-Ty_n^{(2)}\| \\ \|y_n^{(2)}-Ty_n^{(1)}\| \leq \|x_n-Ty_n^{(1)}\|+\beta_n^{(2)}\|x_n-Tx_n\|\end{array}\right\} \tag{3.13}$$

According to our assumption, we have $\beta_n^{(2)} \to 0$. Therefore, from (3.13) we get

$$\left\{\begin{array}{l}\|y_n^{(1)}-y_n^{(2)}\| \leq \beta_n^{(1)}\|x_n-Ty_n^{(2)}\| \\ \|y_n^{(1)}-Ty_n^{(1)}\| \leq \|x_n-Ty_n^{(1)}\|+\beta_n^{(1)}\|x_n-Ty_n^{(2)}\| \\ \|y_n^{(2)}-Ty_n^{(2)}\| \leq \|x_n-Ty_n^{(2)}\| \\ and \\ \|y_n^{(1)}-Ty_n^{(2)}\| \leq \left(1-\beta_n^{(1)}\right)\|x_n-Ty_n^{(2)}\| \\ \|y_n^{(2)}-Ty_n^{(1)}\| \leq \|x_n-Ty_n^{(1)}\|\end{array}\right\} \tag{3.14}$$

Now, combining (3.12) and (3.14), we get

$$\begin{aligned}\|Ty_n^{(1)}-Ty_n^{(2)}\| &\leq kmax\{c_1\beta_n^{(1)}\|x_n-Ty_n^{(2)}\|, \|x_n-Ty_n^{(1)}\|+\beta_n^{(1)}\|x_n-Ty_n^{(2)}\| \\ &\quad +\|x_n-Ty_n^{(2)}\|, \left(1-\beta_n^{(1)}\right)\|x_n-Ty_n^{(2)}\|+\|x_n-Ty_n^{(1)}\|\} \\ &\leq kmax\{c_1\beta_n^{(1)}\|x_n-Ty_n^{(2)}\|, \|x_n-Ty_n^{(1)}\|+\left(1+\beta_n^{(1)}\right)\|x_n-Ty_n^{(2)}\|, \\ &\quad \left(1-\beta_n^{(1)}\right)\|x_n-Ty_n^{(2)}\|+\|x_n-Ty_n^{(1)}\|\}\end{aligned}$$

Since, $c_1, k \geq 0, k < 1$ and $0 \leq \beta_n^{(1)} \leq 1$, hence we can write

$$\|Ty_n^{(1)} - Ty_n^{(2)}\| \leq k\|x_n - Ty_n^{(1)}\| + k\left(1 + \beta_n^{(1)}\right)\|x_n - Ty_n^{(2)}\|$$

and this satisfies the second inequality of (3.2).

Similarly, if we continue the above process, then we can easily obtain the rest inequalities of (3.2). Therefore, we can say that the contractive definition (3.2) is satisfied.

This completes our proof. ♦

In [10] S.A. Naimpally and K.L. Sing used the following contractive definitions to extend some fixed point theorems of B.E. Rhoades:

Let X be a complete Banach space and E be a non-empty bounded closed convex subset of X. Let T be a self-map on E . For some constants $1 \leq a_1 < 2, \frac{1}{2} \leq b_1 < \frac{2}{3}, 1 \leq c_1 < \frac{3}{2}, 0 \leq k < 1$ and for each pair of points $x, y \in X$, we can write

$$\|x - Tx\| + \|y - Ty\| \leq a_1\|x - y\| \tag{3.15}$$

$$\|x - Tx\| + \|y - Ty\| \leq b_1[\|x - Ty\| + \|y - Tx\| + \|x - y\|] \tag{3.16}$$

$$\|x - Tx\| + \|y - Ty\| + \|Tx - Ty\| \leq c_1[\|x - Ty\| + \|y - Tx\|] \tag{3.17}$$

$$\|Tx - Ty\| \leq kmax\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, [\|x - Ty\| + \|y - Tx\|]/2\} \tag{3.18}$$

Now, we state and prove our corollary 3.2 to extend the above context for multi-step iterative schemes defined by (2.1).

Corollary 3.2. *Let X be a complete Banach space and E be a non-empty bounded closed convex subset of X . Let T be a self-map on E satisfying the contractive definitions (3.15) to (3.18). If the multi-step iterative schemes defined by (2.1), and with $\{\alpha_n\}$ bounded away from zero, converges to a point p, then p is a fixed point of T .*

Proof. By the context of our Theorem 3.1 to proof this corollary, it is sufficient to prove that the contractive definitions (3.1) and (3.2) are obtained from the contractive definition (3.15) to (3.18). If we replace x and y by x_n and y_n respectively in (3.15) to (3.18) and combined these replaced inequalities, then we get

$$\|Tx_n - Ty_n\| \leq max\left\{a_1 + 1, \frac{2(1 + b_1)}{1 - b_1}, \frac{1 + 2c_1}{2 - c_1}, \frac{k}{1 - k}\right\}\|x_n - Ty_n\| \tag{3.19}$$

Let $\lambda = max\left\{a_1+1, \frac{2(1+b_1)}{1-b_1}, \frac{1+2c_1}{2-c_1}, \frac{k}{1-k}\right\}$.

Therefore, (3.19) can be written as

$$\|Tx_n - Ty_n\| \leq \lambda \|x_n - Ty_n\| \tag{3.20}$$

Now, put $y_n = y_n^{(1)}$ in (3.20), we get

$$\|Tx_n - Ty_n^{(1)}\| \leq \lambda \|x_n - Ty_n^{(1)}\| \tag{3.21}$$

Since $1 \leq a_1 < 2, \frac{1}{2} \leq b_1 < \frac{2}{3}, 1 \leq c_1 < \frac{3}{2}, 0 \leq k < 1$, hence from (3.21) we can say that, the first inequality of (3.2) is satisfied.

Now, put $x_n = y_n^{(1)}$ and $y_n = y_n^{(2)}$ in (3.20), we get

$$\begin{aligned}\|Ty_n^{(1)} - Ty_n^{(2)}\| &\leq \lambda \|y_n^{(1)} - Ty_n^{(2)}\| \\ &\leq \lambda \left\{\|y_n^{(1)} - Ty_n^{(1)}\| + \|Ty_n^{(1)} - Ty_n^{(2)}\|\right\}\end{aligned}$$

This satisfies the second inequality of (3.2).

Similarly, if we continue the above process, then we can easily obtain the rest inequalities of (3.2) by contractive definitions (3.15) to (3.18).

Therefore, we can say that the contractive definition (3.2) is satisfied.

Now, we have to show that the contractive definition (3.1) is satisfied by contractive definitions (3.15) to (3.18).

By the triangle inequality, we have

$$\left\{\begin{array}{l}\|Tx_n - Tp\| - \|x_n - Tp\| \leq \|Tx_n - x_n\| \\ \\ \|Tx_n - Tp\| - \|p - Tx_n\| \leq \|p - Tp\|\end{array}\right\} \tag{3.22}$$

If we replace x and y by x_n and p respectively in (3.15), then we get

$$\|x_n - Tx_n\| + \|p - Tp\| \leq a_1 \|x_n - p\| \tag{3.23}$$

Now, combining (3.22) and (3.23), we obtain

$$\|Tx_n - Tp\| \leq \frac{1}{2}\left[a_1\|x_n - p\| + \|x_n - Tp\| + \|p - Tx_n\|\right] \tag{3.24}$$

If we replace x and y by x_n and p respectively in (3.16), then we get

$$\|x_n - Tx_n\| + \|p - Tp\| \leq b_1\left[\|x_n - Tp\| + \|p - Tx_n\| + \|x_n - p\|\right] \tag{3.25}$$

Now, combining (3.22) and (3.25), we obtain

$$\|Tx_n - Tp\| \leq \frac{b_1+1}{2}\left[\|x_n - Tp\| + \|p - Tx_n\|\right] + \frac{b_1}{2}\|x_n - p\| \tag{3.26}$$

If we replace x and y by x_n and p respectively in (3.17), then we get

$$\|x_n - Tx_n\| + \|p - Tp\| + \|Tx_n - Tp\| \leq c_1\left[\|x_n - Tp\| + \|p - Tx_n\|\right] \tag{3.27}$$

Now, combining (3.22) and (3.27), we obtain

$$\|Tx_n - Tp\| \leq \frac{c_1+1}{3}\left[\|x_n - Tx_n\| + \|p - Tx_n\|\right] \tag{3.28}$$

If we replace x and y by x_n and p respectively in (3.18), then we get

$$\|Tx_n - Tp\| \leq kmax\{\|x_n - p\|, \|x_n - Tx_n\|, \|p - Tp\|, [\|x_n - Tp\| + \|p - Tx_n\|]/2\} \tag{3.29}$$

Now, combining (3.24), (3.26), (3.28) and (3.29), we obtain

$$\begin{aligned}\|Tx_n - Tp\| \leq max\left\{\frac{a_1}{2}, \frac{b_1}{2}, k\right\}\|x_n - p\| + max\left\{\frac{c_1+1}{3}, k\right\}\|x_n - Tx_n\| \\ + max\left\{\frac{1}{2}, \frac{b_1+1}{2}, \frac{c_1+1}{3}, \frac{k}{2}\right\}\|p - Tx_n\| \\ + max\left\{\frac{1}{2}, \frac{b_1+1}{2}, k\right\} max\{\|p - Tp\|, \|x_n - Tp\|\}.\end{aligned}$$

This satisfies the contractive definition (3.1).

This completes our proof. ♦

In [11] L.A. Khan used the following contractive definitions which are defined on the metric linear spaces for Ishikawa iterative schemes:

If r is a $F-$ norm defined on a metric linear space X , then for some constants $0 \leq a_1 < 2, 0 \leq k < 1$ and for each pair of points $x, y \in X$, we can write

$$r(Tx - Ty) \leq kmax\{r(x - y), r(x - Tx), r(y - Ty), r(x - Ty) + r(y - Tx)\} \tag{3.30}$$

$$r(Tx - Ty) + r(x - Tx) + r(y - Ty) \leq a_1\{r(x - Ty) + r(y - Tx)\} \tag{3.31}$$

Now, we state and prove our corollary 3.3 to extend the above context for multi-step iterative schemes defined by (2.1).

Corollary 3.3. *Let X be a metric linear space and E be a non-empty closed convex subset of X. Let T be a self-map on E satisfying at least one of the contractive definitions (3.30) and (3.31).*

If for some $x_0 \in E$ *, the multi-step iterative schemes defined by (2.1), converges to a point* p*, with* $\lim_{n\to\infty} \alpha_n > 0$ *, then* p *is a fixed point of* T *.*

Proof. By the context of our Theorem 3.1 to proof this corollary, it is sufficient to prove that the contractive definitions (3.1) and (3.2) are obtained from the contractive definition (3.30) and (3.31).

If we replace x and y by x_n and y_n respectively in (3.30), then as in [11, page 59] we get

$$r(Tx_n - Ty_n) \leq k\left[2r(x_n - Ty_n) + r(Tx_n - Ty_n)\right].$$

That is,

$$r(Tx_n - Ty_n) \leq \frac{2k}{1-k} r(x_n - Ty_n) \tag{3.32}$$

If we replace x and y by x_n and y_n respectively in (3.31), then as in [11, page 60] we get

$$3r(Tx_n - Ty_n) \leq (1+a_1)\left[2r(x_n - Ty_n) + r(Ty_n - Tx_n)\right].$$

That is,

$$r(Tx_n - Ty_n) \leq \frac{2(1+a_1)}{2-a_1}\left[r(x_n - Ty_n)\right] \tag{3.33}$$

Now, combining (3.32) and (3.33), we obtain

$$r(Tx_n - Ty_n) \leq max\left\{\frac{2k}{1-k}, \frac{2(1+a_1)}{2-a_1}\right\} r(x_n - Ty_n) \tag{3.34}$$

If we put $y_n = y_n^{(1)}$ in (3.34), then we get

$$r(Tx_n - Ty_n^{(1)}) \leq max\left\{\frac{2k}{1-k}, \frac{2(1+a_1)}{2-a_1}\right\} r(x_n - Ty_n^{(1)}).$$

This satisfies the first inequality of (3.2).

If we put $x_n = y_n^{(1)}$ and $y_n = y_n^{(2)}$ in (3.34), then we get

$$\begin{aligned} r(Ty_n^{(1)} - Ty_n^{(2)}) &\leq max\left\{\frac{2k}{1-k}, \frac{2(1+a_1)}{2-a_1}\right\} r(y_n^{(1)} - Ty_n^{(2)}) \\ &\leq max\left\{\frac{2k}{1-k}, \frac{2(1+a_1)}{2-a_1}\right\}\left[r(y_n^{(1)} - Ty_n^{(1)}) + r(Ty_n^{(1)} - Ty_n^{(2)})\right]. \end{aligned}$$

This satisfies the second inequality of (3.2).

Similarly, if we continue the above process, then we can easily obtain the rest inequalities of (3.2) by contractive definitions (3.30) and (3.31).

Therefore, we can say that the contractive definition (3.2) is satisfied.

Now, we have to show that the contractive definition (3.1) is satisfied by contractive definitions (3.30) and (3.31).

If we replace x and y by x_n and p respectively in (3.30), then we get

$$r(Tx_n - Tp) \leq kmax\{r(x_n - p), r(x_n - Tx_n), r(p - Tp), r(x_n - Tp) + r(p - Tx_n)\}$$
$$\leq kr(x_n - p) + kr(x_n - Tx_n) + kr(p - Tx_n) + kmax\{r(p - Tp), r(x_n - Tp)\} \tag{3.35}$$

If we replace x and y by x_n and p respectively in (3.31), then we get

$$r(Tx_n - Tp) + r(x_n - Tx_n) + r(p - Tp) \leq a_1\{r(x_n - Tp) + r(p - Tx_n)\}$$
$$\Rightarrow 3r(Tx_n - Tp) \leq (1 + a_1)[r(x_n - Tp) + r(p - Tx_n)] \tag{3.36}$$
$$\Rightarrow r(Tx_n - Tp) \leq \frac{(1 + a_1)}{3}[r(x_n - Tp) + r(p - Tx_n)]$$

Now, combining (3.35) and (3.36), we get

$$r(Tx_n - Tp) \leq kr(x_n - p) + kr(x_n - Tx_n) + max\left\{k, \frac{1 + a_1}{3}\right\} r(p - Tx_n)$$
$$+ max\left\{k, \frac{1 + a_1}{3}\right\} max\{r(p - Tp), r(x_n - Tp)\}.$$

This satisfies the contractive definition (3.1).

This completes our proof. ♦

4. Conclusion

Our Theorem 3.1 is an extension of the result of some well known authors in literature. See for instance, the articles of B.E. Rhoades [2-4], S.A. Naimpally and K.L. Singh [10], and L.A. Khan [11]. Here we have replaced the Ishikawa (two-step) iterative schemes by multi-step fixed point iterative schemes defined by (2.1). Our Corollary 3.1, Corollary 3.2, and Corollary 3.3 are extension of the corollaries of B.E. Rhoades [3], because here we replaced Ishikawa (two-step) iterative schemes by multi-step fixed point iterative schemes defined by (2.1). In the preliminary section, we have shown that the multi-step fixed point iterative schemes are the general iterative schemes comparing with other fixed point iterative schemes and we have established our Theorem 3.1 and Corollary 3.1, Corollary 3.2, and Corollary 3.3 for the general contractive definition. Our result is a generalization of the result of M.S. Khan, M. Imdad and

S. Sessa [13], W. R. Mann [6], S. Ishikawa [7], M. A. Noor [8], B.E. Rhoades [2-4], S.A. Naimpally and K.L. Singh [10] and L.A. Khan [11], because from our result it is possible to find all above mention results. So, we can conclude that, our result will play a vital role in the development of fixed point theory as well as fixed point iterative approximation theory.

Conflict of Interests

The authors declare that there is no conflict of interests.

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