Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 4, 520-527
ISSN: 1927-6303

FIXED POINT RESULTS FOR P-1 COMPATIBLE IN FUZZY MENGER SPACE

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Abstract: In this paper we have proved fixed point theorem in Fuzzy Menger Space for P-1 compatible mappings which is introduced by Sevet Kutukcu and Sushil Sharma.

Keywords: fuzzy Menger space; P-1 compatible mappings; common fixed point.
2010 AMS Subject Classification: 47H10 and 54H24.

1. Introduction and Preliminaries

Menger [5] in 1942 introduced the notation of the probabilistic metric space. The probabilistic generalization of metric space appears to be well adopted for the investigation of physical quantities and physiological thresholds. Schweizer and Sklar [7] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [8]. Sessa [9] introduced weakly commuting maps in metric spaces. Jungck [2] enlarged this concept to compatible maps. The notion of compatible mapsin Menger spaces has been introduced by Mishra [6]. Cho [1] et al. and Sharma [10] gave fuzzy version of compatible maps and proved common fixed point theorems for compatible maps in fuzzy metric spaces. So many works have been done in fuzzy and menger space [3], [4] and [12]. Sevet Kutukcu and Sushil Sharma introduce the concept of compatible maps of type (P-1) and type (P-2), show that they are equivalent to compatible maps under certain conditions and prove a common fixed point theorem

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Received January 22, 2015

for such maps in Menger spaces. Rajesh Shrivastav [11] et al. have given the definition of fuzzy probabilistic metric space and proved fixed point theorem for such space.

In this paper we prove fixed point results for fuzzy probabilistic space with compatible P-1.

Definition 1.1.1: A fuzzy probabilistic metric space (FPM space) is an ordered pair (X,F_α) consisting of a nonempty set X and a mapping F_α from XxX into the collections of all fuzzy distribution functions $F_\alpha \in R$ for all $\alpha. \in [0,1]$. For x, y $\in$ X we denote the fuzzy distribution function F_α (x,y) by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(u)$ is the value of $F_{\alpha(x,y)}$ at u in R.

The functions $F_{\alpha(x,y)}$ for all $\alpha. \in [0,1]$ assumed to satisfy the following conditions:

(a) $F_{\alpha(x,y)}(u) = 1 \ \forall\ u > 0$ iff $x = y$,

(b) $F_{\alpha(x,y)}(0) = 0 \ \forall\ x, y$ in X,

(c) $F_{\alpha(x,y)} = F_{\alpha(y,x)} \ \forall\ x, y$ in X,

(d) If $F_{\alpha(x,y)}(u) = 1$ and $F_{\alpha(y,z)}(v) = 1 \Rightarrow F_{\alpha(x,z)}(u+v) = 1 \ \forall\ x, y, z \in X$ and $u, v > 0$.

Definition 1.1.2: A commutative, associative and non-decreasing mapping $t: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm if and only if $t(a,1)= a \ \forall a\in[0,1]$, $t(0,0)=0$ and $t(c,d) \geq t(a,b)$ for $c \geq a$, $d \geq b$.

Definition 1.1.3: A Fuzzy Menger space is a triplet (X,F_α,t), where (X,F_α) is a FPM-space, t is a t-norm and the generalized triangle inequality

$$F_{\alpha(x,z)}(u+v) \geq t\,(F_{\alpha(x,z)}(u), F_{\alpha(y,z)}(v))$$

holds for all x, y, z in X u, v > 0 and $\alpha. \in [0,1]$.

The concept of neighborhoods in Fuzzy Menger space is introduced as

Definition 1.1.4: Let (X,F_α,t) be a Fuzzy Menger space. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0,1)$, then (ε,λ) - neighborhood of x, called $U_x(\varepsilon,\lambda)$, is defined by

$$U_x(\varepsilon,\lambda) = \{y\in X: F_{\alpha(x,y)}(\varepsilon)>(1-\lambda)\}.$$

An (ε,λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon,\lambda): x \in X, \varepsilon > 0, \alpha \in [0,1]$ and $\lambda\in(0,1)\}$ of neighborhood.

Remark: If t is continuous, then Fuzzy Menger space (X,F_α,t) is a Housdroff space in (ε,λ)-topology.

Let (X,F_α,t) be a complete Fuzzy Menger space and $A\subset X$. Then A is called a bounded set if

$$\lim_{u\to\infty} \inf_{x,y\in A} F_{\alpha(x,y)}(u) = 1$$

Definition 1.1.5: A sequence $\{x_n\}$ in (X,F_α,t) is said to be convergent to a point x in X if for every $\varepsilon>0$and $\lambda>0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $x_n \in U_x(\varepsilon,\lambda)$ $\forall n \geq N$ or equivalently $F_\alpha(x_n, x; \varepsilon) > 1-\lambda$ for all $n \geq N$ and $\alpha\in[0,1]$.

Definition 1.1.6: A sequence $\{x_n\}$ in (X,F_α, t) is said to be cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that for all $\alpha\in[0,1]$ $F_\alpha(x_n,x_m; \varepsilon) > 1-\lambda$ $\forall$ n, m $\geq$ N.

Definition 1.1.7: A Fuzzy Menger space (X,F_α,t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all $\alpha\in[0,1]$.

Following lemmas is selected from [8] and [12] respectively in fuzzy menger space.

Lemma 1: Let $\{x_n\}$ be a sequence in a Menger space $(X, F_\alpha,*)$ with continuous t-norm $*$ and $t * t \geq t$. If there exists a constant $k \in (0, 1)$ such that

$$F_{\alpha(x_n,x_{n+1})}(kt) \geq F_{\alpha(x_{n-1},x_n)}(t) \text{ for all } t > 0 \text{ and } n = 1, 2, \ldots,$$

then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2: ([12])**.** Let $(X, F_\alpha, *)$ be a Menger space. If there exists $k \in (0, 1)$ such that

$F_{\alpha(x,y)}(kt) \geq F_{\alpha(x,y)}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

Definition 1.1.8: Self maps A and B of a Menger space $(X, F_\alpha,*)$ are said to be compatible of type (P) if $F_{\alpha(ABx_n,BBx_n)}(t) \to 1$ and $F_{\alpha(BAx_n,AAx_n)}(t) \to 1$ $\forall t > 0$, whenever$\{x_n\}$ is a sequence in X such that $Ax_n,Bx_n \to z$ for some $z \in X$ as $n \to \infty$.

Definition 1.1.9: Self maps A and B of a Menger space (X, F_α, *) are said to be compatible of type (P-1) if $F_{\alpha(ABx_nBBx_n)}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.

2. Main Results

Theorem 1. Let A, P, Q, and S be self maps on a complete Menger space (X, F_α, $*$) with continuous t-norm $*$ and $t * t \geq t$, for all $t \in [0, 1]$, satisfying:

(1.1) $P(X) \subseteq S(X)$, $Q(X) \subseteq A(X)$,

(1.2) there exists a constant $k \in (0, 1)$ such that

$$F_\alpha(Px,Qy)(kt) \geq F_\alpha(Ax,Sy)(t) * F_\alpha(Px,Ax)(t) * F_\alpha(Qy,Sy)(t) * F_\alpha(Px,Sy)(\beta t) * F_\alpha(Qy,Ax)((2-\beta)t)$$

$$F_{\alpha(Px,Qy)}(kt) \geq F_{\alpha(ABx,Sty)}(t) * F_{\alpha(Px,ABx)}(t) * F_{\alpha(Qy,Sty)}(t) * F_{\alpha(Px,Sty)}(\beta t) * F_{\alpha(Qy,ABx)}((2-\beta)t)]$$

$$\forall x, y \in X, \beta \in (0, 2) \text{ and } t > 0,$$

(1.3) either P or A is continuous,

(1.4) the pairs (P,A) and (Q, S) are compatible of type (P-1).

Then A, P, Q and S have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X. By (1.1) there exists $x_1, x_2 \in X$ such that

$$Px_0 = Sx_1 = y_0 \text{ and } Qx_1 = Ax_1 = y_1.$$

Inductively, we can construct sequences$\{x_n\}$ and $\{y_n\}$ in X such that

$$Px_{2n} = Sx_{2n+1} = y_{2n} \text{ and } Qx_{2n+1} = Ax_{2n+2} = y_{2n+1} \text{ for } n = 0, 1, 2, \ldots.$$

Step 1. By taking $x = x_{2n}$, $y = x_{2n+1}$ for all $t > 0$ and $\beta = 1 - q$ with $q \in (0, 1)$ in (1.2), we have

$$F_{\alpha(Px_{2n},Qx_{2n+1})}(kt) = F_{\alpha(y_{2n},y_{2n+1})}(kt)$$

$$\geq F_{\alpha(y_{2n-1},y_{2n})}(t) * F_{\alpha(y_{2n},y_{2n-1})}(t) * F_{\alpha(y_{2n+1},y_{2n})}(t) * F_{\alpha(y_{2n},y_{2n})}((1-q)t) * F_{\alpha(y_{2n+1},y_{2n-1})}((1+q)t)$$

$$\geq F_{\alpha(y_{2n-1},y_{2n})}(t) * F_{\alpha(y_{2n-1},y_{2n})}(t) * F_{\alpha(y_{2n},y_{2n+1})}(t) * 1 * F_{\alpha(y_{2n-1},y_{2n})}(t) * F_{\alpha(y_{2n},y_{2n+1})}(qt)$$

$$\geq F_{\alpha(y_{2n-1},y_{2n})}(t) * F_{\alpha(y_{2n},y_{2n+1})}(t) * F_{\alpha(y_{2n},y_{2n+1})}(qt).$$

Since t-norm is continuous, letting $q \to 1$, we have

$$\geq F_{\alpha(y_{2n},y_{2n+1})}(kt) * F_{\alpha(y_{2n-1},y_{2n})}(t) * F_{\alpha(y_{2n},y_{2n+1})}(t).$$

Similarly, we also have

$$F_{\alpha(y_{2n+1},y_{2n+2})}(kt) * F_{\alpha(y_{2n},y_{2n+1})}(t) * F_{\alpha(y_{2n+1},y_{2n+2})}(t).$$

In general, for all n even or odd, we have

$F_{\alpha(y_n,y_{n+1})}(kt) * F_{\alpha(y_{n-1},y_n)}(t) * F_{\alpha(y_n,y_{n+1})}(t).$

Consequently, for p = 1, 2, . . ., it follows that,

$F_{\alpha(y_n,y_{n+1})}(kt) * F_{\alpha(y_{n-1},y_n)}(t) * F_{\alpha(y_n,y_{n+1})}(\frac{t}{k^p}).$

By nothing that $F_{\alpha(y_n,y_{n+1})}(\frac{t}{k^p}) \to 1 \quad as \quad p \to \infty$

we have

$F_{\alpha(y_n,y_{n+1})}(kt) \geq F_{\alpha(y_{n-1},y_n)}(t)$

for $k \in (0, 1)$ all $n \in N$ and $t > 0$. Hence, by Lemma1, $\{y_n\}$ is a Cauchy sequence in X. Since (X, F, ∗) is complete, it converges to a point z in X. Also its subsequences converge as follows: $\{Px_{2n}\} \to z$, $\{Ax_{2n}\} \to z$, $\{Qx_{2n+1}\} \to z$ and $\{Sx_{2n+1}\} \to z$.

Case I. A is continuous and (P,A) and (Q, S) are compatible of type(P-1).

Since A is continuous, $AAx_{2n} \to Az$ and $APx_{2n} \to Az$. Since (P,A) is compatible of type (P-1), $PPx_{2n} \to Az$.

Step 2. By taking $x = Px_{2n}$, $y = x_{2n+1}$ with β= 1 in (1.2), we have

$F_{\alpha(PPx_{2n},Qx_{2n+1})}(kt) \geq F_{\alpha(ABPx_{2n},Sx_{2n+1})}(t) * F_{\alpha(PPx_{2n},APx_{2n})}(t) * F_{\alpha(Qx_{2n+1},Sx_{2n+1})}(t) * F_{\alpha(PPx_{2n},Sx_{2n+1})}(t) * F_{\alpha(Qx_{2n+1},APx_{2n})}(t).$

This implies that, as $n \to \infty$

$F_{\alpha(z,Az)}(kt) \geq F_{\alpha(z,Az)}(t) * F_{\alpha(Az,Az)}(t) * F_{\alpha(z,z)}(t) * F_{\alpha(z,Az)}(t) * F_{\alpha(z,Az)}(t)$

$= F_{\alpha(z,Az})(t) * 1 * 1 * F_{\alpha(z,Az)}(t) * F_{\alpha(z,Az)}(t) * F_{\alpha(z,Az)}(t).$

Thus, by Lemma 2, it follows that z = Az.

Step 3. By taking x = z, $y = x_{2n+1}$ with β= 1 in (1.2), we have

$F_{\alpha(Pz,Qx_{2n+1})}(kt) \geq F_{\alpha(Az,Sx_{2n+1})}(t) * F_{\alpha(Pz,Az)}(t) * F_{\alpha(Qx_{2n+1},Sx_{2n+1})}(t) * F_{\alpha(Pz,Sx_{2n+1})}(t) * F_{\alpha(Qx_{2n+1},Az)}(t).$

This implies that, as $n \to \infty$

$F_{\alpha(z,Pz)}(kt) \geq F_{\alpha(z,z)}(t) * F_{\alpha(z,Pz)}(t) * F_{\alpha(z,z)}(t) * F_{\alpha(z,Pz)}(t) * F_{\alpha(z,z)}(t)$

$= 1 * F_{\alpha(z,Pz)}(t) * 1 * F_{\alpha(z,Pz)}(t) * 1$

$\geq F\alpha(z,Pz)(t).$

Thu by Lemma 2, it follows that z = Pz. Therefore, z = Az = Pz.

Step 4. Since $P(X) \subset S(X)$, there exists $w \in X$ such that $z = Pz = Sw$. By taking $x = x_{2n}$, $y = w$ with $\beta = 1$ in (1.2), we have

$$F_{\alpha(Px_{2n},Qw)}(kt) \geq F_{\alpha(Ax_{2n},Sw)}(t) * F_{\alpha(Px_{2n},Ax_{2n})}(t) * F_{\alpha(Qw,Sw)}(t) * F_{\alpha(Px_{2n},Sw)}(t) * F_{\alpha(Qw,Ax_{2n})}(t)$$

which implies that, as $n \to \infty$

$$F_{\alpha(z,Qw)}(kt) \geq F_{\alpha(z,z)}(t) * F_{\alpha(z,z)}(t) * F_{\alpha(z,Qw)}(t) * F_{\alpha(z,z)}(t) * F_{\alpha(z,Qw)}(t)$$

$$= 1 * 1 * F_{\alpha(z,Qw)}(t) * 1 * F_{\alpha(z,Qw)}(t)$$

$$\geq F_{\alpha(z,Qw)}(t).$$

Thus, by Lemma 2, we have $z = Qw$. Hence, $Sw = z = Qw$. Since (Q, S) is compatible of type (P-1), we have $QSw = SSw$. Thus, $Sz = Qz$.

Step 5. By taking $x = x_{2n}$, $y = z$ with $\beta = 1$ in (1.2) and using Step 4, we have

$$F_{\alpha(Px_{2n},Qz)}(kt) \geq F_{\alpha(Ax_{2n},Sz)}(t) * F_{\alpha(Px_{2n},Ax_{2n})}(t) * F_{\alpha(Qz,Sz)}(t) * F_{\alpha(Px_{2n},Sz)}(t) * F_{\alpha(Qz,Ax_{2n})}(t)$$

which implies that, as $n \to \infty$

$$F_{\alpha(z,Qz)}(kt) \geq F_{\alpha(z,Qz)}(t) * F_{\alpha(z,z)}(t) * F_{\alpha(Qz,Qz)}(t) * F_{\alpha(z,Qz)}(t) * F_{\alpha(z,Qz)}(t)$$

$$= F_{\alpha(z,Qz)}(t) * 1 * 1 * F_{\alpha(z,Qz)}(t) * 1 * F_{\alpha(z,Qz)}(t)$$

$$\geq F_{\alpha(z,Qz)}(t).$$

Thus, by Lemma 2, we have $z = Qz$. Since $Sz = Qz$, we have $z = Sz$.

Therefore, $z = Az = Pz = Qz = Sz$.

Case II. P is continuous, and (P,A) and (Q, S) are compatible of type (P-1). Since P is continuous, $PPx_{2n} \to Pz$ and $PAx_{2n} \to Pz$. Since (P,A) is compatible of type (P-1), $AAx_{2n} \to Pz$.

Step 6. By taking $x = Ax_{2n}$, $y = x_{2n+1}$ with $\beta = 1$ in (1.2), we have

$$F_{\alpha(PAx_{2n},Qx_{2n+1})}(kt) \geq F_{\alpha(AAx_{2n},Sx_{2n+1})}(t) * F_{\alpha(PAx_{2n},AAx_{2n})}(t) * F_{\alpha(Qx_{2n+1},Sx_{2n+1})}(t) * F_{\alpha(PAx_{2n},Sx_{2n+1})}(t) * F_{\alpha(Qx_{2n+1},AAx_{2n})}(t).$$

This implies that, as $n \to \infty$

$$F_{\alpha(z,Pz)}(kt) \geq F_{\alpha(z,Pz)}(t) * F_{\alpha(Pz,Pz)}(t) * F_{\alpha(z,z)}(t) * F_{\alpha(z,Pz)}(t) * F_{\alpha(z,Pz)}(t)$$

$$= F_{\alpha(z,Pz)}(t) * 1 * 1 * F_{\alpha(z,Pz)}(t) * F_{\alpha(z,Pz)}(t)$$

$$\geq F_{\alpha(z,Pz)}(t).$$

Thus, by Lemma 2, it follows that $z = Pz$. Now using Step 4and 5, we have $z = Qz = Sz$.

Step 9. Since $Q(X) \subset A(X)$, there exists $w \in X$ such that $z = Qz = Aw$. By taking $x = w$, $y = x_{2n+1}$ with $\beta = 1$ in (1.2), we have

$$F_{\alpha(Pw,Qx_{2n+1})}(kt) \geq F_{\alpha(Aw,Sx_{2n+1})}(t) * F_{\alpha(Pw,Aw)}(t) * F_{\alpha(Qx_{2n+1},Sx_{2n+1})}(t) * F_{\alpha(Pw,Sx_{2n+1})}(t) * F_{\alpha(Qx_{2n+1},Aw)}(t)$$

which implies that, as $n \to \infty$

$$F_{\alpha(z,Pw)}(kt) \geq F_{\alpha(z,z)}(t) * F_{\alpha(z,Pw)}(t) * F_{\alpha(z,z)}(t) * F_{\alpha(z,Pw)}(t) * F_{\alpha(z,z)}(t)$$
$$= 1 * F_{\alpha(z,Pw)}(t) * 1 * F_{\alpha(z,Pw)}(t) * 1$$
$$\geq F_{\alpha(z,Pw)}(t).$$

Thus, by Lemma 2, we have z = Pw. Since z = Qz = Aw, Pw = Aw.

Since (P,A) is compatible of type (P-1), we have Pz = Az. Thus, z = Az = Pz. Hence, z is the common fixed point of the four maps.

Step 10. For uniqueness, let v ($v \neq z$) be another common fixed point of A, P, Q, and S . Taking x = z, y = v with $\beta = 1$ in (1.2), we have

$$F_{\alpha(Pz,Qv)}(kt) \geq F_{\alpha(Az,Sv)}(t) * F_{\alpha(Pz,Az)}(t) * F_{\alpha(Qv,Sv)}(t) * F_{\alpha(Pz,Sv)}(\beta t) * F_{\alpha(Qv,Az)}((2-\beta)t)$$

which implies that

$$F_{\alpha(z,v)}(kt) \geq F_{\alpha(z,v)}(t) * F_{\alpha(z,z)}(t) * F_{\alpha(v,v)}(t) * F_{\alpha(z,v)}(t) * F_{\alpha(v,z)}(t)$$
$$= F_{\alpha(z,v)}(t) * 1 * 1 * F_{\alpha(z,v)}(t) * F_{\alpha(z,v)}(t)$$
$$\geq F_{\alpha(z,v)}(t).$$

Thus, by Lemma 2, we have z = v.

This completes the proof of the theorem.

If we take A = S = IX (the identity map on X) in Theorem 1, we have the following:

Corollary. Let P and Q be self maps on a complete Fuzzy Menger space $(X,F_\alpha,*)$ with continuous t-norm * and $t*t \geq t$ for all $t \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that

$$F_{\alpha(Px,Qy)}(kt) \geq F_{\alpha(x,y)}(t) * F_{\alpha(x,Px)}(t) * F_{\alpha(y,Qy)}(t) * F_{\alpha(y,Px)}(\beta t) * F_{\alpha(x,Qy)}((2-\beta)t)$$

for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$, then P and Q have a unique common fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Y. J. Cho, H. K. Pathak, S. M. Kang, and J. S. Jung, Common fixed points of compatible maps of type (β) on fuzzy metric spaces, Fuzzy Sets and Systems 93 (1998), 99-111.

[2] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Sci. 9(1986), 771-779.

[3] S.Kutukcu, S. Sharma and H.Tokgoz, A Fixed Point Theorem in Fuzzy metric Spaces, Int. Journal of Math. Analysis, 1 (2007), no. 18, 861– 872.

[4] S. Kutukcu, A. Tuna, and A. T. Yakut, Generalized contraction mapping principle in intuitionistic Menger spaces and an application to differential equations, Appl. Math.Mech. 28 (2007), no. 6, 779-809.

[5] K. Menger, Statistical metric, Proc. Nat. Acad. 28 (1942), 535-537.

[6] S. N. Mishra, Common fixed points of compatible mappings in PM-spaces, Math. Japon.36 (1991), 283-289.

[7] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North-Holland, Amsterdam,1983.

[8] V. M. Sehgal and A. T. Bharucha-Reid, Fixed point of contraction mapping on PM spaces, Math. Systems Theory 6 (1972), 97-100.

[9] S. Sessa, On weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. Beograd 32 (1982), 149-153.

[10] S. Sharma, Common fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 127 (2002), 345-352.

[11] R.Shrivastav, V.Patel and V.B.Dhagat, Fixed point theorem in fuzzy menger spaces satisfying occasionally weakly compatible mappings, Int. J. of Math. Sci. & Engg. Appls. , Vol.6 No.VI, 2012, 243-250.

[12] B. Singh and S. Jain, A fixed point theorem in Menger space through weak compatibility,J. Math. Anal. Appl. 301 (2005), 439-448.