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CONSTRUCTION OF COMMON FIXED POINTS OF A FINITE FAMILY OF ASYMPTOTICALLY ϕ-DEMICONTRACTIVE MAPS IN ARBITRARY BANACH SPACES

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Abstract. In this paper, common fixed points of a finite family of asymptotically ϕ-demicontractive maps in arbitrary Banach spaces are investigated. Strong convergence theorems of common fixed points are established.

Keywords: Asymptotically ϕ-demicontractive; Composite implicit iteration; Fixed point, Banach spaces.

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1. Introduction

Let K be a nonempty subset of an arbitrary real Banach space E and J the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 : \|x\|^2 = \|f\|^2\},$$

where E^* denotes the dual space of E and $\langle , \rangle$ denotes the generalized duality paring. If E^* is strictly convex, then J is single-valued. In the sequel, we shall denote single-valued duality mapping by j. A mapping $T : K \longrightarrow K$ is said to be *uniformly* $L-$ *Lipschitian* mapping with

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contant $L \geq 1$ if $\|T^n x - T^n y\| \leq L\|x-y\|$ $\forall n \in N$. T is *$r-$strictly asymptotically pseudocontractive* (See for example [1]) with sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ if $\forall x,y \in K$ $\exists$ $j(x-y) \in J(x-y)$ and a contant $r \in (0,1)$ such that

$$\langle (I-T^n)x-(I-T^n)y, j(x-y)\rangle \geq \frac{1}{2}(1-r)\|(I-T^n)x-(I-T^n)y\|^2 - \frac{1}{2}(k_n^2-1)\|x-y\|^2, \forall n \in N. \tag{1}$$

T is said be *asymptotically demicontractive* with sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and $\forall x \in K$, $p \in F(T)$, $\exists$ $j(x-p) \in J(x-p)$ such that

$$\langle x-T^n x, j(x-p)\rangle \geq \frac{1}{2}(1-r)\|x-T^n x\|^2 - \frac{1}{2}(k_n^2-1)\|x-p\|^2, \forall n \in N. \tag{2}$$

The class of $r-$strictly asymptotically pseudocontractive maps and the class of asymptotically demicontractive maps were first introduced in Hilbert spaces by Qihou [14]. Clearly, an $r-$strictly asymptotically pseudocontractive map with a nonempty fixed point set is asymptotically demicontractive. An example of a $r-$strictly asymptotically pseudocontractive map is given in [13] while an example of an asymptotically demicontractive map is given in [12].

A mapping $T : K \longrightarrow K$ is said to be *asymptotically $\phi-$demicontractive* with sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ if $F(T) \neq \emptyset$ and $\exists$ a strictly increasing continuous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle x-T^n x, j(x-p)\rangle \geq \phi\left(\|x-T^n x\|\right) - \frac{1}{2}(k_n^2-1)\|x-p\|^2, \forall x \in K, p \in F(T), n \in \mathbb{N}. \tag{3}$$

The class of asymptotically $\phi-$demicontractive maps was first introduced in arbitrary Banach spaces by Osilike and Isiogugu [12]. In [12], it is shown that the class of asymptotically demicontractive map is a proper subclass of the class of asymptotically $\phi-$demicontractive maps. Observe from (2) and (3) that every asymptotically demicontractive map is asymptotically $\phi-$demicontractive with $\phi : [0,\infty) \longrightarrow [0,\infty)$ given by

$$\phi(t) = \frac{1}{2}(1-r)t.$$

These classes of operators have been studied by several authors (See for example [3, 4, 5, 6, 7, 10, 11, 12, 13, 14]. Osilike and Isiogugu proved the convergence of the modified averaging iteration process of Mann [8] to the fixed points of asymptotically $\phi-$demicontractive maps. Specifically they proved the following.

Theorem 1.1. [12] *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $T: K \longrightarrow K$ be a completely continuous uniformly $L-$Lipschitzian asymptotically $\phi-$demicontractive mapping with a sequence $\{a_n\}_{n=1}^{\infty} \subseteq [1,\infty) \ni \sum(a_n^2-1) < \infty$. Let $\{a_n\}$ be a real sequence satisfying (i) $0 < \alpha_n \leq 1$ (ii) $\sum \alpha_n = \infty$ (iii) $\sum \alpha_n^2 < \infty$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from arbitrary $x_1 \in K$ by the modified averaging Mann iteration process*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, n \geq 1 \tag{4}$$

converges strongly to a common fixed point of T.

Similarly, in [6] using the modified averaging implicit iteration scheme of Sun [16], generated from an $x_1 \in K$, by $x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_n$, $n \geq 1$, where $1 \leq n = (k-1)N + i, i \in I = \{1,2,3,...,N\}$, Igbokwe and Udofia proved that under certain conditions on the iteration sequence $\{\alpha_n\}$, the above iteration process $\{x_n\}$ converges strongly to the common fixed point of the family $\{T_i\}_{i=1}^{N}$ of N uniformly L_i-Lipschitzian asymptotically $\phi-$demicontractive self maps of nonempty closed convex subset of a Banach space E.

Recently, Su and Li [15] introduced the following iteration scheme and called it Composite Implicit Iteration Process. From $x_1 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by

$$\begin{aligned} x_n = &\quad \alpha_n x_{n-1} + (1-\alpha_n)T_i y_n, \\ y_n = &\quad \beta_n x_{n-1} + (1-\beta_n)T_i^k x_n, n \geq 1, \end{aligned} \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1], T_n = T_{nmodN}$.

Motivated by the results of Su and Li [15], Igbokwe and Ini [4] modified the iteration process (5) and applied the modified iteration process for the approximation of common fixed points of a finite family of $r-$strictly asymptotically pseudocontractive maps. In compact form, the modified composite implicit iteration process is expressed as follows:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_i^k y_n, \\ y_n &= \beta_n x_{n-1} + (1-\beta_n)T_i^k x_n, n \geq 1 \end{aligned} \tag{6}$$

where $1 \leq n = (k-1)N + i$, $i = \{1, 2, ..., N\}$, $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$.

Observe that, if $T : K \longrightarrow K$ is uniformly $L-$Lipschitzian asymptotically $\phi-$demicontractive map with sequence $\{a_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} a_n = 1$, then for every fixed $u \in K$ and $t \in (\frac{L}{1+L}, 1)$, the operator $S_{t,s,n} : K \longrightarrow K$ defined for all $x \in K$ by $S_{t,s,n}x = tu + (1-t)T^n(su+(1-s)T^n x)$ satisfies $\|S_{t,s,n}x - S_{t,s,n}y\| \leq (1-t)(1-s)L^2\|x-y\| \; \forall\, x, y \in K$. Thus, the composite implicit iteration process (6) is defined in K for the family $\{T_i\}_{i=1}^{N}$ of N uniformly L-Lipschitzian asymptotically ϕ-demicontractive mappings of nonempty closed convex subset K of a real Banach space provided that $\{\alpha_n\}, \{\beta_n\} \subseteq (\eta, 1)$ for all $n \geq 1$, where $\eta = \frac{L}{1+L}$ and $L = \max_{1\leq i\leq N}\{L_i\}$.

2. Preliminaries

In this paper, we prove that the iteration process (6) converges to the common fixed points of the finite family of N uniformly $L-$Lipschitzian asymptotically $\phi-$demicontractive mappings in arbitrary real Banach spaces. Our results generalize Theorem 1.1 and extend the recent result of Igbokwe and Ini [4] from $r-$strictly asymptotically pseudocontractive maps to the much more general class of asymptotically $\phi-$demicontractive maps. Moreso, the theorem of Qihou [14], a result of Osilike [9], Osilike and Aniagbsor [10] and several others in the literature are special cases of our results.

In the sequel, we need the following.

Lemma 2.1 [11] *Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq (1+\delta_n)a_n + b_n, n \geq 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.*

Definition 2.1. *Let K be a closed subset of a real Banach space E and $T : K \longrightarrow K$ be a mapping. T is said to be semicompact (see for example* [1]*) if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - T_n x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \longrightarrow x^* \in K$.*

Definition 2.2. [1]: *A bounded convex subset K of a real Banach space E is said to have normal structure if every nontrivial convex subset C of K contains at least one nondimetrial point. That* *is, there exists* $x_0 \in E$ *such that* $sup\{\|x_0 - x\| : x \in C\} < sup\{\|y - x\| : x, y \in C = d(C)\}$ *where* $d(C)$ *is the diameter of* C.

Every uniformly convex Banach space and every compact convex subset K of a Banach space E has normal structure. For the definition of modulus of convexity of E and the characteristic of convexity ε_0 of E, see [1].

Theorem 2.1. [13] *Let* E *be a real Banach space with normal structure* $N(E) > max(1, \varepsilon_0)$, $\varepsilon_0 > 0$, *K a nonempty closed convex subset of E and* $T : K \longrightarrow K$ *a uniformly* $L-$*Lipschitzian mapping with* $L < \alpha, \alpha > 1$. *Then* T *has a fixed point.*

3. Main results

Lemma 3.1. *Let E be a normed space and K a nonempty convex subset of E. Let* $\{T_i\}_{i=1}^N$ *be N uniformly* L_i-*Lipschitzian self mappings of K such that* $L = \max\{L_i\}, L_i$ *the Lipschitzian constant of* $T_i, i = 1, 2, ..., N$. *Let* $\{\alpha_n\}, \{\beta_n\}$ *be sequences in* $(0, 1]$ *such that* (i) $\sum_{n=1}^{\infty}(1-\alpha_n) = +\infty$ (ii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < +\infty$ (iii) $\sum_{n=1}^{\infty}(1-\beta_n) < +\infty$. *For arbitrary* $x_1 \in K$, *generate the sequence* $\{x_n\}$ *by* $y_n = \beta_n x_{n-1} + (1-\beta_n)T_i^k x_n$, $x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k y_n$, $n \geq 1$, *where* $1 \leq n = (k-1)N + i$, $i = \{1, 2, ..., N\}$. *Then* $\|T_i x_n - x_n\| \leq 2[1 + L^2 + 2L^3(2+L)]\|T_i^k x_n - x_n\| + 2L\|T_i^{k-1}x_{n-1} - x_{n-1}\| + 4(1-\alpha_n)L^2[1 + 2L(1+L)]\|x_n - x_{n-1}\|$.

Proof. Putting $\lambda_{in} = \|x_n - T_i^k x_n\|$, we have

$$\begin{aligned}
\|x_n - T_i x_n\| &\leq \lambda_{in} + \alpha_n L \lambda_{in-1} + (1-\alpha_n)L^2\|T_i^k y_n - T_i^k x_n + T_i^k x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)L^2\|T_i y_n - T_i x_n + T_i x_n - x_{n-1}\| \\
&\leq \lambda_{in} + \alpha_n L \lambda_{in-1} + 2(1-\alpha_n)L^3\|y_n - x_n\| + (1-\alpha_n)L^2\|T_i^k x_n - x_n + x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)L^2\|T_i x_n - x_n + x_n - x_{n-1}\| \\
&\leq [1 + (1-\alpha_n)L^2]\lambda_{in} + L\lambda_{in-1} + 2(1-\alpha_n)L^2\|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n)L^2\|T_i x_n - x_n\| + 2(1-\alpha_n)L^3\|y_n - x_n\|,
\end{aligned}$$

$$
\begin{aligned}
[1-(1-\alpha_n)L^2]\|T_ix_n-x_n\| \quad \leq \quad & [1+L^2]\lambda_{in}+L\lambda_{in-1}+2(1-\alpha_n)L^2\|x_n-x_{n-1}\| \\
& +2(1-\alpha_n)L^3\|y_n-x_n\|. \qquad (7)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|y_n-x_n\| &\leq \alpha_n\|x_{n-1}-y_n\|+(1-\alpha_n)\|T_i^k y_n-y_n\| \\
&\leq \alpha_n\|x_{n-1}-y_n\|+(1-\alpha_n)\|T_i^k y_n-x_{n-1}\|+(1-\alpha_n)\|y_n-x_{n-1}\| \\
&\leq (1-\beta_n)\|T_i^k x_n-x_{n-1}\|+(1-\alpha_n)\|T_i^k y_n-x_{n-1}\| \\
&\leq (1-\beta_n)\|T_i^k x_n-x_n\|+(1-\beta_n)\|x_n-x_{n-1}\|+(1-\alpha_n)L\|y_n-x_{n-1}\| \\
&\quad +(1-\alpha_n)\|T_i^k x_{n-1}-x_{n-1}\| \\
&\leq (1-\beta_n)\lambda_{in}+(1-\beta_n)\|x_n-x_{n-1}\|+(1-\alpha_n)(1-\beta_n)L\|T_i^k x_n-x_n+x_n-x_{n-1}\| \\
&\quad +(1-\alpha_n)\|T_i^k x_{n-1}-T_i^k x_n+T_i^k x_n-x_{n-1}\| \\
&\leq (1-\beta_n)\lambda_{in}+(1-\beta_n)\|x_n-x_{n-1}\|+(1-\alpha_n)(1-\beta_n)L\lambda_{in} \\
&\quad +(1-\alpha_n)(1-\beta_n)L\|x_n-x_{n-1}\|+(1-\alpha_n)L\|x_n-x_{n-1}\|+(1-\alpha_n)\|T_i^k x_n-x_{n-1}\| \\
&\leq (1-\beta_n)\lambda_{in}+(1-\beta_n)\|x_n-x_{n-1}\|+(1-\alpha_n)(1-\beta_n)L\lambda_{in} \\
&\quad +(1-\alpha_n)(1-\beta_n)L\|x_n-x_{n-1}\|+(1-\alpha_n)L\|x_n-x_{n-1}\|+(1-\alpha_n)\lambda_{in} \\
&\quad +(1-\alpha_n)\|x_n-x_{n-1}\| \\
&\leq \lambda_{in}+(1-\beta_n)\|x_n-x_{n-1}\|+L\lambda_{in} \\
&\quad +(1-\beta_n)L\|x_n-x_{n-1}\|+(1-\alpha_n)L\|x_n-x_{n-1}\|+\lambda_{in}+(1-\alpha_n)\|x_n-x_{n-1}\| \\
&\leq (2+L)\lambda_{in}+2(1+L)\|x_n-x_{n-1}\|. \qquad (8)
\end{aligned}
$$

Substituting (8) into (7), we have

$$\begin{aligned}[1-(1-\alpha_n)L^2]\|T_ix_n-x_n\| &\leq [1+L^2]\lambda_{in}+L\lambda_{in-1}+2(1-\alpha_n)L^2\|x_n-x_{n-1}\| \\ &+2(1-\alpha_n)L^3\{(2+L)\lambda_{in}+2(1+L)\|x_n-x_{n-1}\|\} \\ &\leq [1+L^2]\lambda_{in}+L\lambda_{in-1}+2(1-\alpha_n)L^2\|x_n-x_{n-1}\| \\ &+2L^3(2+L)\lambda_{in}+4(1-\alpha_n)L^3(1+L)\|x_n-x_{n-1}\|,\end{aligned}$$

$$\begin{aligned}\|T_ix_n-x_n\| &\leq \frac{1}{[1-(1-\alpha_n)L^2]}\Big\{[2L^3(2+L)+(1+L^2)]\|T_i^kx_n-x_n\| \\ &+L\|T_i^{k-1}x_{n-1}-x_{n-1}\|+2(1-\alpha_n)L^2[1+2L(1+L)]\|x_n-x_{n-1}\|\Big\}.\end{aligned}$$

From condition (ii) $\lim_{n\to\infty}(1-\alpha_n)=0$, we find that there exists an $N_1>0$ such that $\forall n\geq N_1$, $1-(1-\alpha_n)L^2\geq\frac{1}{2}$. Therefore,

$$\begin{aligned}\|T_ix_n-x_n\| &\leq 2[1+L^2+2L^3(2+L)]\|T_i^kx_n-x_n\| \\ &+2L\|T_i^{k-1}x_{n-1}-x_{n-1}\|+4(1-\alpha_n)L^2[1+2L(1+L)]\|x_n-x_{n-1}\|.\end{aligned}$$

This completes the proof.

Theorem 3.1. *Let E be a real Banach space with normal structure $N(E)>max(1,\varepsilon_0)$, $\varepsilon_0>0$, and K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L_i-Lipschitzian asymptotically $\phi-$demicontractive self maps of K with sequence $\{a_{in}\}\in[1,\infty)$ such that $\sum_{n=1}^{\infty}(a_{in}-1)<\infty$ for all $i\in I$ and $F=\bigcap_{i=1}^N F(T_i)\neq\emptyset$ where $F(T_i)=\{x\in K: T_ix=x\}$. Let one member of the family $\{T_i\}_{i=1}^N$ be semicompact. Let $\{\alpha_n\}\subset(0,1),\{\beta_n\}\subset(0,1]$ be two real sequences satisfying the conditions; (i) $\sum_{n=1}^{\infty}(1-\alpha_n)=+\infty$ (ii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2<+\infty$ (iii) $\sum_{n=1}^{\infty}(1-\beta_n)<+\infty$. $(1-\alpha_n)(1-\beta_n)L^2<1$ $\forall n\geq1$ where $L\geq1$ is the common Lipschitz constant of $\{T_i\}_{i=1}^N$. For $x_1\in K$, let $\{x_n\}_{n=1}^{\infty}$ be the modified implicit iteration sequence defined by*

$$\begin{aligned} x_n &= \alpha_nx_{n-1}+(1-\alpha_n)T_i^ky_n, \\ y_n &= \beta_nx_{n-1}+(1-\beta_n)T_i^kx_n, n\geq1, \end{aligned} \tag{9}$$

where $n=(k-1)N+i$, $i=\{1, 2, ..., N\}$, then (a) $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F$. (b) $\liminf_{n\to\infty}\|x_n-T_ix_n\|=0$. (c) $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of the mapping $\{T_i\}_{i=1}^N$ if there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges strongly to p.

Proof. It is well known (See, for example, Chang [2]) that the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{10}$$

holds for all $x, y \in E$ and $j(x-y) \in J(x-y)$. The existence of fixed points for each T_i follows from Theorem 1.2. Let $p \in F$, then, using (9) and (10), we have

$$\begin{aligned}
\|x_n - p\|^2 &\leq \alpha_n^2\|x_{n-1}-p\|^2 + 2(1-\alpha_n)\langle T_i^k y_n - p, j(x_n-p)\rangle \\
&= \alpha_n^2\|x_{n-1}-p\|^2 + 2(1-\alpha_n)\langle T_i^k y_n - T_i^k x_n, j(x_n-p)\rangle \\
&\quad +2(1-\alpha_n)\langle T_i^k x_n - p, j(x_n-p)\rangle \\
&\leq \alpha_n^2\|x_{n-1}-p\|^2 + 2L(1-\alpha_n)\|y_n - x_n\|\|x_n-p\| + 2(1-\alpha_n)\langle x_n - p, j(x_n-p)\rangle \\
&\quad +2(1-\alpha_n)\langle T_i^k x_n - x_n, j(x_n-p)\rangle \\
&= \alpha_n^2\|x_{n-1}-p\|^2 + 2L(1-\alpha_n)\|y_n - x_n\|\|x_n-p\| + 2(1-\alpha_n)\|x_n-p\|^2 \\
&\quad -2(1-\alpha_n)\langle x_n - T_i^k x_n, j(x_n-p)\rangle. \qquad (11)
\end{aligned}$$

Now, $T_i : K \longrightarrow K$ is asymptotically $\phi-$demicontractive. For each T_i, we have

$$\langle x_n - T_i^k x_n, j(x_n-p)\rangle \geq \phi_i(\|x_n - T_i^k x_n\|) - \frac{1}{2}(a_{in}^2 - 1)\|x_n - p\|^2.$$

Choosing $\phi(t) = \min_{1\leq i\leq N}\{\phi_i(t)\}$ so that

$$\begin{aligned}
\|x_n - p\|^2 &\leq \alpha_n^2\|x_{n-1}-p\|^2 + 2L(1-\alpha_n)\|y_n - x_n\|\|x_n-p\| + 2(1-\alpha_n)\|x_n-p\|^2 \\
&\quad -2(1-\alpha_n)\{\phi(\|x_n - T_i^k x_n\|) - \frac{1}{2}(a_{in}^2-1)\|x_n-p\|^2\}, \qquad (12)
\end{aligned}$$

$$\|y_n - x_n\| \leq \beta_n(1-\alpha_n)\|T_i^k y_n - x_{n-1}\| + (1-\beta_n)\|x_n - T_i^k x_n\|, \tag{13}$$

and

$$\begin{aligned}
\|T_i^k y_n - x_{n-1}\| &\leq \|T_i^k y_n - p\| + \|x_{n-1} - p\| \\
&\leq L\|(\beta_n(x_{n-1}-p) + (1-\beta_n)(T_i^k x_n - p)\| + \|x_{n-1}-p\| \\
&\leq (L\beta_n + 1)\|x_{n-1}-p\| + L^2(1-\beta_n)\|x_n - p\|. \qquad (14)
\end{aligned}$$

Substituting (14) into (13), we obtain

$$\begin{aligned}\|y_n - x_n\| &\le \beta_n(1-\alpha_n)\{(L\beta_n+1)\|x_{n-1}-p\| + L^2(1-\beta_n)\|x_n - p\|\}\\ &\quad +(1-\beta_n)\|x_n - T_i^k x_n\|\\ &= \beta_n(1-\alpha_n)(L\beta_n+1)\|x_{n-1}-p\| + L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_n-p\|\\ &\quad +(1-\beta_n)\|x_n - T_i^k x_n\|\\ &\le \beta_n(1-\alpha_n)(L\beta_n+1)\|x_{n-1}-p\| + [\beta_n(1-\alpha_n)(1-\beta_n)L^2\\ &\quad +(1-\beta_n)(L+1)]\|x_n-p\|. \qquad (15)\end{aligned}$$

Substituting (15) into (12), we obtain

$$\begin{aligned}\|x_n-p\|^2 &\le \alpha_n^2\|x_{n-1}-p\|^2 + 2L(1-\alpha_n)\{\beta_n(1-\alpha_n)(L\beta_n+1)\|x_{n-1}-p\|\\ &\quad +[\beta_n(1-\alpha_n)(1-\beta_n)L^2 + (1-\beta_n)(L+1)]\|x_n-p\|\}\|x_n-p\|\\ &\quad +2(1-\alpha_n)\|x_n-p\|^2 - 2(1-\alpha_n)\{\phi(\|x_n - T_i^k x_n\|) - \frac{1}{2}(a_{in}^2-1)\|x_n-p\|^2\}\\ &= \alpha_n^2\|x_{n-1}-p\|^2 + 2L\beta_n(1-\alpha_n)^2(L\beta_n+1)\|x_{n-1}-p\|\|x_n-p\| +\\ &\quad +[2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3 + 2(1-\alpha_n)(1-\beta_n)(L+1)L\\ &\quad +2(1-\alpha_n) + (1-\alpha_n)(a_{in}^2-1)]\|x_n-p\|^2 - 2(1-\alpha_n)\phi(\|x_n - T_i^k x_n\|),\end{aligned}$$

$$\begin{aligned}&[1-2\beta_n(1-\beta_n)^2(1-\beta_n)L^3 - 2(1-\alpha_n)(1-\beta_n)(L+1)L - 2(1-\alpha_n)\\ &-(1-\alpha_n)(a_{in}^2-1)]\|x_n-p\|^2 \le \alpha_n^2\|x_{n-1}-p\|^2\\ &+2L\beta_n(1-\alpha_n)^2(L\beta_n+1)\|x_{n-1}-p\|\|x_n-p\| - 2(1-\alpha_n)\phi(\|x_n - T_i^k x_n). \qquad (16)\end{aligned}$$

$$\begin{aligned}\|x_n-p\|^2 &\le \frac{\alpha_n^2}{\kappa_n}\|x_{n-1}-p\|^2 + \frac{2L\beta_n(1-\alpha_n)^2(L\beta_n+1)}{\kappa_n}\|x_{n-1}-p\|\|x_n-p\|\\ &\quad -\frac{2(1-\alpha_n)}{\kappa_n}\phi(\|x_n - T_i^k x_n\|),\end{aligned}$$

where $\kappa_n = 1-2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3-2(1-\alpha_n)(1-\beta_n)(L+1)L-2(1-\alpha_n)-(1-\alpha_n)(a_{in}^2-1)$,

$$\begin{aligned}\|x_n-p\|^2 &\leq (1+\frac{\lambda_n}{\kappa_n})\|x_{n-1}-p\|^2+\frac{2L\beta_n(1-\alpha_n)^2(L\beta_n+1)}{\kappa_n}\|x_{n-1}-p\|\|x_n-p\| \\ &- \frac{2(1-\alpha_n)}{\kappa_n}\phi(\|x_n-T_i^kx_n\|), \qquad (17)\end{aligned}$$

where $\lambda_n = \alpha_n^2-1+2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3+2(1-\alpha_n)(1-\beta_n)(L+1)L+2(1-\alpha_n)+(1-\alpha_n)(a_{in}^2-1)$. Since $1-2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3-2(1-\alpha_n)(1-\beta_n)(L+1)L-2(1-\alpha_n)-(1-\alpha_n)(a_{in}^2-1) = 1-(1-\alpha_n)[2\beta_n(1-\alpha_n)(1-\beta_n)L^3+2(1-\beta_n)(L+1)L+2+(a_{in}^2-1)]$, and condition (ii), we have $\lim_{n\to\infty}(1-\alpha_n)=0$. So there exists a natural number N_2 such that $\forall\ n \geq N_2$,

$$1-2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3-2(1-\alpha_n)(1-\beta_n)(L+1)L-2(1-\alpha_n)-(1-\alpha_n)(a_{in}^2-1)\geq\frac{1}{2}.$$

It follows that

$$\begin{aligned}\|x_n-p\|^2 &\leq [1+2[\alpha_n^2-1+2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3 \\ &+2(1-\alpha_n)(1-\beta_n)(L+1)L+2(1-\alpha_n)+(1-\alpha_n)(a_{in}^2-1)]]\|x_{n-1}-p\|^2 \\ &+4L\beta_n(1-\alpha_n)^2(L\beta_n+1)\|x_{n-1}-p\|\|x_n-p\|-2(1-\alpha_n)\phi\left(\|x_n-T_i^kx_n\|\right) \\ &= [1+2[(1-\alpha_n)^2+2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3 \\ &+2(1-\alpha_n)(1-\beta_n)(L+1)L+(1-\alpha_n)(a_{in}^2-1)]]\|x_{n-1}-p\|^2 \\ &+4L\beta_n(1-\alpha_n)^2(L\beta_n+1)\|x_{n-1}-p\|\|x_n-p\|-2(1-\alpha_n)\phi\left(\|x_n-T_i^kx_n\|\right). \qquad (18)\end{aligned}$$

Considering the second term on the right hand side of (18), we have

$$\begin{aligned}\|x_n-p\|^2 &= \alpha_n\langle x_{n-1}-p, j(x-p)\rangle+(1-\alpha_n)\langle T_i^ky_n-p, j(x-p)\rangle \\ &= \alpha_n\langle x_{n-1}-p, j(x-p)\rangle+(1-\alpha_n)\langle T_i^ky_n-T_i^kx_n, j(x-p)\rangle \\ &+(1-\alpha_n)\langle T_i^kx_n-p, j(x-p)\rangle \\ &\leq \alpha_n\|x_{n-1}-p\|\|x_n-p\|+L(1-\alpha_n)\|y_n-x_n\|\|x_n-p\| \\ &+L(1-\alpha_n)\|x_n-p\|^2. \qquad (19)\end{aligned}$$

Substituting (15) into (19), we obtain

$$\begin{aligned}
&\|x_n - p\|^2 \\
\leq\ & \alpha_n\|x_{n-1}-p\|\|x_n-p\| + L(1-\alpha_n)\{\beta_n(1-\alpha_n)(L\beta_n+1)\|x_{n-1}-p\| \\
&+[\beta_n(1-\alpha_n(1-\beta_n)L^2+(1-\beta_n)(L+1)]\|x_n-p\|\}|\|x_n-p\|+L(1-\alpha_n)\|x_n-p\|^2 \\
=\ & \alpha_n\|x_{n-1}-p\|\|x_n-p\| + L\beta_n(1-\alpha_n)^2(L\beta_n+1)\|x_{n-1}-p\|\|x_n-p\| \\
&+[L^3\beta_n(1-\alpha_n)^2(1-\beta_n)+L(1-\alpha_n)(1-\beta_n)(L+1)+L(1-\alpha_n)]\|x_n-p\|^2,
\end{aligned}$$

$$\begin{aligned}
&[1-L^3\beta_n(1-\alpha_n)^2(1-\beta_n)-L(1-\alpha_n)(1-\beta_n)(L+1)-L(1-\alpha_n)]\|x_n-p\|^2 \leq \\
&\quad \alpha_n\|x_{n-1}-p\|\|x_n-p\|+L\beta_n(1-\alpha_n)^2(L\beta_n+1)\|x_{n-1}-p\|\|x_n-p\|,
\end{aligned}$$

and

$$\begin{aligned}
&[1-L^3\beta_n(1-\alpha_n)^2(1-\beta_n)-L(1-\alpha_n)(1-\beta_n)(L+1)-L(1-\alpha_n)]\|x_n-p\|^2 \leq \\
&\quad \{\alpha_n+L\beta_n(1-\alpha_n)^2(L\beta_n+1)\}\|x_{n-1}-p\|\|x_n-p\|.
\end{aligned}$$

Hence, we have

$$\|x_n-p\|^2 \leq \frac{\alpha_n+L\beta_n(1-\alpha_n)^2(L\beta_n+1)}{w_n}\|x_{n-1}-p\|\|x_n-p\|, \tag{20}$$

where

$$w_n = 1-L^3\beta_n(1-\alpha_n)^2(1-\beta_n)-L(1-\alpha_n)(1-\beta_n)(L+1)-L(1-\alpha_n).$$

Since $\lim_{n\to\infty}(1-\alpha_n)=0$, we see that there exists a natural number N_3 such that $\forall\ n \geq N_3$,

$$\begin{aligned}
&1-L^3\beta_n(1-\alpha_n)^2(1-\beta_n)-L(1-\alpha_n)(1-\beta_n)(L+1)-L(1-\alpha_n) = \\
&\quad 1-(1-\alpha_n)\{L^3\beta_n(1-\beta_n)+L(1-\beta_n)(L+1)+L\} \geq \frac{1}{2}.
\end{aligned}$$

It follows that

$$\|x_n-p\| \leq 2\{\alpha_n+L\beta_n(1-\alpha_n)^2(L\beta_n+1)\}\|x_{n-1}-p\|. \tag{21}$$

Substituting (22) into (18), we obtain

$$\begin{aligned}
\|x_n-p\|^2 &\leq [1+2[(1-\alpha_n)^2+2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3 \\
&\quad +2(1-\alpha_n)(1-\beta_n)(L+1)L+(1-\alpha_n)(a_{in}^2-1)]]\|x_{n-1}-p\|^2 \\
&\quad +4L\beta_n(1-\alpha_n)^2(L\beta_n+1)\|x_{n-1}-p\|\{2\{\alpha_n+L\beta_n(1-\alpha_n)^2(L\beta_n+1)\}\|x_{n-1}-p\|\} \\
&\quad -2(1-\alpha_n)\phi\left(\|x_n-T_i^k x_n\|\right) \\
&= [1+2[(1-\alpha_n)^2+2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3+2(1-\alpha_n)(1-\beta_n)(L+1)L \\
&\quad +(1-\alpha_n)(a_{in}^2-1)]]\|x_{n-1}-p\|^2 \\
&\quad +8L\beta_n(1-\alpha_n)^2(L\beta_n+1)\{\alpha_n+L\beta_n(1-\alpha_n)^2(L\beta_n+1)\}\|x_{n-1}-p\|^2 \\
&\quad -2(1-\alpha_n)\phi\left(\|x_n-T_i^k x_n\|\right) \\
&= [1+2[(1-\alpha_n)^2+2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3+2(1-\alpha_n)(1-\beta_n)(L+1)L \\
&\quad +(1-\alpha_n)(a_{in}^2-1) \\
&\quad +4L\beta_n(1-\alpha_n)^2(L\beta_n+1)\{\alpha_n+L\beta_n(1-\alpha_n)^2(L\beta_n+1)\}]]\|x_{n-1}-p\|^2 \\
&\quad -2(1-\alpha_n)\phi\left(\|x_n-T_i^k x_n\|\right), \qquad (22)
\end{aligned}$$

$$\|x_n-p\|^2 \leq [1+\delta_{in}]\|x_{n-1}-p\|^2-2(1-\alpha_n)\phi\left(\|x_n-T_i^k x_n\|\right), \qquad (23)$$

where

$$\begin{aligned}
\delta_{in} &= 2[(1-\alpha_n)^2+2\beta_n(1-\alpha_n)^2(1-\beta_n)L^3+2(1-\alpha_n)(1-\beta_n)(L+1)L \\
&\quad +(1-\alpha_n)(a_{in}^2-1)+4L\beta_n(1-\alpha_n)^2(L\beta_n+1)\{\alpha_n+L\beta_n(1-\alpha_n)^2(L\beta_n+1)\}].
\end{aligned}$$

From conditions (ii) and (iii), we have $\sum_{n=1}^{\infty}\delta_{in}<\infty$. Thus using Lemma 2.1, it follows that $\lim_{n\to\infty}\|x_n-p\|$ exists and $\{x_n\}$ is bounded. This completes the proof of (a). Since $\{x_n\}$ is bounded, we have there exists $M>0$ such that $\|x_n-p\|^2\leq M\ \forall n\geq 1$. It follows from (24) that

$$\begin{aligned}
2(1-\alpha_n)\phi\left(\|x_n - T_i^k x_n\|\right) &\leq [1+\delta_{in}]\|x_{n-1}-p\|^2 - \|x_n - p\|^2 \\
2\sum_{j=1}^{\infty}(1-\alpha_j)\phi\left(\|x_j - T_j^k x_j\|\right) &\leq \sum_{j=1}^{\infty}[\|x_{j-1}-p\|^2 - \|x_j - p\|^2] + \sum_{j=1}^{\infty}\delta_{ij}\|x_j - p\|^2 \\
2\sum_{j=N+1}^{\infty}(1-\alpha_j)\phi\left(\|x_j - T_j^k x_j\|\right) &\leq \|x_N - p\|^2 + M\sum_{j=N+1}^{\infty}\delta_{ij} < \infty \\
\sum_{n=1}^{\infty}(1-\alpha_n)\phi\left(\|x_n - T_j^k x_n\|\right) < \infty.
\end{aligned}$$

Condition (i) implies $\liminf_{n\to\infty}\phi(\|x_n - T_i^k x_n\|) = 0$. Since ϕ is an increasing and continuous function, then $\liminf_{n\to\infty}\|x_n - T_i^k x_n\| = 0$. Since $\{x_n\} \subseteq (0,1)$, $\lim_{n\to\infty}\|x_n - p\|$ exists and $\{x_n\}$ is bounded, by Lemma 3.1, $\liminf_{n\to\infty}\|x_n - T_i x_n\| = 0$. Thus completing the proof of (b). Since one member of the family $\{T_i\}_{i=1}^N$ is semicompact, $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges strongly to p and since $\lim_{n\to\infty}\|x_n - p\|$ exists also, then by Lemma 2.1 $\lim_{n\to\infty}\|x_n - p\| = 0$. This completes the proof.

Remark. (1) Our results complement and generalize the result of Su and Li [12].

(2) Setting $\beta_n = 1$, the iteration scheme (6) takes the non-implicit form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_{n-1}. \tag{24}$$

In the case of $N = 1$, (24) becomes the modified Mann iteration process in [8] given by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T^k x_{n-1}.$$

In such case, the results of Osilike and Isiogugu [12] become special cases of our results.

(3) Theorem 3.1 extends the result of Igbokwe and Ini [4] from $r-$strictly asymptotically pseudocontractive maps to the much more general class of asymptotically $\phi-$demicontractive maps.

(4) In general, Theorem 3.1 extends several results in the literature from asymptotically demicontractive maps to the more general class of asymptotically ϕ-demicontractive maps (see for example [3, 9, 10, 11, 14]).

Conflict of Interests

The authors declare that there is no conflict of interests.

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