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# GENERALIZED SYSTEMS OF VARIATIONAL INCLUSIONS INVOLVING $(A, \eta)$-MONOTONE MAPPINGS

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**Abstract.** In this paper, we introduce a generalized system of nonlinear relaxed cocoercive variational inclusions involving $(A, \eta)$-monotone mappings in the framework of real Hilbert spaces. Based on the generalized resolvent operator technique associated with $(A, \eta)$-monotonicity, we consider the approximation solvability of solutions.

**Keywords**: $(A, \eta)$-monotone mapping; nonexpansive mappings; $A$-monotone mappings; $H$-monotone mappings; Hilbert space.

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## 1. Introduction

Variational inclusions problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inclusions problems have been generalized and extended in different directions using the novel and innovative techniques. Various kinds of iterative algorithms to solve

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the variational inequalities and variational inclusions have been developed by many authors. There exists a vast literature [1-31] on the approximation solvability of nonlinear variational inequalities as well as nonlinear variational inclusions using projection type methods, resolvent operator type methods or averaging techniques. In most of the resolvent operator methods, the maximal monotonicity has played a key role, but more recently introduced notions of $A$-monotonicity [20] and $H$-monotonicity [8,9] have not only generalized the maximal monotonicity, but gave a new edge to resolvent operator methods. Recently Verma [19] generalized the recently introduced and studied notion of $A$-monotonicity to the case of $(A, \eta)$-monotonicity. Furthermore, these developments added a new dimension to the existing notion of the maximal monotonicity and its applications to several other fields such as convex programming and variational inclusions. Inspired and motivated by the recent research going on in this area, in this paper, we explore the approximation solvability of a generalized system of nonlinear variational inclusion problems based on $(A, \eta)$-resolvent operator technique in the framework Hilbert spaces.

## 2. Preliminaries

In this section, we explore some basic properties derived from the notion of $(A, \eta)$-monotonicity. Let $H$ denote a real Hilbert space with the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $\eta : H \times H :\to H$ be a single-valued mapping. The map $\eta$ is called $\tau$-Lipschitz continuous if there is a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \leq \tau \|y - v\|, \quad \forall u, v \in H.$$

Let $M$ be a multivalued mapping from a Hilbert space $H$ to $2^H$, the power set of $H$. Recall following definition:

(i) The set $D(M)$ defined by

$$D(M) = \{u \in H : M(u) \neq \emptyset\},$$

is called the effective domain of $M$.

(ii) The set $R(M)$ defined by

$$R(M) = \bigcup_{u \in H} M(u),$$

is called the range of $M$.

(iii) The set $G(M)$ defined by

$$G(M) = \{(u, v) \in H \times H : u \in D(M), v \in M(u)\},$$

is the graph of $M$.

**Definition 2.1.** Let $\eta : H \times H \to H$ be a single-valued mapping and let $M : H \to 2^H$ be a multivalued mapping on $H$.

(i) The map $M$ is said to be $(r, \eta)$-strongly monotone if

$$\langle u^* - v^*, \eta(u, v)\rangle \geq r\|u - v\|, \quad \forall (u, u^*), (v, v^*) \in \mathrm{G}(M).$$

(ii) $\eta$-pseudomonotone if $\langle v^*, \eta(u, v)\rangle \geq 0$ implies

$$\langle u^*, \eta(u, v)\rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in \mathrm{G}(M).$$

(iii) $(m, \eta)$-relaxed monotone if there exists a positive constant $m$ such that

$$\langle u^* - v^*, \eta(u, v)\rangle \geq -m\|u - v\|^2, \quad \forall (u, u^*), (v, v^*) \in \mathrm{G}(M).$$

**Definition 2.2** [8,9]. Let $H : X \to X$ be a nonlinear mapping and $M : X \to 2^X$ a multivalued mapping. The mapping $M$ is said to be $H$-monotone if $(H + \rho M)X = X$ for $\rho > 0$.

**Definition 2.3** [20]. Let $A : H \to H$ be a nonlinear mapping and $M : H \to 2^H$ a multivalued mapping. The mapping $M$ is said to be $A$-monotone if

(i) $M$ is $m$-relaxed monotone.

(ii) $A + \rho M$ is maximal monotone for $\rho > 0$.

**Definition 2.4** [19]. A mapping $M : H \to 2^H$ is said to be maximal $(m, \eta)$-relaxed monotone if

(i) $M$ is $(m,\eta)$-relaxed monotone,

(ii) for $(u,u^*) \in H \times H$ and

$$\langle u^* - v^*, \eta(u,v)\rangle \geq -m\|u-v\|^2, \quad (v,v^*) \in \text{graph}(M),$$

we have $u^* \in M(u)$.

**Definition 2.5** [19]. Let $A : H \to H$ and $\eta : H \times H \to H$ be two single-valued mappings. The map $M : H \to 2^H$ is said to be $(A,\eta)$-monotone if

(i) $M$ is $(m,\eta)$-relaxed monotone,

(ii) $R(A + \rho M) = H$ for $\rho > 0$.

Note that alternatively, the mapping $M : H \to 2^H$ is said to be $(A,\eta)$-monotone if

(i) $M$ is $(m,\eta)$-relaxed monotone,

(ii) $A + \rho M$ is $\eta$-pseudomonotone for $\rho > 0$.

**Definition 2.6.** Let $A : H \to H$ be an $(r,\eta)$-strong monotone mapping and $M : H \to H$ an $(A,\eta)$-monotone mapping. Then the generalized resolvent operator $J^{A,\eta}_{M,\rho} : H \to H$ is defined by

$$J^{A,\eta}_{M,\rho}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in H,$$

where $\rho > 0$ is a constant.

**Definition 2.7.** The mapping $N : H \to H$ is said to be relaxed $(\beta,\gamma)$-cocoercive with respect to $A$ if there exists two positive constants $\alpha, \beta$ such that

$$\langle Nx - Ny, Ax - Ay\rangle \geq (-\beta)\|Nx - Ny\|^2 + \gamma\|x-y\|^2,$$

for all $(x,y,u) \in H \times H \times H$.

**Proposition 2.8** [8]. Let $H : X \to X$ be a strictly monotone mapping and $M : X \to 2^X$ an $H$-monotone mapping. Then the operator $(H + \rho M)^{-1}$ is single-valued.

**Proposition 2.9** [20]. Let $A : H \to H$ be an $r$-strongly monotone mapping and $M : H \to 2^H$ an $A$-monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.

**Proposition 2.10** [19]. Let $\eta : H\times \to H$ a single-valued mapping, $A : H \to H$ $(r,\eta)$-strongly monotone mapping and $M : H \to 2^H$ an $(A,\eta)$-monotone mapping. Then the mapping $(A+\rho M)^{-1}$ is single-valued.

## 3. Algorithm

Let $N_1, N_2 : H \to H$, $\eta_1, \eta_2 : H\times H \to H$ $g_1, g_2 : H \to H$ be six nonlinear mappings. Let $M_1 : H \to 2^H$ be an $(A,\eta)$-monotone mapping and $M_2 : H \to 2^H$ an $(A_2,\eta_2)$-monotone mapping, respectively. Then the nonlinear system of variational inclusion (NSVI) problem: determine elements $u, v \in H$ such that

$$0 \in A_1 g_1(u) - A_1 g_1(v) + \rho_1[N_1 v + M_1 g_1(u)], \tag{3.1}$$

$$0 \in A_2 g_2(v) - A_2 g_2(u) + \rho_2[N_2 u + M_2 g_2(v)]. \tag{3.2}$$

Next, we consider some special cases of NSVI problem (3.1)-(3.2).

(I) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $\eta_1 = \eta_2 = \eta$, $g_1 = g_2 = g$ and $N_1 = N_2 = N$, then NSVI problem (3.1)-(3.2) is reduced to the following NSVI problem: find $u, v \in H$ such that

$$0 \in Ag(u) - Ag(v) + \rho_1[Nv + Mg(u)], \tag{3.3}$$

$$0 \in Ag(v) - Ag(u) + \rho_2[Nu + Mg(v)]. \tag{3.4}$$

(II) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $\eta_1 = \eta_2 = \eta$, $g_1 = g_2 = I$ and $N_1 = N_2 = N$, then NSVI problem (3.1)-(3.2) is reduced to the following NSVI problem: find $u, v \in H$ such that

$$0 \in Au - Av + \rho_1(Nv + Mu), \tag{3.5}$$

$$0 \in Av - Au + \rho_2(Nu + Mv). \tag{3.6}$$

(III) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $N_1 = N_2 = N$, $u = v$, $\eta_1 = \eta_2 = \eta$, $g_1 = g_2 = g$ and $\rho_1 = \rho_2 = \rho$ in NSVI (3.1)-(3.2), we have the following NVI problem: find an element $u \in H$ such that

$$0 \in Nu + Mg(u), \tag{3.7}$$

(IV) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $N_1 = N_2 = N$, $u = v$, $\eta_1 = \eta_2 = \eta$, $g_1 = g_2 = I$ and $\rho_1 = \rho_2 = \rho$ in NSVI (3.1)-(3.2), we have the following NVI problem: find an element $u \in H$ such that

$$0 \in Nu + Mu, \tag{3.8}$$

In order to prove our main results, we need the following lemmas.

**Lemma 3.1.** *Let $H$ be a real Hilbert space and let $\eta : H \times H \to H$ be a $\tau$-Lipschitz continuous nonlinear mapping. Let $A : H \to H$ be a $(r, \eta)$-strongly monotone and let $M : H \to 2^H$ be $(A, \eta)$-monotone. Then the generalized resolvent operator $J^{A,\eta}_{M,\rho} : H \to H$ is $\tau/(r - \rho m)$, that is,*

$$\|J^{A,\eta}_{M,\rho}(x) - J^{A,\eta}_{M,\rho}(y)\| \leq \frac{\tau}{r - \rho m}\|x - y\|, \quad \forall x, y \in H.$$

**Lemma 3.2.** *Let $H$ be a real Hilbert space. Let $A_i : H \to H$ be a $(r_i, \eta_i)$-strongly monotone mapping, $M_i : H \to 2^H$ an $(A_i, \eta_i)$-monotone mapping and $\eta_i : H \times H \to H$ an $\tau_i$-Lipschitz continuous nonlinear mapping for each $i = 1, 2$. Then $(u, v)$ is the solution of NSVI (3.1)-(3.2) if and only if it satisfies*

$$g_1(u) = J^{A_1,\eta_1}_{M_1,\rho_1}[A_1 g_1(v) - \rho_1 N_1 v], \tag{3.9}$$

$$g_2(v) = J^{A_2,\eta_2}_{M_2,\rho_2}[A_2 g_2(u) - \rho_2 N_2 u]. \tag{3.10}$$

Next, we construct the following iterative algorithms based on (3.9)-(3.10).

**Algorithm 3.1.** For any $u_0, v_0 \in H$, compute the sequences $\{u_n\}$ and $\{v_n\}$ by the iterative process:

$$\begin{cases} u_{n+1} = u_n - g(u_n) + J^{A_1,\eta_1}_{M_1,\rho_1}[A_1 g(v_n) - \rho_1 N_1 v_n], \quad n \geq 0 \\ g(v_n) = J^{A_2,\eta_2}_{M_2,\rho_2}[A_2 g(u_n) - \rho_2 N_2 u_n], \quad n \geq 0. \end{cases}$$

(I) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $\eta_1 = \eta_2 = \eta$, $g_1 = g_2 = g$ and $N_1 = N_2 = N$ in Algorithm 3.1, then we have the following algorithm:

**Algorithm 3.2.** For any $u_0, v_0 \in H$, compute the sequence $\{u_n\}$ and $\{v_n\}$ by the iterative process:

$$\begin{cases} u_{n+1} = u_n - g(u_n) + J_{M,\rho_1}^{A,\eta}[Ag(v_n) - \rho_1 N v_n], \quad n \geq 0 \\ g(v_n) = J_{M,\rho_2}^{A,\eta}[Ag(u_n) - \rho_2 N u_n], \quad n \geq 0. \end{cases}$$

**Remark 3.1.** Algorithm 3.2 gives the approximate solution to the NSVI (3.3)-(3.4).

(II) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $\eta_1 = \eta_2 = \eta$, $g_1 = g_2 = I$ and $N_1 = N_2 = N$ in Algorithm 3.1, then we have the following algorithm:

**Algorithm 3.3.** For any $u_0, v_0 \in H$, compute the sequence $\{u_n\}$ by the iterative processes:

$$\begin{cases} u_{n+1} = J_{M,\rho_1}^{A,\eta}[Av_n - \rho_1 N v_n], \quad n \geq 0, \\ v_n = J_{M,\rho_2}^{A,\eta}[Au_n - \rho_2 N u_n], \quad n \geq 0. \end{cases}$$

**Remark 3.2.** Algorithm 3.3 gives the approximate solution to the NSVI (3.5)-(3.6).

(III) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $N_1 = N_2 = N$, $u = v$, $\eta_1 = \eta_2 = \eta$, $g_1 = g_2 = g$ and $\rho_1 = \rho_2 = \rho$ in Algorithm 3.1, then we have the following algorithm:

**Algorithm 3.4.** For any $u_0 \in H$, compute the sequence $\{u_n\}$ by the iterative processes:

$$u_{n+1} = u_n - g(u_n) + J_{M,\rho}^{A,\eta}[Ag(u_n) - \rho N u_n], \quad n \geq 0.$$

**Remark 3.3.** Algorithm 3.4 gives the approximate solution to the NVI (3.7).

(IV) If $A_1 = A_2 = A$, $M_1 = M_2 = M$, $N_1 = N_2 = N$, $u = v$, $\eta_1 = \eta_2 = \eta$, $g_1 = g_2 = I$ and $\rho_1 = \rho_2 = \rho$ in Algorithm 3.1, then we have the following algorithm:

**Algorithm 3.5.** For any $u_0 \in H$, compute the sequence $\{u_n\}$ by the iterative processes:

$$u_{n+1} = J_{M,\rho}^{A,\eta}[Au_n - \rho N u_n], \quad n \geq 0.$$

**Remark 3.4.** Algorithm 3.5 gives the approximate solution to the NVI (3.8).

## 4. Results on algorithmic convergence analysis

**Theorem 4.1.** *Let $H$ be a real Hilbert space. Let $A_i : H \times H$ be a $(r_i, \eta_i)$-strongly monotone and $s_i$-Lipschitz continuous mapping and $M_i : H \to 2^H$ an $(A_i, \eta_i)$-monotone mapping. Let $\eta_i : H \times H \to H$ be a $\tau_i$-Lipschitz continuous mapping and $N_i : H \times H \to H$ a relaxed $(\alpha_i, \beta_i)$-cocoercive (with respect to $A_i g_i$) and $\mu_i$-Lipschitz continuous mapping. Let $g_i : H \to H$ be relaxed $(\gamma_i, \delta_i)$-cocoercive and $\sigma_i$-Lipschitz for $i = 1, 2$. Let $(u^*, v^*)$ be the solution of NSVI problem (3.1)-(3.2). Let $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.1. Suppose that the following condition is satisfied:*

$$\tau_1 \tau_2 \theta_1 \theta_2 < (1 - \theta_3)(1 - \theta_4)(r_1 - \rho_1 m_1)(r_2 - \rho_2 m_2),$$

*where*

$$\theta_1 = \sqrt{\sigma_1^2 s_1^2 - 2\rho_1 \beta_1 + 2\rho_1 \alpha_1 \mu_1^2 + \rho_1^2 \mu_1^2}, \theta_2 = \sqrt{\sigma_2^2 s_2^2 - 2\rho_2 \beta_2 + 2\rho_2 \alpha_2 \mu_2^2 + \rho_2^2 \mu_2^2},$$

$$\theta_3 = \sqrt{1 + 2\sigma_2^2 \gamma_2 - 2\delta_2 + \sigma_2^2}$$

*and*

$$\theta_4 = \sqrt{1 + 2\sigma_1^2 \gamma_1 - 2\delta_1 + \sigma_1^2}.$$

*Then the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to $u^*$ and $v^*$, respectively.*

**Proof.** Letting $(u^*, v^*) \in H$ be the solution of NSVI problem (3.1)-(3.2), we have

$$\begin{cases} u^* = u^* - g_1(u^*) + J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v^*) - \rho_1 N_1 v^*], \\ g_2(v^*) = J_{M_2,\rho_2}^{A_2,\eta_2}[A_2 g_2(u^*) - \rho_2 N_2 u^*]. \end{cases}$$

It follows that

$$\begin{aligned} \|u_{n+1} - u^*\| &= \|u_n - g_1(u_n) + J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v_n) - \rho_1 N_1 v_n] - u^*\| \\ &= \|u_n - u^* - \big(g_1(u_n) + g_1(u^*)\big) + J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v_n) - \rho_1 N_1 v] \\ &\quad - J_{M_1,\rho_1}^{A_1,\eta_1}\big(A_1 g_1(v^*) - \rho_1 N_1 v^*\big)\| \\ &\le \|u_n - u^* - \big(g_1(u_n) - g_1(u^*)\big)\| \qquad (4.1) \\ &\quad + \|J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v_n) - \rho_1 N_1 v_n] - J_{M_1,\rho_1}^{A_1,\eta_1}[A_1 g_1(v^*) - \rho_1 N_1 v^*]\| \\ &\le \|u_n - u^* - \big(g_1(u_n) - g_1(u^*)\big)\| \\ &\quad + \frac{\tau_1}{r_1 - \rho_1 m_1}\|A_1 g_1(v_n) - A_1 g_1(v^*) - \rho_1(N_1 v_n - N_1 v^*)\|. \end{aligned}$$

It follows from relaxed $(\alpha_1, \beta_1)$-cocoercive monotonicity and $\mu_1$-Lipschitz continuity of $N_1$, $A_1$ is $s_1$-Lipschitz continuous and $g_1$ is $\sigma_1$-Lipschitz continuous that

$$\begin{aligned} &\|A_1g_1(v_n) - A_1g_1(v^*) - \rho_1(N_1v_n - N_1v^*)\|^2 \\ &= \|A_1g_1(v_n) - A_1g_1(v^*)\|^2 - 2\rho_1\langle N_1v_n - N_1v^*, A_1g_1(v_n) - A_1g_1(v^*)\rangle \\ &\quad + \rho_1^2\|N_1v_n - N_1v^*\|^2 \\ &\leq \theta_1^2\|v_n - v^*\|^2, \end{aligned} \tag{4.2}$$

where

$$\theta_1 = \sqrt{\sigma_1^2 s_1^2 - 2\rho_1\beta_1 + 2\rho_1\alpha_1\mu_1^2 + \rho_1^2\mu_1^2}.$$

On the other hand, we have

$$\begin{aligned} \|g_2(v_n) - g_2(v^*)\| &= \|J_{M_2,\rho_2}^{A_2,\eta_2}[A_2g(u_n) - \rho_2N_2u_n] - J_{M_2,\rho_2}^{A_2,\eta_2}[A_2g_2(u^*) - \rho_2N_2u^*]\| \\ &\leq \frac{\tau_2}{r_2 - \rho_2m_2}\|A_2g(u_n) - A_2g_2(u^*) - \rho_2[N_2u_n - N_2u^*]\|. \end{aligned} \tag{4.3}$$

It follows from relaxed $(\alpha_2, \beta_2)$-cocoercive monotonicity and $\mu_2$-Lipschitz continuity of $N_2$, $A_2$ is $s_2$-Lipschitz continuous and $g_2$ is $\sigma_2$-Lipschitz continuous that

$$\begin{aligned} &\|A_2g_2(u_n) - A_2g_2(u^*) - \rho(N_2u_n - N_2u^*)\|^2 \\ &= \|A_2g_2(u_n) - A_2g_2(u^*)\|^2 - 2\rho_2\langle N_2u_n - N_2u^*, A_2g_2(u_n) - A_2g_2(u^*)\rangle \\ &\quad + \rho_2^2\|N_2u_n - N_2u^*\|^2 \\ &\leq \theta_2^2\|u_n - u^*\|^2, \end{aligned} \tag{4.4}$$

where

$$\theta_2 = \sqrt{\sigma_2^2 s_2^2 - 2\rho_2\beta_2 + 2\rho_2\alpha_2\mu_2^2 + \rho_2^2\mu_2^2}.$$

Substituting (4.4) into (4.3), we obtain that

$$\|g_2(v_n) - g_2(v^*)\| \leq \frac{\tau_2\theta_2}{r_2 - \rho_2m_2}\|u_n - u^*\|. \tag{4.5}$$

Note that

$$\|v_n - v^*\| \leq \|v_n - v^* - \big(g_2(v_n) - g_2(v^*)\big)\| + \|g_2(v_n) - g_2(v^*)\|. \tag{4.6}$$

From the relaxed $(\gamma_2, \delta_2)$-cocoercive monotonicity and $\sigma_2$-Lipschitz continuity of $g_2$ that

$$
\begin{aligned}
&\|v_n - v^* - \big(g_2(v_n) - g_2(v^*)\big)\|^2 \\
&= \|v_n - v^*\|^2 - 2\langle g_2(v_n) - g_2(v^*), v_n - v^*\rangle + \|g_2(v_n) - g_2(v^*)\|^2 \\
&\le \|v_n - v^*\|^2 - 2\big(-\gamma_2\|g_2(v_n) - g_2(v^*)\|^2 + \delta_2\|v_n - v^*\|^2\big) + \|g_2(v_n) - g_2(v^*)\|^2 \quad (4.7) \\
&\le \|v_n - v^*\|^2 + 2\sigma_2^2\gamma_2\|v_n - v^*\|^2 - 2\delta_2\|v_n - v^*\|^2 + \sigma_2^2\|v_n - v^*\|^2 \\
&= \theta_3^2\|v_n - v^*\|^2,
\end{aligned}
$$

where

$$\theta_3 = \sqrt{1 + 2\sigma_2^2\gamma_2 - 2\delta_2 + \sigma_2^2}.$$

Substituting (4.5) and (4.7) into (4.6) yields that

$$\|v_n - v^*\| \le \theta_3\|v_n - v^*\| + \frac{\tau_2\theta_2}{r_2 - \rho_2 m_2}\|u_n - u^*\|.$$

It follows that

$$\|v_n - v^*\| \le \frac{\tau_2\theta_2}{(1-\theta_3)(r_2 - \rho_2 m_2)}\|u_n - u^*\|. \tag{4.8}$$

Substituting (4.8) into (4.2), we arrive at

$$
\begin{aligned}
&\|A_1 g_1(v_n) - A_1 g_1(v^*) - \rho_1(N_1 v_n - N_1 v^*)\| \\
&\le \frac{\tau_2\theta_2\theta_1}{(1-\theta_3)(r_2 - \rho_2 m_2)}\|u_n - u^*\|,
\end{aligned} \tag{4.9}
$$

On the other hand, it follows from relaxed $(\gamma_1, \delta_1)$-cocoercive monotonicity and $\sigma_1$-Lipschitz continuity of $g_1$ that

$$
\begin{aligned}
&\|u_n - u^* - g_1(u_n) - g_1(u^*)\|^2 \\
&= \|u_n - u^*\|^2 - 2\langle g_1(u_n) - g_1(u^*), u_n - u^*\rangle + \|g_1(u_n) - g_1(u^*)\|^2 \\
&\le \|u_n - u^*\|^2 - 2\big(-\gamma_1\|g_1(u_n) - g_1(u^*)\|^2 + \delta_1\|u_n - u^*\|^2\big) + \|g_1(u_n) - g_1(u^*)\|^2 \quad (4.10) \\
&\le \|u_n - u^*\|^2 + 2\sigma_1^2\gamma_1\|u_n - u^*\|^2 - 2\delta_1\|u_n - u^*\|^2 + \sigma_1^2\|u_n - u^*\|^2 \\
&= \theta_4^2\|u_n - u^*\|^2,
\end{aligned}
$$

where

$$\theta_4 = \sqrt{1 + 2\sigma_1^2\gamma_1 - 2\delta_1 + \sigma_1^2}.$$

Substituting (4.9) and (4.10) into (4.1), we obtain that

$$\|u_{n+1}-u^*\| \leq [\theta_4 + \frac{\tau_1\tau_2\theta_1\theta_2}{(r_1-\rho_1 m_1)(1-\theta_3)(r_2-\rho_2 m_2)}]\|u_n-u^*\|. \tag{3.21}$$

In view of the condition

$$\tau_1\tau_2\theta_1\theta_2 < (1-\theta_3)(1-\theta_4)(r_1-\rho_1 m_1)(r_2-\rho_2 m_2),$$

we can obtain the desired conclusion. This completes the proof.

From Theorem 4.1, we have the following results immediately.

**Corollary 4.2.** *Let $H$ be a real Hilbert space. Let $A: H\times H$ be a $(r,\eta)$-strongly monotone and $s$-Lipschitz continuous mapping and $M: H\to 2^H$ an $(A,\eta)$-monotone mapping. Let $\eta: H\times H\to H$ be a $\tau$-Lipschitz continuous mapping and $N: H\times H\to H$ a relaxed $(\alpha,\beta)$-cocoercive (with respect to $Ag$) and $\mu$-Lipschitz continuous mapping. Let $g: H\to H$ be relaxed $(\gamma,\delta)$-cocoercive and $\sigma$-Lipschitz. Let $(u^*,v^*)$ be the solution of NSVI problem (3.3)-(3.4). Let $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.2. Suppose that the following condition is satisfied:*

$$\tau\theta < (1-\theta')(r-\rho m),$$

*where*

$$\theta = \sqrt{\sigma^2 s^2 - 2\rho\beta + 2\rho\alpha\mu^2 + \rho^2\mu^2}$$

*and*

$$\theta' = \sqrt{1+2\sigma^2\gamma - 2\delta + \sigma^2}.$$

*Then the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to $u^*$ and $v^*$, respectively.*

**Corollary 4.3.** *Let $H$ be a real Hilbert space. Let $A: H\times H$ be a $(r,\eta)$-strongly monotone and $s$-Lipschitz continuous mapping and $M: H\to 2^H$ an $(A,\eta)$-monotone mapping. Let $\eta: H\times H\to H$ be a $\tau$-Lipschitz continuous mapping and $N: H\times H\to H$ a relaxed $(\alpha,\beta)$-cocoercive (with respect to $A$) and $\mu$-Lipschitz continuous mapping. Let $(u^*,v^*)$ be the solution of NSVI problem (3.5)-(3.6). Let $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.3. Suppose that the following condition is satisfied:*

$$\tau\sqrt{s^2 - 2\rho\beta + 2\rho\alpha\mu^2 + \rho^2\mu^2} < (r-\rho m).$$

*Then the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to $u^*$ and $v^*$, respectively.*

**Corollary 4.4.** *Let $H$ be a real Hilbert space. Let $A : H \times H$ be a $(r, \eta)$-strongly monotone and $s$-Lipschitz continuous mapping and $M : H \to 2^H$ an $(A, \eta)$-monotone mapping. Let $\eta : H \times H \to H$ be a $\tau$-Lipschitz continuous mapping and $N : H \times H \to H$ a relaxed $(\alpha, \beta)$-cocoercive (with respect to $Ag$) and $\mu$-Lipschitz continuous mapping. Let $g : H \to H$ be relaxed $(\gamma, \delta)$-cocoercive and $\sigma$-Lipschitz. Let $u^*$ be the solution of NVI problem (3.7). Let $\{u_n\}$ be a sequence generated by Algorithm 3.4. Suppose that the following condition is satisfied:*

$$\tau\theta < (1 - \theta')(r - \rho m),$$

*where*

$$\theta = \sqrt{\sigma^2 s^2 - 2\rho\beta + 2\rho\alpha\mu^2 + \rho^2\mu^2}$$

*and*

$$\theta' = \sqrt{1 + 2\sigma^2\gamma - 2\delta + \sigma^2}.$$

*Then the sequence $\{u_n\}$ converges strongly to $u^*$.*

**Corollary 4.5.** *Let $H$ be a real Hilbert space. Let $A : H \times H$ be a $(r, \eta)$-strongly monotone and $s$-Lipschitz continuous mapping and $M : H \to 2^H$ an $(A, \eta)$-monotone mapping. Let $\eta : H \times H \to H$ be a $\tau$-Lipschitz continuous mapping and $N : H \times H \to H$ a relaxed $(\alpha, \beta)$-cocoercive (with respect to $A$) and $\mu$-Lipschitz continuous mapping. Let $u^*$ be the solution of NVI problem (3.8). Let $\{u_n\}$ be a sequence generated by Algorithm 3.5. Suppose that the following condition is satisfied:*

$$\tau\sqrt{s^2 - 2\rho\beta + 2\rho\alpha\mu^2 + \rho^2\mu^2} < (r - \rho m).$$

*Then the sequence $\{u_n\}$ converges strongly to $u^*$.*

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