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ON THE CONVERGENCE OF THE MULTI-STEP NOOR FIXED POINT ITERATIVE SCHEME WITH ERRORS FOR ZAMFIRESCU OPERATORS

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Abstract. The purpose of this paper is to establish a general theorem to approximate fixed points of Zamfirescu operators on an arbitrary normed space through the multi-step Noor fixed point iterative scheme with errors in the sense of Plubtieng and Wangkeeree [S. Plubtieng and R. Wangkeeree, Strong convergence theorem for multi-step Noor iterations with errors in Banach spaces, J. Math. Anal. Appl. 321 (2006), 10-23, 2006]. Our result generalizes and improves the corresponding results of Rafiq [A. Rafiq, A Convergence Theorem for Mann Fixed Point Iteration Procedure, Appl. Math. E-Notes, 6, 289-293], Xu [Y. Xu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl. 224 (1998), 91-101], Liu [L. S. Liu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl. 194 (1995), 114-125], Osilike [M. O. Osilike, Ishikawa and Mann iteration methods with errors for nonlinear equations of the accretive type, J. Math. Anal. Appl. 213 (1997), 91-105] and several authors in literature.

Keywords: Noor iterative scheme with errors; multi-step Noor fixed point iterative scheme with errors; Zamfirescu operators.

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1. Introduction and Preliminaries

Throughout this paper $\mathbf{N}$ will denote the set of all positive integers. Let B be a nonempty subset of an arbitrary normed space X and let T be a mapping from B into itself. The point $x \in B$ is called a fixed point of the mapping T if and only if $Tx = x$ and the set of fixed points of an mapping T is denoted by $F(T) = \{p \in X : T(p) = p\}$.

In 2006, Plubtieng and Wangkeeree [1] introduced the following multi-step iterative scheme: For a given, $u_0 \in B$ and a fixed $m \in \mathbf{N}$, compute the iterative sequences

$$\left\{u_n^{(1)}\right\}, \left\{u_n^{(2)}\right\}, \left\{u_n^{(3)}\right\}, \cdots, \left\{u_n^{(m-1)}\right\}, \left\{u_n^{(m)}\right\}$$

defined by

$$\left\{\begin{array}{l} u_n^{(1)} = a_n^{(1)} T u_n + b_n^{(1)} u_n + c_n^{(1)} v_n^{(1)} \\ u_n^{(2)} = a_n^{(2)} T u_n^{(1)} + b_n^{(2)} u_n + c_n^{(2)} v_n^{(2)} \\ u_n^{(3)} = a_n^{(3)} T u_n^{(2)} + b_n^{(3)} u_n + c_n^{(3)} v_n^{(3)} \\ \vdots \\ \vdots \\ u_n^{(m-1)} = a_n^{(m-1)} T u_n^{(m-2)} + b_n^{(m-1)} u_n + c_n^{(m-1)} v_n^{(m-1)} \\ u_{n+1} = u_n^{(m)} = a_n^{(m)} T u_n^{(m-1)} + b_n^{(m)} u_n + c_n^{(m)} v_n^{(m)}, n \in \mathbf{N} \end{array}\right\} \tag{1.1}$$

where, $\left\{v_n^{(1)}\right\}, \left\{v_n^{(2)}\right\}, \left\{v_n^{(3)}\right\}, \cdots, \left\{v_n^{(m-1)}\right\}, \left\{v_n^{(m)}\right\}$ are bounded sequences in B and $\left\{a_n^{(i)}\right\}$, $\left\{b_n^{(i)}\right\}, \left\{c_n^{(i)}\right\}$ are appropriate real sequences in $[0,1]$ such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for each $i \in \{1, 2, \cdots, m\}$.

Iterative scheme (1.1) is known multi-step Noor fixed point iterative scheme with errors. The Mann iterative scheme, Ishikawa iterative scheme, Noor iterative scheme, Mann iterative scheme with errors, Ishikawa iterative scheme with errors and Noor iterative scheme with errors all are special case of the iterative schemes (1.1), which will be trustworthy by the following discussion: If $m = 3$ and $b_n^{(i)} = 1 - a_n^{(i)} - c_n^{(i)}$ for all $i = 1, 2, 3$ then (1.1) reduces to the following

Noor iterative scheme with errors defined by Cho et al. [2]:

$$\left\{\begin{array}{l} u_n^{(1)} = a_n^{(1)} T u_n + \left(1 - a_n^{(1)} - c_n^{(1)}\right) u_n + c_n^{(1)} v_n^{(1)} \\ u_n^{(2)} = a_n^{(2)} T u_n^{(1)} + \left(1 - a_n^{(2)} - c_n^{(2)}\right) u_n + c_n^{(2)} v_n^{(2)} \\ u_{n+1} = u_n^{(3)} = a_n^{(3)} T u_n^{(2)} + \left(1 - a_n^{(3)} - c_n^{(3)}\right) u_n + c_n^{(3)} v_n^{(2)}, \, n \in \mathbf{N} \end{array}\right\} \tag{1.2}$$

where $\left\{a_n^{(i)}\right\}, \left\{c_n^{(i)}\right\}$ are appropriate real sequences in $[0,1]$ for all $i \in \{1,2,3\}$.

If $m = 3$ and $b_n^{(i)} = 1 - a_n^{(i)} - c_n^{(i)}$ for all $i = 1,2,3$ and $c_n^{(1)} = c_n^{(2)} = c_n^{(3)} \equiv 0$, then (1.1) reduces to the following Noor iterative scheme defined by Noor et al. [3-6]:

$$\left\{\begin{array}{l} u_n^{(1)} = a_n^{(1)} T u_n + \left(1 - a_n^{(1)}\right) u_n \\ u_n^{(2)} = a_n^{(2)} T u_n^{(1)} + \left(1 - a_n^{(2)}\right) u_n \\ u_{n+1} = u_n^{(3)} = a_n^{(3)} T u_n^{(2)} + \left(1 - a_n^{(3)}\right) u_n, \, n \in \mathbf{N} \end{array}\right\} \tag{1.3}$$

where $\left\{a_n^{(i)}\right\}$ are appropriate real sequences in $[0,1]$ for all $i \in \{1,2,3\}$.

If $m = 2$, then (1.1) reduces to the following Ishikawa iterative scheme with errors defined by Xu [7]:

$$\left\{\begin{array}{l} u_n^{(1)} = a_n^{(1)} T u_n + b_n^{(1)} u_n + c_n^{(1)} v_n^{(1)} \\ u_{n+1} = u_n^{(2)} = a_n^{(2)} T u_n^{(1)} + b_n^{(2)} u_n + c_n^{(2)} v_n^{(2)}, \, n \in \mathbf{N} \end{array}\right\} \tag{1.4}$$

where, $\left\{a_n^{(i)}\right\}, \left\{b_n^{(i)}\right\}, \left\{c_n^{(i)}\right\}$ are appropriate real sequences in $[0,1]$ for all $i \in \{1,2\}$.

If $m = 2$ and $c_n^{(1)} = c_n^{(2)} \equiv 1$, then (1.1) reduces to the following Ishikawa iterative scheme with errors defined by Lu [8]:

$$\left\{\begin{array}{l} u_n^{(1)} = a_n^{(1)} T u_n + b_n^{(1)} u_n + v_n^{(1)} \\ u_{n+1} = u_n^{(2)} = a_n^{(2)} T u_n^{(1)} + b_n^{(2)} u_n + v_n^{(2)}, \, n \in \mathbf{N} \end{array}\right\} \tag{1.5}$$

where, $\left\{a_n^{(i)}\right\}, \left\{b_n^{(i)}\right\}$ are appropriate real sequences in $[0,1]$ for all $i \in \{1,2\}$.

If $m = 2$, $b_n^{(i)} = 1 - a_n^{(i)} - c_n^{(i)}$ for all $i = 1,2$ and $c_n^{(1)} = c_n^{(2)} \equiv 0$, then (1.1) reduces to the following Ishikawa iterative scheme defined by Ishikawa [9]:

$$\left\{\begin{array}{l} u_n^{(1)} = a_n^{(1)} T u_n + \left(1 - a_n^{(1)}\right) u_n \\ u_{n+1} = u_n^{(2)} = a_n^{(2)} T u_n^{(1)} + \left(1 - a_n^{(2)}\right) u_n, \, n \in \mathbf{N} \end{array}\right\} \tag{1.6}$$

where $\left\{a_n^{(i)}\right\}$ are appropriate real sequences in $[0,1]$ for all $i \in \{1,2\}$.

If $m=1$, then (1.1) reduces to the following Mann iterative scheme with errors defined by Y. Xu [7]:

$$u_{n+1} = u_n^{(1)} = a_n^{(1)}Tu_n + b_n^{(1)}u_n + c_n^{(1)}v_n^{(1)}, n \in \mathbf{N} \tag{1.7}$$

where, $\left\{a_n^{(1)}\right\}, \left\{b_n^{(1)}\right\}, \left\{c_n^{(1)}\right\}$ are appropriate real sequences in $[0,1]$.

If $m=1$ and $c_n^{(1)}=1$, then (1.1) reduces to the following Mann iterative scheme with errors defined by Lu [8]:

$$u_{n+1} = u_n^{(1)} = a_n^{(1)}Tu_n + b_n^{(1)}u_n + v_n^{(1)}, n \in \mathbf{N}, \tag{1.8}$$

where $\left\{a_n^{(1)}\right\}, \left\{b_n^{(1)}\right\}$ are appropriate real sequences in $[0,1]$.

If $m=1$ and $b_n^{(1)} = 1-a_n^{(1)}-c_n^{(1)}$ and $c_n^{(1)} \equiv 0$, then (1.1) reduces to the following Mann iterative scheme defined by Mann [10]:

$$u_{n+1} = u_n^{(1)} = a_n^{(1)}Tu_n + \left(1-a_n^{(1)}\right)u_n, n \in \mathbf{N}, \tag{1.9}$$

where $\left\{a_n^{(1)}\right\}$ are appropriate real sequences in $[0,1]$.

If $m=1$ and $b_n^{(1)} = 1-a_n^{(1)}-c_n^{(1)}$, $c_n^{(1)} \equiv 0$ and $a_n^{(1)} = \lambda \in (0,1)$, then (1.1) reduces to the following Krasnoselskij iterative scheme defined by Krasnoselskij [11]:

$$u_{n+1} = u_n^{(1)} = \lambda Tu_n + (1-\lambda)u_n, n \in \mathbf{N}. \tag{1.10}$$

Therefore, it is clear from above discussion that multi-step Noor fixed point iterative scheme with errors (1.1) is a general iterative scheme among the analogous iterative schemes. From this point of view, here we have established convergence theorem for multi-step Noor fixed point iterative scheme with errors, which generates the convergence theorem of other relevant iterative schemes.

The celebrated Banachs fixed point theorem is one of the most useful results in metric fixed point theory. It can be briefly stated as follow.

Theorem 1.1. *Let (X,d) be a complete metric space and let $T: X \longrightarrow X$ be a contraction mapping on X, i.e. a map satisfying*

$$d(Tx,Ty) \leq \alpha d(x,y), for all\, x,y \in X, \tag{1.11}$$

where $0 < \alpha < 1$ is a real constant. Then the mapping T admit a unique fixed point $p \in X$ and the Picard iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, n \in \boldsymbol{N} \tag{1.12}$$

converges to p, for any $x_0 \in X$.

Theorem 1.1 has many advantages to solving nonlinear problems, but it has one disadvantage - the contractive condition (1.11) forces to continuous mapping on X.

In 1968, Kannan [12], obtained a fixed point theorem which extends the Theorem 1.1 to mappings that need not be continuous, by considering instead of (1.11) the condition: there exists $\beta \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right], for all\, x, y \in X, \tag{1.13}$$

If a mapping T satisfies (1.3), then it is known as Kannan mapping. By applying Kannan's theorem, a lot of papers were committed to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T , see for instance, Rus [13] and references therein. One of them is Chatterjea's fixed point theorem, and due to Chatterjea's fixed point theorem [14] we can write, there exists $\gamma \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)], for all\, x, y \in X, \tag{1.14}$$

and the mapping which satisfies (1.14) is known as Chatterjea mapping.

In 1972, Zamfirescu [15] combined the above three contractive definitions, defined by (1.11), (1.13) and (1.14), and obtained a very interesting result as follow.

Theorem 1.2. [15]*Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a mapping on X for which there exist the real numbers α, β, γ satisfying $\alpha \in (0, 1), \beta, \gamma \in \left(0, \frac{1}{2}\right)$ such that for each pair $x, y \in X$, at least one of the following is true:*

$$(z_1)\ d(Tx, Ty) \leq \alpha d(x, y),$$

$$(z_2)\ d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right],$$

$$(z_3)\ d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)].$$

Then the mapping T has a unique fixed point $p \in X$ and the Picard iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, n \in \boldsymbol{N}$$

converges to p, for any $x_0 \in X$.

One of the most general contraction condition for which the unique fixed point can be approximated by means of Picard iteration, has been obtained by Ciric [16] in 1974 as follows: there exists $0 < h < 1$ such that

$$d(Tx,Ty) \leq h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \tag{1.15}$$

for all $x, y \in X$.

A mapping satisfying (1.15) is commonly called quasi-contractive operator. It is obvious that each of the conditions (1.11) (1.13), (1.14) and (z_1) to (z_3) implies (1.15). An operator T which satisfies the contractive conditions of the theorem 1.2 will be called a Zamfirescu operator.

One of the most studied classes of quasi contractive type operators is that of Zamfirescu operators, for which all important fixed point iteration schemes, i.e., the Picard [15], Mann [10], Ishikawa [9], Noor [3-6] iterative schemes are known to converge to the unique fixed point of T. Zamfirescu showed in [15] that an operator satisfying conditions in Theorem 1.2 has a unique fixed point that can be approximated using the Picard iteration scheme. Later, Rhoades [17], [18] proved that the Mann and Ishikawa iterative schemes can also be used to approximate fixed points of Zamfirescu operators.

The class of operators satisfying the conditions (z_1) to (z_2) is independent to the class of strictly pseudocontractive operators, see for instance Rhoades [17] and the references therein, which have extensively studied by several authors in the last years. For a recent survey and a comprehensive bibliography, we refer to the Berindes monograph [19]. Recently, Berinde [20-22] proved the convergence theorems in arbitrary Banach spaces of the Mann and Ishikawa iterative associated to Zamfirescu operators for extending the results of Rhoades [17]. In 2006, Rafiq [23] extends the result of Berinde [20-22]. In 2013, first and third authors of this paper extend the result of Berinde [20-22] and Rafiq [23] for Noor iterative scheme using Zamfirescu operator as follows, see for instance [24].

Theorem 1.3. [24] *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T : B \longrightarrow B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T, where $F(T)$ denotes the set of fixed points of T . Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iterative scheme defined by (1.3) and $x_0 \in B$. Then the Noor iterative scheme $\{x_n\}_{n=0}^{\infty}$ strongly converges to the fixed point $p \in F(T)$.*

There is a certain gap in the above described results. Actually in the above described results, different types of fixed point iterative schemes associated with Zamfirescu operators have been considered without errors. To fill up this gap in 2005 Rafiq [25] established a result for Mann iterative scheme with errors in the sense of Liu [8] using Zamfirescu operator as follows.

Theorem 1.4. [25] *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \longrightarrow C$ be a Zamfirescu operator. Let $\{x_n\}$ be defined by Mann iterative scheme with errors (1.8). If $F(T) \neq \phi, \sum_{n=1}^{\infty} a_n = \infty$, and $\|u_n\| = 0(a_n)$, then $\{x_n\}$ converges strongly to a fixed point of T.*

From this continuation, here we have established a general convergence theorem to approximate fixed point of Zamfirescu operators on a arbitrary normed space through the multi-step Noor fixed point iterative scheme with errors in the sense of Plubtieng and Wangkeeree [1], which generates the rest. So, the main objective of our present paper is to recognized a convergence theorem for muti-step Noor fixed point iterative scheme with errors defined by (1.1) in the class of Zamfirescu operator on arbitrary normed spaces. Our result generalizes and improves the corresponding results of Rafiq [25].

Now we state a lemma which is proved by Osilike [26].

Lemma 1.5. [26] *Let $\{r_n\},\{s_n\},\{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1-s_n)\, r_n + s_n t_n + k_n$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty} s_n = \infty, \lim_{n\longrightarrow\infty} ?t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $\lim_{n\longrightarrow\infty} r_n = 0$.

2. Main result

In this section we state and prove our main result with some appropriate corollaries and finally give some remarks.

Theorem 2.1. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ be a Zamfirescu operator. Let $\left\{u_n^{(k))}\right\}$ be a sequence defined by multi-step Noor fixed point iterative scheme with errors defined by (1.1), for each $k = 1,2,3,\cdots,m$ and $n \in \mathbf{N}$. If $F(T) \neq \phi, \sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0\left(a_n^{(k)}\right)$, for each $k = 1,2,3,\cdots,m$ and $n \in \mathbf{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Proof. According to our assumption T is a Zamfirescu operator, so by Theorem 1.2, we know that T has a unique fixed point in B, say p,

$$i.e., Tp = p. \tag{2.1}$$

Now, we combine the Zamfirescu conditions according to the approach of Berinde [20-22]. Since T is a Zamfirescu operator, hence T is satisfied at least one of the Zamfirescu conditions $(z_1),(z_2)$ and (z_3) defined by the theorem 1.2 . If T satisfies (z_2), then for all $x,y \in B$ we have

$$\begin{aligned}\|Tx - Ty\| &\leq \beta\left[\|x - Tx\| + \|y - Ty\|\right] \\ &\leq \beta\left[\|x - Tx\| + \|y - x\| + \|x - Tx\| + \|Tx - Ty\|\right],\end{aligned}$$

which implies

$$\|Tx - Ty\| \leq \frac{\beta}{1-\beta}\|x - y\| + \frac{2\beta}{1-\beta}\|x - Tx\|. \tag{2.2}$$

If T satisfies (z_3), then for all $x,y \in B$ similarly we obtain

$$\|Tx - Ty\| \leq \frac{\gamma}{1-\gamma}\|x - y\| + \frac{2\gamma}{1-\gamma}\|x - Tx\|. \tag{2.3}$$

Now, if we take

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}. \tag{2.4}$$

Then we have $0 \leq \delta < 1$ and in view of (z_1) and (2.2) to (2.4), we obtained the following inequality.

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\|. \tag{2.5}$$

If we suppose $\left\{u_n^{(k)}\right\}$ be a multi-step Noor fixed point iterative scheme with errors defined by (1.1) and $u_0 \in B$ arbitrary, then we have

$$\|u_{n+1}-p\| = \|a_n^{(m)}Tu_n^{(m-1)} + b_n^{(m)}u_n + c_n^{(m)}v_n^{(m)} - p\|. \tag{2.6}$$

Since $a_n^{(m)} + b_n^{(m)} + c_n^{(m)} = 1$, hence from (2.6) we have

$$\begin{aligned} \|u_{n+1}-p\| &= \|(1-a_n^{(m)})(u_n-p) + a_n^{(m)}(Tu_n^{(m-1)}-p) + c_n^{(m)}(v_n^{(m)}-u_n)\| \\ &\le (1-a_n^{(m)})\|u_n-p\| + a_n^{(m)}\|Tu_n^{(m-1)}-p\| + c_n^{(m)}\|v_n^{(m)}-u_n\|. \end{aligned} \tag{2.7}$$

But according to our assumption $\|v_n^{(m)}-u_n\| = 0(a_n^{(m)})$, hence from (2.7) we have

$$\|u_{n+1}-p\| \le (1-a_n^{(m)})\|u_n-p\| + a_n^{(m)}\|Tu_n^{(m-1)}-p\| + c_n^{(m)}0(a_n^{(m)}). \tag{2.8}$$

Now, if we put $x = u_n^{(m-1)}$ and $y = p$ in (2.5), then we have

$$\|Tu_n^{(m-1)} - Tp\| \le \delta\|u_n^{(m-1)} - p\|, \tag{2.9}$$

where δ is given by (2.4). Combining (2.8) and (2.9), we obtain

$$\|u_{n+1}-p\| \le (1-a_n^{(m)})\|u_n-p\| + a_n^{(m)}\delta\|u_n^{(m-1)}-p\| + c_n^{(m)}0(a_n^{(m)}). \tag{2.10}$$

Further by the definition of multi-step Noor fixed point iterative scheme with errors (1.1), we have

$$\|u_n^{(m-1)}-p\| = \|a_n^{(m-1)}Tu_n^{(m-2)} + b_n^{(m-1)}u_n + c_n^{(m-1)}v_n^{(m-1)} - p\| \tag{2.11}$$

Since$a_n^{(m-1)} + b_n^{(m-1)} + c_n^{(m-1)} = 1$, hence from (2.11) we have

$$\|u_n^{(m-1)}-p\| \le (1-a_n^{(m-1)})\|u_n-p\| + a_n^{(m-1)}\|Tu_n^{(m-2)}-p\| + c_n^{(m-1)}\|v_n^{(m-1)}-u_n\|. \tag{2.12}$$

But according to our assumption $\|v_n^{(m-1)}-u_n\| = 0(a_n^{(m-1)})$, hence from (2.12) we have

$$\|u_n^{(m-1)}-p\| \le (1-a_n^{(m-1)})\|u_n-p\| + a_n^{(m-1)}\|Tu_n^{(m-2)}-p\| + c_n^{(m-1)}0(a_n^{(m-1)}). \tag{2.13}$$

Now, if we put $x = u_n^{(m-2)}$ and $y = p$ in (2.5), then we have

$$\|Tu_n^{(m-2)} - Tp\| \le \delta\|u_n^{(m-2)} - p\|, \tag{2.14}$$

where δ is given by (2.4). Combining (2.13) and (2.14), we obtain

$$\|u_n^{(m-1)}-p\| \le (1-a_n^{(m-1)})\|u_n-p\| + a_n^{(m-1)}\delta\|u_n^{(m-2)}-p\| + c_n^{(m-1)}0(a_n^{(m-1)}). \tag{2.15}$$

From (2.10) and (2.15), we have

$$\begin{aligned}\|u_{n+1}-p\| &\leq (1-a_n^{(m)})\|u_n-p\|+a_n^{(m)}\delta[(1-a_n^{(m-1)})\|u_n-p\| \\ &+a_n^{(m-1)}\delta\|u_n^{(m-2)}-p\|]+c_n^{(m)}0(a_n^{(m)})+a_n^{(m)}c_n^{(m-1)}0(a_n^{(m-1)}) \\ &=(1-a_n^{(m)}+a_n^{(m)}\delta(1-a_n^{(m-1)}))\|u_n-p\|+\delta^2 a_n^{(m)}a_n^{(m-1)}\|u_n^{(m-2)}-p\| \\ &+c_n^{(m)}0(a_n^{(m)})+a_n^{(m)}c_n^{(m-1)}0(a_n^{(m-1)}).\end{aligned} \tag{2.16}$$

Further by the definition of multi-step Noor fixed point iterative scheme with errors (1.1), we have

$$\|u_n^{(m-2)}-p\|=\|a_n^{(m-2)}Tu_n^{(m-3)}+b_n^{(m-2)}u_n+c_n^{(m-2)}v_n^{(m-2)}-p\|. \tag{2.17}$$

Since, $a_n^{(m-2)}+b_n^{(m-2)}+c_n^{(m-2)}=1$, hence from (2.17) we have

$$\|u_n^{(m-2)}-p\|\leq(1-a_n^{(m-2)})\|u_n-p\|+a_n^{(m-2)}\|Tu_n^{(m-3)}-p\|+c_n^{(m-2)}\|v_n^{(m-2)}-u_n\|. \tag{2.18}$$

But according to our assumption $\|v_n^{(m-2)}-u_n\|=0(a_n^{(m-2)})$, hence from (2.18) we have

$$\|u_n^{(m-2)}-p\|\leq(1-a_n^{(m-2)})\|u_n-p\|+a_n^{(m-2)}\|Tu_n^{(m-3)}-p\|+c_n^{(m-2)}0(a_n^{(m-2)}). \tag{2.19}$$

Now, if we put $x=u_n^{(m-3)}$ and $y=p$ in (2.5), then we have

$$\|Tu_n^{(m-3)}-Tp\|\leq\delta\|u_n^{(m-3)}-p\|, \tag{2.20}$$

where δ is given by (2.4). Combining (2.19) and (2.20), we obtain

$$\|u_n^{(m-2)}-p\|\leq(1-a_n^{(m-2)})\|u_n-p\|+a_n^{(m-2)}\delta\|u_n^{(m-3)}-p\|+c_n^{(m-2)}0(a_n^{(m-2)}). \tag{2.21}$$

From (2.16) and (2.21), we have

$$\begin{aligned}\|u_{n+1}-p\| &\leq \left(1-a_n^{(m)}\right)\|u_n-p\|+a_n^{(m)}\delta[\left(1-a_n^{(m-1)}\right)\|u_n-p\| \\ &+a_n^{(m-1)}\delta[\left(1-a_n^{(m-2)}\right)\|u_n-p\|+a_n^{(m-2)}\delta\|u_n^{(m-3)}-p\|]]+c_n^{(m)}0(a_n^{(m)}) \\ &+a_n^{(m)}c_n^{(m-1)}0(a_n^{(m-1)})+\delta^2 a_n^{(m)}a_n^{(m-1)}c_n^{(m-2)}0(a_n^{(m-2)}) \\ &=(1-a_n^{(m)}+\delta a_n^{(m)}(1-a_n^{(m-1)})+\delta^2 a_n^{(m)}a_n^{(m-1)}(1-a_n^{(m-2)}))\|u_n-p\| \\ &+\delta^3 a_n^{(m)}a_n^{(m-1)}a_n^{(m-2)}\|u_n^{(m-3)}-p\|+c_n^{(m)}0(a_n^{(m)}) \\ &+a_n^{(m)}c_n^{(m-1)}0(a_n^{(m-1)})+\delta^2 a_n^{(m)}a_n^{(m-1)}c_n^{(m-2)}0(a_n^{(m-2)}).\end{aligned} \tag{2.22}$$

Now if we continue the above process until the initial equation of multi-step Noor fixed point iterative scheme with errors defined by (1.1) have been used, then the inequality (2.22) can be written as follows.

$$\begin{aligned}
\|u_{n+1}-p\| \le {} & [1-a_n^{(m)}+\delta a_n^{(m)}\left(1-a_n^{(m-1)}\right)+\delta^2 a_n^{(m)}a_n^{(m-1)}\left(1-a_n^{(m-2)}\right)+\cdots \\
& +\delta^{m-1}a_n^{(m)}a_n^{(m-1)}\cdots a_n^{(3)}a_n^{(2)}\left(1-a_n^{(1)}\right)]\|u_n-p\| \\
& +\delta^m a_n^{(m)}a_n^{(m-1)}\cdots a_n^{(3)}a_n^{(2)}a_n^{(1)}\|u_n-p\|+c_n^{(m)}0\left(a_n^{(m)}\right) \\
& +a_n^{(m)}c_n^{(m-1)}0\left(a_n^{(m-1)}\right)+\delta^2 a_n^{(m)}a_n^{(m-1)}c_n^{(m-2)}0\left(a_n^{(m-2)}\right) \\
& +\cdots+\delta^{m-1}a_n^{(m)}a_n^{(m-1)}\cdots a_n^{(3)}a_n^{(2)}c_n^{(1)}0\left(a_n^{(1)}\right) \\
& =(1-a_n^{(m)}+\delta a_n^{(m)}\left(1-a_n^{(m-1)}\right)+\delta^2 a_n^{(m)}a_n^{(m-1)}\left(1-a_n^{(m-2)}\right)+\cdots \\
& +\delta^{m-1}a_n^{(m)}a_n^{(m-1)}\cdots a_n^{(3)}a_n^{(2)}\left(1-a_n^{(1)}\right)+\delta^m a_n^{(m)}a_n^{(m-1)}\cdots a_n^{(3)}a_n^{(2)}a_n^{(1)}) \\
& \|u_n-p\|+c_n^{(m)}0\left(a_n^{(m)}\right)+a_n^{(m)}c_n^{(m-1)}0\left(a_n^{(m-1)}\right)+\delta^2 a_n^{(m)}a_n^{(m-1)} \\
& c_n^{(m-2)}0\left(a_n^{(m-2)}\right)+\cdots+\delta^{m-1}a_n^{(m)}a_n^{(m-1)}\cdots a_n^{(3)}a_n^{(2)}c_n^{(1)}0\left(a_n^{(1)}\right),
\end{aligned}$$

which yields that

$$\begin{aligned}
\|u_{n+1}-p\| \le {} & [1-(1-\delta)a_n^{(m)}(1-\delta a_n^{(m-1)})(1-\delta a_n^{(m-2)})(1-\delta a_n^{(m-3)})\cdots \\
& (1-\delta a_n^{(1)})]\|u_n-p\|+c_n^{(m)}0(a_n^{(m)})+a_n^{(m)}c_n^{(m-1)}0(a_n^{(m-1)})+\delta^2 a_n^{(m)}a_n^{(m-1)} \\
& c_n^{(m-2)}0(a_n^{(m-2)})\cdots+\delta^{m-1}a_n^{(m)}a_n^{(m-1)}\cdots a_n^{(3)}a_n^{(2)}c_n^{(1)}0(a_n^{(1)}).
\end{aligned} \tag{2.23}$$

But it is clear that,

$$[1-(1-\delta)a_n^{(m)}(1-\delta a_n^{(m-1)})(1-\delta a_n^{(m-2)})(1-\delta a_n^{(m-3)})\cdots(1-\delta a_n^{(1)})]\le\left[1-(1-\delta)^m a_n^{(m)}\right].$$

Hence form (2.23), we obtain

$$\begin{aligned}
\|u_{n+1}-p\| \le {} & \left[1-(1-\delta)^m a_n^{(m)}\right]\|u_n-p\|+c_n^{(m)}0(a_n^{(m)})+a_n^{(m)}c_n^{(m-1)}0(a_n^{(m-1)}) \\
& +\delta^2 a_n^{(m)}a_n^{(m-1)}c_n^{(m-2)}0(a_n^{(m-2)}) \\
& +\cdots+\delta^{m-1}a_n^{(m)}a_n^{(m-1)}\cdots a_n^{(3)}a_n^{(2)}c_n^{(1)}0(a_n^{(1)}),\ n\in\mathbf{N}.
\end{aligned} \tag{2.24}$$

By (2.24) inductively we obtain

$$\begin{aligned}\|u_{n+1}-p\| \leq \prod_{r=0}^{n}\left[1-(1-\delta)^m a_r^{(m)}\right]\|u_0-p\| + \delta^2 a_n^{(m)} a_n^{(m-1)} c_n^{(m-2)} 0(a_n^{(m-2)}) + \cdots \\ + \delta^{m-1} a_n^{(m)} a_n^{(m-1)} \cdots a_n^{(3)} a_n^{(2)} c_n^{(1)} 0(a_n^{(1)}),\ n \in \mathbf{N}.\end{aligned} \tag{2.25}$$

Now since $0 \leq \delta < 1$, $a_n^{(m)} \in (0,1)$ and $\sum_{n=1}^{\infty} a_n^{(m)} = \infty$, hence by lemma 1.5 we can write

$$\lim_{n\longrightarrow\infty} \prod_{r=0}^{n}\left[1-(1-\delta)^m a_r^{(m)}\right] = 0. \tag{2.26}$$

Taking limit as $n \longrightarrow \infty$ on both sides of (2.25) and using (2.26), we get

$$\lim_{n\longrightarrow\infty} \|u_{n+1}-p\| = 0,$$

which implies that $\left\{u_n^{(k)}\right\}$, $k = 1,2,3,\cdots,m$ converges strongly to $p \in F(T)$. This completes our proof.

Corollary 2.2. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ satisfies the Kannan's contractive conditions defined by (1.13). Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by multi-step Noor fixed point iterative scheme with errors defined by (1.1), for each $k = 1,2,3,\cdots,m$ and $n \in \mathbf{N}$. If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, for each $k = 1,2,3,\cdots,m$ and $n \in \mathbf{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Corollary 2.3. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ satisfies the Chatterjea's contractive conditions defined by (1.14). Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by multi-step Noor fixed point iterative scheme with errors defined by (1.1), for each $k = 1,2,3,\cdots,m$ and $n \in \mathbf{N}$. If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, for each $k = 1,2,3,\cdots,m$ and $n \in \mathbf{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Corollary 2.4. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by Noor iterative scheme with errors defined by (1.2), for each $k = 1,2,3$ and $n \in \mathbf{N}$. If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, for each $k = 1,2,3$ and $n \in \mathbf{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Corollary 2.5. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by Noor iterative scheme defined by (1.3), for each $k = 1,2,3$ and $n \in \boldsymbol{N}$. If $F(T) \neq \phi, \sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, for each $k = 1,2,3$ and $n \in \boldsymbol{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Corollary 2.6. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by Ishikawa iterative scheme with errors defined by (1.4), for each $k = 1,2$ and $n \in \boldsymbol{N}$. If $F(T) \neq \phi, \sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, for each $k = 1,2$ and $n \in \boldsymbol{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Corollary 2.7. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by Ishikawa iterative scheme with errors defined by (1.5), for each $k = 1,2$ and $n \in \boldsymbol{N}$. If $F(T) \neq \phi, \sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, for each $k = 1,2$ and $n \in \boldsymbol{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Corollary 2.8. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by Ishikawa iterative scheme defined by (1.6), for each $k = 1,2$ and $n \in \boldsymbol{N}$. If $F(T) \neq \phi, \sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, for each $k = 1,2$ and $n \in \boldsymbol{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Corollary 2.9. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by Mann iterative scheme with errors defined by (1.7), for each $k = 1$ and $n \in \boldsymbol{N}$. If $F(T) \neq \phi, \sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, for each $k = 1$ and $n \in \boldsymbol{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.*

Corollary 2.10. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T : B \longrightarrow B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by Mann iterative scheme with errors defined by (1.8), for each $k = 1$ and $n \in \boldsymbol{N}$. If $F(T) \neq \phi, \sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and*

$\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, *for each* $k = 1$ *and* $n \in \boldsymbol{N}$. *Then* $\left\{u_n^{(k)}\right\}$ *converges strongly to a fixed point of* T.

Corollary 2.11. *Let* B *be a nonempty closed convex subset of an arbitrary normed space* X. *Let* $T : B \longrightarrow B$ *be a Zamfirescu operator. Let* $\left\{u_n^{(k)}\right\}$ *be a sequence defined by Mann iterative scheme defined by (1.9), for each* $k = 1$ *and* $n \in \boldsymbol{N}$. *If* $F(T) \neq \phi$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and* $\|v_n^{(k)} - u_n\| = 0(a_n^{(k)})$, *for each* $k = 1$ *and* $n \in \boldsymbol{N}$. *Then* $\left\{u_n^{(k)}\right\}$ *converges strongly to a fixed point of* T.

Remark 2.12. (1) The contractive condition defined by (1.11) makes the mapping T a continuous function on X while this is not the case with the contractive conditions defined by (1.13) to (1.15) and (2.5).

(2) The Kannan's and the Chatterjea's contractive conditions defined by (1.13) and (1.14) respectively are both included in the class of Zamfirescu operators and so their convergence theorems for the multi-step Noor fixed point iterative scheme with errors are obtained in Corollary 2.2 and Corollary 2.3 respectively.

(3) Theorem 3 of Rafiq [25] in the context of Mann iterative scheme with errors on a closed convex normed space has been obtained in Corollary 2.10.

(4) Theorem 3 of Rafiq [23] in the context of Noor iterative scheme on a closed convex normed space has been obtained in Corollary 2.5.

(5) In Corollary 2.8, Theorem 2 of Berinde [21] is generalized to the setting of normed spaces.

(6) In Corollary 2.11, Theorem 2 of Berinde [20] and Theorem 2.1 of Berinde [22] are generalized to the setting of normed spaces.

(7) In Corollary 2.4, the result of Cho, Zhou, Guo [2] is generalized to the class of Zamfirescu operators.

(8) Corollary 2.6 and Corollary 2.9 are used to generalize the result of Xu [7] to the setting for the class of Zamfirescu operators.

(9) Corollary 2.6 and Corollary 2.10 are used to generalize the result of Liu [8] to the setting for the class of Zamfirescu operators.

(10) Our Theorem 2.1 also generalized the result of Osilike [26-28].

3. Conclusions

Our Theorem 2.1 improves the Theorem 3 of Rafiq [25] by extending it from Mann iterative scheme with errors to multi-step Noor fixed point iterative scheme with errors. Since the iterative schemes defined by (1.2) to (1.10) are special cases of the iterative scheme defined by (1.1), therefore our Theorem 2.1 made by the iterative scheme defined by (1.1) associated with Zamfirescu operator generalizes all Theorems made by the iterative schemes defined by (1.2) to (1.10) associated with Zamfirescu operator. Furthermore, several results in literature are extended and generalized by our result in the following way:

(1) The fixed point theorems of Kannan's operators [12] and Chatterjea's operators [14] are extended to the larger class of Zamfirescu operators associated with multi-step Noor fixed point iterative scheme with errors.

(2) The fixed point theorems of Berinde [20-22] are extended from the Mann and Ishikawa iterative scheme to multi-step Noor fixed point iterative scheme with errors.

(3) The fixed point theorem of Rafiq [23] is extended from the Mann, Ishikawa and Noor iterative scheme to multi-step Noor fixed point iterative scheme with errors.

(4) The fixed point theorem of Cho, Zhou, Guo [2] is generalized and extended from three-step iterations with errors in asymptotically nonexpansive mapping multi-step Noor fixed point iterative scheme with errors in Zamfirescu operator.

(5) The fixed point theorems of Xu [7] and Liu [8] are generalized from the Mann and Ishikawa iterative scheme with errors to multi-step Noor fixed point iterative scheme with errors.

(6) The fixed point theorems of Osilike [26-28] are generalized from the Mann and Ishikawa iterative scheme with errors to multi-step Noor fixed point iterative scheme with errors.

Conflict of Interests

The authors declare that there is no conflict of interests.

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