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STRONG CONVERGENCE THEOREM FOR MONOTONE OPERATORS AND STRICT PSEUDO-NONSPREADING MAPPING

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Abstract. In this paper, based on the recent results of Osilike *et al.* [9] and motivated by the results of Liu *et al.* [10] and Takahashi et al. [13], we introduce an iterative sequence and prove that the sequence converges strongly to a common element of the set of fixed points of strict pseudo-non spreading mapping, T and the set of zeros of sum of an $\alpha-$inverse strongly monotone mapping A and a maximal monotone operator B in a real Hilbert space. Our results improve and generalize many recent important results.

Keywords: monotone operators; strict-pseudo nonspreading mappings.

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1. Introduction

Throughout this paper, we assume that H is a real Hilbert space, C is a nonempty subset of H. We denote by $x_n \rightharpoonup x$ and $x_n \to x$ weak and strong convergence of a sequence $\{x_n\}$, respectively and by $F(T)$ the set of fixed points of a mapping $T : C \to C$. $B^{-1}(0)$ will stand for the zero point of a mapping B.

Let $T : C \to C$ be a mapping. T is said to be nonexpansive if

$$(1) \qquad ||Tx - Ty|| \leq ||x - y|| \ \forall \ x, y \in C.$$

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The mapping T is said to be firmly nonexpansive if

(2) $$||Tx-Ty||^2 \leq \langle x-y, Tx-Ty\rangle \ \forall\, x,y \in C.$$

(see e.g., Browder [7], Goebel and Kirk [19]). It is said to be quasi-nonexpansive if $F(T)$ is nonempty and

(3) $$||Tx-p|| \leq ||x-p|| \ \forall\, x \in C, p \in F(T).$$

T is called nonspreading if

(4) $$2||Tx-Ty||^2 \leq ||Tx-y||^2 + ||Ty-x||^2 \ \forall\, x,y \in C$$

It can be shown, (see e.g. Iemoto and Takahashi [12]) that T is nonspreading if and only

$$||Tx-Ty||^2 \leq ||x-y||^2 + 2\langle x-Tx, y-Ty\rangle \ \forall\, x,y \in C$$

T is said to be $k-$strictly pseudo-nonspreading (see e.g. [9]), if there exists a constant $k \in [0,1)$ such that

(5) $$||Tx-Ty||^2 \leq ||x-y||^2 + k||x-Tx-(y-Ty)||^2 + 2\langle x-Tx, y-Ty\rangle \ \forall\, x,y \in C.$$

It is shown in [9] that the class of k-strictly pseudo-nonspreading mappings is more general than the class of nonspreading mappings.

Let C be a nonempty closed convex subset of H.
A set-valued mapping $A : D(A) \subset H \to H$ is said to be monotone if for any $x,y \in D(A)$ and $x^* \in Ax, y^* \in Ay$, the following holds;

(6) $$\langle x-y, x^*-y^*\rangle \geq 0.$$

A monotone operator A on H is said to be maximal if A has no monotone extension, that is, its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator A on H and $r > 0$, the single-valued operator $J_r = (I+rA)^{-1} : 2^H \to D(A)$, is called the resolvent of A. It is known (see for instance [13]), that J_r is firmly nonexpansive, hence, it is nonexpansive. For a constant $\alpha > 0$, a mapping $A : C \to H$ is said to be $\alpha-$ inverse strongly monotone if for all $x, y \in C$,

$$\langle x-y, Ax-Ay\rangle \geq \alpha||Ax-Ay||^2. \tag{7}$$

For solving the problem of approximating fixed points of nonexpansive mappings, Mann [11] in 1953 introduced the following iteration process:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{8}$$

where the initial guess $x_1 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $(0,1)$. It is known that under appropriate conditions, the sequence $\{x_n\}$ converges weakly to a fixed point of T. However, even in a Hilbert space, Mann iteration may fail to converge strongly, see for instance [2].

Some attempts to construct iteration method guaranteeing strong convergence to fixed points of nonexpansive mappings have been made. For example, Halpern [3] proposed the following so-called Halpern iteration:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n \tag{9}$$

where $u, x_1 \in C$ are arbitrary and $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying appropriate conditions. Halpern proved that the sequence $\{x_n\}$ generated by (9) converges strongly to a fixed point of T, where T is nonexpansive.

Finding a point $x^* \in F(T) \cap (A+B)^{-1}(0)$ where T is some nonlinear operator, and A, B are monotone operators is of interest in applications and have been studied extensively by many authors (see for instance, [8, 13, 14, 15, 18] and the references therein).

Recently, in the case where $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an $\alpha-$ inverse strongly monotone mapping, and $B \subset H \times H$ is a maximal monotone operator, Takahashi *et al.* [13] proved a strong convergence theorem for finding a point of $F(T) \cap (A+B)^{-1}(0)$, where $(A+B)^{-1}(0)$ is the set of zero points of $(A+B)$.

More recently, Liu *et al.* [10] extended the result of Takahashi *et al.* [13] to the case when T is a nonspreading mapping. In fact, they proved the following result.

Theorem 1.1 (Liu *et al.* [10]) Let C be a nonempty closed convex subset of a real Hilbert space $H, A : C \to H$ be $\alpha-$ inverse strongly monotone mapping and let $B : D(B) \subseteq C \to 2^H$ be maximal monotone. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$, and let $T : C \to C$ be a nonspreading mapping. Assume that $F := F(T) \cap (A+B)^{-1}(0) \neq \emptyset$. Define

$$
\begin{aligned}
x_1 &= x \in C, \text{arbitrarily}, \\
z_n &= J^B_{\lambda_n}(I - \lambda_n A)x_n \\
y_n &= \sum_{i=1}^{n-1} T^i z_n \\
x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n
\end{aligned}
\tag{10}
$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions;
(i) $\alpha_n \to 0,\ n \to \infty$, (ii) $\sum_{n=1}^{\infty} \alpha_n = +\infty$, and there exist $a, b \in \mathbb{R}$ with (iii) $0 < a \leq \lambda_n \leq b < 2\alpha\ \forall\ n \in \mathbb{N}$. Then the sequence $\{x_n\}$ constructed by algorithm (10) converges strongly to $P_F u$, where P is the metric projection of H onto F.

In 2010, Kurokawa and Takahashi [4] obtained a weak mean ergodic theorem of Baillon's type [5] for nonspreading mappings in Hilbert spaces. They further proved a strong convergence theorem somewhat related to Halpern's type for this class of mappings using the idea of mean convergence in Hilbert spaces.

In 2011, Osilike *et al.* [9] first introduced the concept of $k-$strictly pseudo nonspreading mapping and proved a weak mean convergence theorem of Baillon's type similar to the ones obtained in [4]. Furthermore, using the idea of mean convergence,a strong-convergence theorem similar to the one obtained in [4] was also proved which extends and improves the main theorems of [4] and an affirmative answer given to an open problem posed by Kurokawa and Takahashi [4] for the case where the mapping T is averaged.

Motivated by the results of Takahashi *et al.* [13], Liu *et al.* [10], Osilike *et al.* [9], Kurokawa and Takahashi [4], we introduce a new algorithm and prove strong convergence of the sequence of the algorithm to a common element of the set of fixed points of $k-$strictly pseudo nonspreading mapping and the set of zero points of sum of $\alpha-$inverse strongly monotone operator A and a maximal operator B in a real Hilbert space.

Our results improve and generalize those of Osilike *et al.* [9], Takahashi *et al.* [13], Liu *et al.* [10] and a host of other recent important results.

2. Preliminaries

In the sequel, we shall make use of the following Lemmas. First, we give the following definition.

Definition 2.1 Let X be a normed linear space and K a nonempty subset of X. A mapping $T: K \to K$ is said to be demiclosed at $y \in K$ if for any sequence $\{x_n\} \subset K$ which converges weakly to $x \in K$, strong convergence of the sequence $\{Tx_n\}$ to y in K implies that $Tx = y$.

Lemma 2.1 [1] Let H be a real Hilbert space. Then the following holds:
For all $x, y \in H$ $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle$.

Lemma 2.2 [16] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \; n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\sigma_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ satisfy the conditions:
(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$,
(iii) $\gamma_n \geq 0 (n \geq 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.3 [19] Assume that T is a nonexpansive self mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $(I-T)$ is demiclosed at zero.

Lemma 2.4 [9] Let H be a real Hilbert space, C be a nonempty and closed convex subset of H, and $T: C \to C$ be a $k-$strictly pseudo-nonspreading mapping.
(i) If $F(T) \neq \emptyset$, then $F(T)$ is closed and convex;
(ii) $I-T$ is demiclosed at zero.

Lemma 2.5 [17] Let C be a nonempty closed convex subset of H and $T: C \to C$ be a $k-$ strictly pseudo-nonspreading mapping with $F(T) \neq \emptyset$. Let $T_\beta = \beta I + (1-\beta)T, \beta \in [k,1)$. Then the following conclusions hold:

(i) $F(T) = F(T_\beta)$;

(ii) $I - T_\beta$ is demiclosed at zero;

(iii) $||T_\beta x - T_\beta y||^2 \leq ||x-y||^2 + \frac{2}{(1-\beta)}\langle x - T_\beta x, y - T_\beta y\rangle$;

(iv) T_β is a quasi-nonexpansive mapping.

Lemma 2.6 [8] Let $A : C \to H$ be an $\alpha-$inverse strongly monotone mapping, and let B be a maximal monotone operator on H with $D(B) \subset C$. Then, for any $\sigma > 0$, the following holds; $(A+B)^{-1}(0) = F(J^B_\sigma(I - \sigma A))$.

Lemma 2.7 [14] Let B be a maximal monotone operator on H. Then, for any $s,t \in \mathbb{R}$ with $s,t > 0$. and for any $x \in H$, the following hold:

(i) $||J^B_s x - J^B_t x|| \leq \frac{|s-t|}{s}||x - J^B_s sx||$.

(ii) $F(J^B_s) = B^{-1}(0)$.

Lemma 2.8 [6] Let $A : C \to H$ be an α-inverse strongly monotone mapping. Then, for any $\sigma \in (0, 2\alpha], (I - \sigma A)$ is nonexpansive.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of H, $A : C \to H$ be $\alpha-$inverse strongly monotone, and $B : D(B) \subseteq C \to 2^H$ be maximal monotone. Let $J_\lambda(I + \lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$, $T : C \to C$ be a $k-$ strictly pseudo nonspreading mapping. Suppose that $\Omega := F(T) \cap (A+B)^{-1}(0) \neq \emptyset$.*

For arbitrary $x_1, u \in C$, we define

$$\begin{aligned} z_n &= J^B_{\sigma_n}(I - \sigma_n A)x_n, \ \forall\, n \geq 1, \\ y_n &= \sum_{i=1}^{n-1} T^i_\beta z_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n \end{aligned} \tag{11}$$

where $T_\beta := \beta I + (1-\beta)T$, $\beta \in [k,1)$, $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions;

(i) $\alpha_n \to 0$, $n \to \infty$, (ii) $\sum_{n=1}^\infty \alpha_n = +\infty$, $\{\sigma_n\}$ is a sequence in $(0,\infty)$ and there exist $a,b \in \mathbb{R}$

with (iii) $0 < a \leq \sigma_n \leq b < 2\alpha \; \forall \; n \in \mathbb{N}$. *Then the sequence* $\{x_n\}$ *constructed by algorithm* (11) *converges strongly to* $P_\Omega u$*, where* P *is the metric projection of* H *onto* Ω.

Proof. We divide the proof into several steps.

STEP I: $\{x_n\}$ is bounded. Observe that by Lemma 2.8, $(I - \sigma_n A)$ is nonexpansive. Also $J^B_{\sigma_n}$ is nonexpansive. Let $p \in \Omega$, then

$$\begin{aligned} ||z_n - p|| &= ||J^B_{\sigma_n}(I - \sigma_n A)x_n - J^B_{\sigma_n}(I - \sigma_n A)p|| \\ &\leq ||x_n - p||. \end{aligned} \tag{12}$$

Since T is $k-$ strictly pseudo nonspreading, then using Lemma 2.5 (iii), we obtain

$$\begin{aligned} ||T_\beta z_n - p||^2 &= ||T_\beta z_n - T_\beta p||^2 \\ &\leq ||z_n - p||^2 + \frac{2}{(1-\beta)} \langle z_n - T_\beta z_n, \; p - T_\beta p \rangle \\ &= ||z_n - p||^2 \end{aligned}$$

This implies that $||T_\beta z_n - p|| \leq ||z_n - p||$. Assume that $||T^r_\beta z_n - p|| \leq ||z_n - p||$ for some $r \geq 1$. Then

$$\begin{aligned} ||T^{r+1}_\beta z_n - p||^2 &= ||T_\beta(T^r_\beta z_n) - p||^2 \\ &= ||T_\beta(T^r_\beta z_n) - T_\beta p||^2 \\ &\leq ||T^r_\beta z_n - p||^2 + \frac{2}{(1-\beta)} \langle T^r_\beta z_n - T^{r+1}_\beta z_n, \; p - T_\beta p \rangle \\ &= ||T^r_\beta z_n - p||^2 \\ &\leq ||z_n - p||^2. \end{aligned}$$

Thus, $||T^{r+1}_\beta z_n - p|| \leq ||z_n - p||$ and so, by induction, we have that

$$||T^i_\beta z_n - p|| \leq ||z_n - p|| \; \forall \; i, n \in \mathbb{N}.$$

Hence,

$$(13) \qquad \begin{aligned} ||y_n - p|| &= \Big|\Big|\frac{1}{n}\sum_{i=0}^{n-1} T_\beta^i z_n - p\Big|\Big| \\ &\leq \frac{1}{n}\sum_{i=0}^{n-1} \Big|\Big|T_\beta^i z_n - p\Big|\Big| \\ &\leq \frac{1}{n}\sum_{i=0}^{n-1} ||z_n - p|| \\ &= ||z_n - p||. \end{aligned}$$

Applying (12) in (13) gives $||y_n - p|| \leq ||x_n - p||$. From (11), we get

$$\begin{aligned} ||x_{n+1} - p|| &\leq \alpha_n ||u - p|| + (1 - \alpha_n)||y_n - p|| \\ &\leq \alpha_n ||u - p|| + (1 - \alpha_n)||z_n - p|| \\ &\leq \alpha_n ||u - p|| + (1 - \alpha_n)||x_n - p||. \end{aligned}$$

Hence by induction, we obtain

$$||x_n - p|| \leq \max\{||u - p||,\ ||x_1 - p||\}\ \forall\ n \geq 1.$$

Therefore, $\{x_n\}_{n\geq 1}$ is bounded and so $\{y_n\}$, $\{z_n\}$ and $\{T_\beta^n z_n\}$ are all bounded.
Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} ||x_{n_j} - p||$ exists.
Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$, say, of $\{x_{n_j}\}$ which we still call $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup w \in C$ as $j \to \infty$.

We prove that $w \in \Omega$. We first prove that $w \in F(T)$.
Since $||x_{n+1} - y_n|| = \alpha_n ||u - y_n||$, replacing n by n_j , we have

$$||x_{n_j+1} - y_{n_j}|| = \alpha_{n_j} ||u - y_{n_j}||.$$

This together with condition (i) and the fact that $\{y_n\}$ is bounded, yield

$$||x_{n_j+1} - y_{n_j}|| \to 0 \text{ as } j \to \infty.$$

Thus, $y_{n_j} \to w$ as $j \to \infty$. Since T is strictly pseudo nonspreading, then for all $\xi \in C$ and for any $k = 0, 1, 2, \cdots, n-1$, we have using Lemma 2.5 (iii),

$$
\begin{aligned}
& ||T_\beta^{k+1} z_n - T_\beta \xi||^2 = ||T_\beta \big(T_\beta^k\big) z_n - T_\beta \xi||^2 \\
\leq\ & ||T_\beta^k z_n - \xi||^2 + \frac{2}{(1-\beta)} \big\langle T_\beta^k z_n - T_\beta^{k+1} z_n, \xi - T_\beta \xi \big\rangle \\
=\ & ||T_\beta^k z_n - T_\beta \xi + T_\beta \xi - \xi||^2 + \frac{2}{(1-\beta)} \big\langle T_\beta^k z_n - T_\beta^{k+1} z_n, \xi - T_\beta \xi \big\rangle \\
=\ & ||T_\beta^k z_n - T_\beta \xi||^2 + ||T_\beta \xi - \xi||^2 + 2\big\langle T_\beta^k z_n - T_\beta \xi, T_\beta \xi - \xi \big\rangle \\
+\ & \frac{2}{(1-\beta)} \big\langle T_\beta^k z_n - T_\beta^{k+1} z_n, \xi - T_\beta \xi \big\rangle. \qquad (14)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
||T_\beta^{k+1} z_n - T_\beta \xi||^2 - ||T_\beta^k z_n - T_\beta \xi||^2 \ \leq\ & ||T_\beta \xi - \xi||^2 + 2\big\langle T_\beta^k z_n - T_\beta \xi, T_\beta \xi - \xi \big\rangle \\
+\ & \frac{2}{(1-\beta)} \big\langle T_\beta^k z_n - T_\beta^{k+1} z_n, \xi - T_\beta \xi \big\rangle. \qquad (15)
\end{aligned}
$$

Summing (15) from $k = 0$ to $n-1$ and dividing by n, we have

$$
\begin{aligned}
\frac{1}{n}\big(||T_\beta^n x_n - T_\beta \xi||^2 - ||z_n - T_\beta \xi||^2\big) \ \leq\ & ||T_\beta \xi - \xi||^2 + 2\big\langle y_n - T_\beta \xi, T_\beta \xi - \xi \big\rangle \\
+\ & \frac{2}{n(1-\beta)} \big\langle z_n - T_\beta^n z_n, \xi - T_\beta \xi \big\rangle. \qquad (16)
\end{aligned}
$$

Replacing n with n_j in (16), we obtain

$$
\begin{aligned}
\frac{1}{n_j}\big(||T_\beta^{n_j} z_{n_j} - T_\beta \xi||^2 - ||z_{n_j} - T_\beta \xi||^2\big) \ \leq\ & ||T_\beta \xi - \xi||^2 + 2\big\langle y_{n_j} - T_\beta \xi, T_\beta \xi - \xi \big\rangle \\
+\ & \frac{2}{n_j(1-\beta)} \big\langle z_{n_j} - T_\beta^{n_j} z_{n_j}, \xi - T_\beta \xi \big\rangle. \qquad (17)
\end{aligned}
$$

Letting $j \to \infty$ in (17) and given the fact that $\{z_n\}$ and $\{T_\beta^n z_n\}$ are bounded, we obtain

$$0 \leq ||T_\beta \xi - \xi||^2 + 2\big\langle w - T_\beta \xi, T_\beta \xi - \xi \big\rangle.$$

In particular for $\xi = w$, we have

$$
\begin{aligned}
0 \ \leq\ & ||T_\beta w - w||^2 + 2\big\langle w - T_\beta w, T_\beta w - w \big\rangle \\
=\ & ||T_\beta w - w||^2 - 2||T_\beta w - w||^2. \qquad (18)
\end{aligned}
$$

This implies that $w = T_\beta w$. That is $w \in F(T_\beta)$, by Lemma 2.5 (i), we have $w \in F(T)$.

STEP II: We prove that $||Ax_{n_j} - Ap|| \to 0$ as $j \to \infty$. From (11) and using the fact that A is $\alpha-$inverse strongly monotone and convexity of $||.||^2$, we have

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||\alpha_n u + (1-\alpha_n)y_n - p||^2 \\
&\leq \alpha_n||u-p||^2 + (1-\alpha_n)||y_n-p||^2 \\
&\leq \alpha_n||u-p||^2 + (1-\alpha_n)||y_n-p||^2 \\
&\leq \alpha_n||u-p||^2 + (1-\alpha_n)||z_n-p||^2 \\
&= \alpha_n||u-p||^2 + (1-\alpha_n)||J^B_{\sigma_n}(I-\sigma_n A)x_n \\
&- J^B_{\sigma_n}(I-\sigma_n A)p||^2 \\
&\leq \alpha_n||u-p||^2 + (1-\alpha_n)||x_n - p - \sigma_n(Ax_n - Ap)||^2 \\
&= \alpha_n||u-p||^2 + (1-\alpha_n)[||x-p||^2 \\
&- 2\sigma_n\langle x_n - p,\, Ax_n - Ap\rangle + \sigma_n^2||Ax_n - Ap||^2] \\
&= \alpha_n||u-p||^2 + (1-\alpha_n)[||x_n-p||^2 \\
&- \sigma_n(2\alpha - \sigma_n)||Ax_n - Ap||^2] \\
&\leq \alpha_n||u-p||^2 + ||x_n-p||^2 \\
&- (1-\alpha_n)\sigma_n(2\alpha-\sigma_n)||Ax_n - Ap||^2
\end{aligned}$$

So,

$$\begin{aligned}
(1-\alpha_n)\sigma_n(2\alpha-\sigma_n)||Ax_n - Ap||^2 &\leq \alpha_n||u-p||^2 \\
&+ ||x_n-p||^2 - ||x_{n+1}-p||^2
\end{aligned}$$

Passing to subsequence and using condition (iii), we obtain

$$\begin{aligned}
(1-\alpha_{n_j})a(2\alpha-b)||Ax_{n_j} - Ap||^2 &\leq \alpha_{n_j}||u-p||^2 \\
&+ ||x_{n_j}+p|| - ||x_{n_j+1}-p||. \qquad (19)
\end{aligned}$$

Applying condition (i) and the fact that $\lim_{j\to\infty}||x_{n_j}-p||$ exists, we obtain the desired conclusion. That is, $||Ax_{n_j}-Ap||\to 0$ as $j\to\infty$.

STEP III: We prove that $||z_{n_j}-x_{n_j}||\to 0,\ j\to\infty$. Since $J^B_{\sigma_n}$ is firmly nonexpansive, we obtain,

$$\begin{aligned}
||z_n-p||^2 &= ||J^B_{\sigma_n}(I-\sigma_n)x_n - J^B_{\sigma_n}(I-\sigma_nA)p||^2\\
&\le \langle z_n-p,\ (I-\sigma_nA)x_n-(I-\sigma_nA)p\rangle\\
&= \frac{1}{2}\{||z_n-p||^2+||(I-\sigma_nA)x_n-(I-\sigma_nA)p||^2\\
&- ||z_n-p-[(I-\sigma_nA)x_n-(I-\sigma_nA)p]||^2\}\\
&\le \frac{1}{2}\{||z_n-p||^2+||x_n-p||^2-||z_n-p-(I-\sigma_nA)x_n\\
&+ (I-\sigma_nA)p||^2\}\\
&= \frac{1}{2}\{||z_n-p||^2+||x_n-p||^2-||z_n-x_n||^2\\
&- 2\sigma_n\langle z_n-x_n,\ Ax_n-Ap\rangle-\sigma_n^2||Ax_n-Ap||^2\} \qquad (20)
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{1}{2}||z_n-p||^2 &\le \frac{1}{2}\{||x_n-p||^2-||z_n-x_n||^2\\
&- 2\sigma_n\langle z_n-x_n,Ax_n-Ap\rangle-\sigma_n^2||Ax_n-Ap||^2\},\ \text{so,}\\
||z_n-p||^2 &\le ||x_n-p||^2-||z_n-x_n||^2\\
&- 2\sigma_n\langle z_n-x_n,Ax_n-Ap\rangle-\sigma_n^2||Ax_n-Ap||^2
\end{aligned}$$

From (11),

$$
\begin{aligned}
||x_{n+1}-p||^2 &\leq \alpha_n||u-p||^2+(1-\alpha_n)||y_n-p||^2\\
&\leq \alpha_n||u-p||^2+(1-\alpha_n)||z_n-p||^2\\
&\leq \alpha_n||u-p||^2+(1-\alpha_n)[||x_n-p||^2-||z_n-x_n||^2\\
&- 2\sigma_n\langle z_n-x_n,Ax_n-Ap\rangle-\sigma_n^2||Ax_n-Ap||^2]\\
||z_n-x_n||^2 &\leq \alpha_n||u-p||^2+||x_n-p||^2-||x_{n+1}-p||^2\\
&- 2\sigma_n\langle z_n-x_n,\ Ax_n-Ap\rangle-\sigma_n^2||Ax_n-Ap||^2\\
&\leq \alpha_n||u-p||^2+||x_n-p||^2-||x_{n+1}-p||^2\\
&- \sigma_n^2||Ax_n-Ap||^2. \qquad (21)
\end{aligned}
$$

Passing to subsequence, applying condition (i) together with the fact that $\lim_{j\to\infty}||x_{n_j}-p||$ exists and conclusion of STEP II in (21), we obtain $||z_{n_j}-x_{n_j}||\to 0,\ j\to\infty$.

STEP IV: We show that $w\in(A+B)^{-1}(0)$. Since $0<a\leq\sigma_n\leq b<2\alpha$, there exists a subsequence $\{\sigma_{n_j}\}$ of $\{\sigma_n\}$ such that $\sigma_{n_j}\to\sigma\ \in[a,b]$. Applying Lemma 2.7, we have

$$
\begin{aligned}
||J_\sigma^B(I-\sigma A)x_n-z_n|| &\leq ||J_\sigma^B(I-\sigma A)x_n-J_\sigma^B(I-\sigma_nA)x_n||+||J_\sigma^B(I-\sigma_nA)x_n-z_n||\\
&\leq ||(I-\sigma A)x_n-(I-\sigma_nA)x_n||\\
&+ ||J_\sigma^B(I-\sigma_nA)x_n-J_{\sigma_n}(I-\sigma_nA)x_n||\\
&\leq |\sigma_n-\sigma|||Ax_n||+\frac{|\sigma_n-\sigma|}{\sigma}||J_\sigma^B(I-\sigma_nA)x_n-(I-\sigma_nA)x_n||\\
&\leq |\sigma_n-\sigma|||Ax_n||+\frac{|\sigma_n-\sigma|}{\sigma}||K \qquad (22)
\end{aligned}
$$

for some $K>0$ such that $K=\sup_n||J_\sigma^B(I-\sigma_nA)x_n-(I-\sigma_nA)x_n||$. Replacing n with n_j in (22) and using boundedness of $\{Ax_n\}$, we get as $j\to\infty$ that

$$
||J_\sigma^B(I-\sigma A)x_{n_j}-z_{n_j}||\to 0 \qquad (23)
$$

But

$$\begin{aligned} ||J_\sigma^B(I-\sigma A)x_n - x_n|| &\leq ||J_\sigma^B(I-\sigma A)x_n - z_n + z_n - x_n|| \\ &\leq ||J_\sigma^B(I-\sigma A)x_n - z_n|| + ||z_n - x_n||. \end{aligned} \tag{24}$$

Replacing n with n_j in (24), using STEP III and conclusion (23) yield

$$||J_\sigma^B(I-\sigma A)x_{n_j} - x_{n_j}|| \to 0 \text{ as } j \to \infty.$$

Thus from definition 2.1, we have that $w \in F\big(J_\sigma^B(I-\sigma A)\big)$. Since by Lemma 2.6, $F\big(J_\sigma^B(I-\sigma A)\big) = (A+B)^{-1}(0)$, it implies that $w \in (A+B)^{-1}(0)$.

STEP V: We show that $x_n \to P_\Omega u$ as $n \to \infty$. Without loss of generality, we may choose a subsequence $\{x_{n_{j_i}+1}\}$ of $\{x_{n_j+1}\}$ which we still call $\{x_{n_j+1}\}$ such that $\limsup_{n\to\infty}\big\langle u - P_\Omega u, x_{n+1} - P_\Omega u\big\rangle = \lim_{j\to\infty}\big\langle u - P_\Omega u, x_{n_j+1} - P_\Omega u\big\rangle$. Since P is the metric projection of H onto Ω and $x_{n_{j+1}} \rightharpoonup w \in \Omega$, we have

$$\lim_{j\to\infty}\big\langle u - P_\Omega u, x_{n_j+1} - P_\Omega u\big\rangle = \big\langle u - P_\Omega u, w - P_\Omega u\big\rangle \leq 0. \tag{25}$$

Hence,

$$\limsup_{n\to\infty}\big\langle u - P_\Omega u, x_{n+1} - P_\Omega u\big\rangle \leq 0. \tag{26}$$

Using Lemma 2.1, we have

$$\begin{aligned} ||x_{n+1} - P_\Omega u||^2 &= ||\alpha_n(u - P_\Omega u) + (1-\alpha_n)\big(y_n - P_\Omega u\big)||^2 \\ &\leq (1-\alpha_n)^2||y_n - P_\Omega u||^2 + 2\alpha_n\big\langle u - P_\Omega u, x_{n+1} - P_\Omega u\big\rangle \\ &\leq (1-\alpha_n)^2||x_n - P_\Omega u||^2 + 2\alpha_n\big\langle u - P_\Omega u, x_{n+1} - P_\Omega u\big\rangle. \end{aligned}$$

Using condition (i) and conclusion (26), we have from Lemma 2.2, that
$x_n \to P_\Omega u$ as $n \to \infty$. This completes the proof.

If in Theorem 3.1, we set $k = 0$, then T is nonspreading and we can take $\beta = 0$ to obtain the following Corollary.

Corollary 3.2. *Let C be a nonempty closed convex subset of H, $A : C \to H$ be $\alpha-$inverse strongly monotone, and $B : D(B) \subseteq C \to 2^H$ be maximal monotone. Let $J_\lambda(I+\lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$, $T : C \to C$ be a nonspreading mapping. Suppose that $\Omega := F(T) \cap (A+B)^{-1}(0) \neq \emptyset$.*
For arbitrary $x_1, u \in C$, we define

$$\begin{aligned} z_n &= J^B_{\sigma_n}(I-\sigma_n A)x_n, \; \forall\, n \geq 1, \\ y_n &= \sum_{i=1}^{n-1} T^i z_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n \end{aligned} \tag{27}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions;
(i) $\alpha_n \to 0,\ n \to \infty$, (ii) $\sum_{n=1}^{\infty} \alpha_n = +\infty$, $\{\sigma_n\}$ is a sequence in $(0,\infty)$ and there exist $a, b \in \mathbb{R}$ with (iii) $0 < a \leq \sigma_n \leq b < 2\alpha\ \forall\, n \in \mathbb{N}$. Then the sequence $\{x_n\}$ constructed by algorithm (27) converges strongly to $P_\Omega u$, where P is the metric projection of H onto Ω.

Remark 3.3. *The results of Osilike et al* [9] *address the problem of approximating fixed point of strict pseudo non spreading mapping in a real Hilbert space. In Theorem 3.1 of this paper, the problem of approximating fixed point of strict pseudo non spreading mapping as well as zeroes of sum of two monotone mappings is solved in a real Hilbert space. Consequently, Theorem 3.1 complements the result of Osilike et al* [9]. *Furthermore, Theorem 3.1 of this paper generalizes the result of Liu et al* [10] *from non spreading mapping to strict pseudo non spreading mapping and many other important results in this direction of research.*

Conflict of Interests

The author declares that there is no conflict of interests.

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