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SOME COMMON FIXED POINT RESULTS FOR COMMUTING k-SET-CONTRACTION MAPPINGS AND THEIR APPLICATIONS

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Abstract. The purpose of this paper is to prove new common fixed point theorems for commuting mappings. We also deduce new classes of k-set contraction mappings and guarantee the existence of their fixed points. As applications, we establish an integral version of these results. Finally, we also introduce and study the resolvability of a new type of integral equations.

Keywords: Meir-Keeler contraction; Set contraction; Common fixed point; Integral equation.

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1. Introduction

Fixed point theory has become a very popular tool in solving existence problems in many branches of mathematical analysis (see e.g. [6-8], [10], [12]). For this reason, many authors are interested in extending fixed point results of nonlinear operators (see e.g. [2-3], [9], [14], [20]). One of the more powerful of these is due to Meir and Keeler [16], they defined a new class of contraction operators which includes the contraction mappings as special case and proved a very

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interesting theorem more general than the Banach contraction theorem. In [17], Par and Bae extended this new class of mappings in order to study the existence of common fixed points.

Recently, motivated by Meir and Keeler Aghajani *et al.* [4] presented a new class of condensing operators and proved a theorem which presents a very nice generalization of the Darbo fixed point theorem.

In this paper, we present theorems that guarantee the existence of common fixed points for commuting set contraction mappings and generalize the work of Jungck [13]. Motivated by Park and Bae [17], we deduce new classes of k-set contraction mappings and study the existence of common fixed points for these classes of mappings, the results obtained extend those given by Aghajani *et al.* As applications, we also study the existence of common fixed points for mappings that satisfy conditions like,

$$\int_0^{\mu(N(A))} \varphi(r)\,dr \leqslant \Phi\left(\int_0^{\mu(H(A))} \varphi(r)\,dr\right),$$

or

$$\varepsilon \leqslant \Phi\left(\int_0^{\mu(HA)} \varphi(r)\,dr\right) < \varepsilon+\delta \implies \int_0^{\mu(N(A))} \varphi(r)\,dr < \varepsilon,$$

where A is a nonempty bounded convex closed subset of a Banach space X, μ is a measure of noncompactness defined on X and N, H are mappings that satisfy some conditions which will be given later.

Finally, we introduce and investigate the resolvability of the following integral equation

$$\int_0^{x(t)} \varphi(r)\,dr = \int_0^{f\left(t,x(t),\int_0^1 K(s,t),x(s)ds\right)} \varphi(r)\,dr.$$

2. Preliminaries

Throughout this paper, the following notation will be used, X denotes a Banach space. A is a closed, bounded and convex subset of X. $\mathscr{B}_X$ is the family of all bounded subsets of X.

Theorem 2.1. [13] *Let H be a continuous self map of a complete metric space (X,d). If there exists $k \in (0,1)$ and a mapping $N : X \to X$ which commutes with H $(NH = HN)$ such that $NX \subseteq HX$ and*

$$d(Nx,Ny) \leqslant kd(Hx,Hy), \forall x,y \in X,$$

then, N and H have a unique common fixed point.

Definition 2.1. (Park and Bae [16]) Let H and N be self maps of a metric space (X,d). A map N is say to be an $(\varepsilon,\delta)-H-$contraction if the following conditions are satisfied.

$i)$ For any $\varepsilon>0$ there exists $\delta>0$, such that

$$\varepsilon\leqslant d\left(Hx,Hy\right)<\varepsilon+\delta \text{ implies that } d\left(Nx,Ny\right)<\varepsilon.$$

$ii)$ $Hx=Hy$ then $Nx=Ny$.

Theorem 2.2. [16] *If H is a continuous self map of a complete metric space X and N is an $(\varepsilon,\delta)-H-$contraction which commutes with H, then H and N have a unique common fixed point in X.*

Theorem 2.3. (Schauder [1]) *Let A be nonempty, convex, compact subset of a Banach space X. Then every continuous self-mapping $T:A\to A$ has at least one fixed point on A.*

The Kuratowski [15] measure of noncompactness for a metric space Y is defined as,

Definition 2.2. A functional $\mu:\mathscr{B}_Y\to\mathbb{R}_+$ such that

$$\mu\left(A\right)=\inf\left\{\varepsilon>0:\ A \text{ is the finite union of subsets } A_i \text{ such that } \sup\left\{d\left(x,y\right):x,y\in A_i\right\}\leqslant\varepsilon,\ \forall i\right\},$$

is called measure of noncompactness.

Lemma 2.1. [9] *A measure of noncompactness μ satisfies the following properties:*

(1) $\mu\left(A\right)=0\Leftrightarrow A$ *is a compact set.*

(2) $A\subset B\Rightarrow\mu\left(A\right)\leqslant\mu\left(B\right),\ \forall A,B\in B_X$.

(3) $\mu\left(A\cup\{x_0\}\right)=\mu\left(A\right)$, *for any* $x_0\in X$.

(4) $\mu\left(A\right)=\mu\left(\overline{A}\right),\ \forall A\in B_X$.

(5) $\mu\left(ConvA\right)=\mu\left(A\right),\ \forall A,B\in B_X$.

(6) $\mu\left(\lambda A+(1-\lambda)B\right)\leqslant\lambda\mu\left(A\right)+(1-\lambda)\mu\left(B\right)$, *for* $\forall A,B\in B_X$ *and* $\lambda\in[0,1]$.

(7) *Let (A_n) be a sequence of closed subsets from B_X such that $A_{n+1}\subseteq A_n$, $(n\geqslant1)$ and $\lim\limits_{n\to\infty}\mu\left(A_n\right)=0$. Then, the intersection set $A_\infty=\bigcap\limits_{n=1}^{\infty}A_n$ is nonempty and A_∞ is compact.*

Definition 2.3. [7] A mapping $N:A\to A$ that satisfies $\mu\left(TA\right)\leqslant k\mu\left(A\right)$, is said to be a k-set-contraction mapping.

Moreover, if $k < 1$, then N is called condensing mapping.

Lemma 2.2. [5] *Let $\psi : R^+ \to R^+$ be a nondecreasing and upper semicontinuous function. Then,*

$$\lim_{n\to\infty} \psi^n(t) = 0 \ \ \forall t > 0 \Leftrightarrow \ \psi(t) < t \ \ \forall t > 0.$$

3. Main results

In this section, we present the main results of this work.

Theorem 3.1. *Let H be a continuous self-mapping defined on X. If there exists a self-mapping $N : A \to A$ such that N commutes with H and*

$$\mu(NA) \leqslant \Phi(\mu(HA)),$$

where, $\Phi : R_+ \to R_+$ is a nondecreasing and upper semicontinuous function such that $\Phi(t) < t$ for all $t > 0$. Then *N and H have at least one common fixed point and the set of common fixed points is compact.*

Proof. Let $(A_n)_{n=0}^{\infty}$ be closed, bounded and convex sequence of subset of X, such that $A_{n+1} = Conv(N(A_n))$. We notice that $A_1 = Conv(N(A_0)) \subseteq A_0$ and $A_2 = Conv(N(A_1)) \subseteq A_1$. By induction, we get

$$...A_{n+1} \subseteq A_n \subseteq ... \subseteq A_0.$$

Moreover, let

$$\begin{aligned} \mu(A_{n+1}) = \mu(Conv(N(A_n))) &= \mu(N(A_n)) \\ &\leqslant \Phi(\mu(H(A_n))). \end{aligned}$$

Since H is a self-mapping, $H(A_n) \subseteq A_n$ which implies that $\mu(H(A_n)) \leqslant \mu(A_n)$. Using the fact that Φ is nondecreasing, we get

$$\mu(A_{n+1}) \leqslant \Phi(\mu(A_n)). \tag{1}$$

Similarly, we obtain

$$\mu\left(A_n\right) \leqslant \Phi\left(\mu\left(A_{n-1}\right)\right). \tag{2}$$

By substituting (2) in (1), we have $\mu\left(A_{n+1}\right) \leqslant \Phi^2\left(\mu\left(A_{n-1}\right)\right)$. Repeating this process n time, we get $\mu\left(A_{n+1}\right) \leqslant \Phi^{n+1}\left(\mu\left(A_0\right)\right)$. Using Lemma 2.2. we have $\lim\limits_{n\to\infty} \Phi^{n+1}\left(\mu\left(A_0\right)\right) = 0$. Consequently, $\lim\limits_{n\to\infty} \mu\left(A_{n+1}\right) = 0$.

In view of Lemma 2.1. $A_\infty = \bigcap\limits_{n=1}^{\infty} A_n$ is compact. Since N is a continuous mapping on a compact A_∞, then by Schaulder Theorem, H has at least one fixed point.

Suppose that x^* is the fixed point of H, that is, $Hx^* = x^*$, then $HNx^* = NHx^* = x^*$, hence Hx^* $(= x^*)$ is also fixed point of N. Consequently, N and H have at least one common fixed point.

Now, let us show that the set of common fixed point is compact. Suppose

$$\mathscr{F} = \{x \in X,\ Nx = Hx = x\}$$

and $\mu\left(\mathscr{F}\right) \neq 0$. Then

$$\mu\left(\mathscr{F}\right) = \mu\left(N\left(\mathscr{F}\right)\right) \leqslant \Phi\left(\mu\left(H\left(\mathscr{F}\right)\right)\right) < \mu\left(H\left(\mathscr{F}\right)\right) \leqslant \mu\left(\mathscr{F}\right).$$

This is a contradiction. Thus, $\mu\left(\mathscr{F}\right) = 0$. Hence $\mathscr{F}$ is compact. This completes the proof.

Corollary 3.1. *Let N and H given as in Theorem 3.1. such that*

$$\mu\left(NA\right) \leqslant k\mu\left(HA\right),\ \ k \in [0,1).$$

Then, N and H have at least one common fixed point and the set of common fixed points is compact.

Proof. It suffices to take $\Phi(t) = kt$ where $k \in [0,1[$ in Theorem 3.1.

Corollary 3.2. [3] *Let A be a nonempty closed, bounded and convex subset of X. If $N : A \to A$ is a continuous mapping such that $\mu\left(NA\right) \leqslant \Phi\left(\mu\left(A\right)\right)$, where $\Phi : R_+ \to R_+$ is a nondecreasing and upper semicontinuous function such that $\Phi(t) < t$ for all $t > 0$. Then, H has at least one fixed point in A.*

Proof. Taking $H = I$ in Theorem 3.1, we obtain the desired conclusion immediately.

Corollary 3.3. (Darbo Theorem [9]) *Let A be a nonempty closed, bounded and convex subset of X. If $N : A \to A$ is a continuous mapping such that*

$$\mu(NA) \leqslant k\mu(A),\ k \in [0,1), \tag{3}$$

then N has a fixed point in A.

Proof. Taking $\Phi(t) = kt$ in Corollary 3.2, we obtain the desired conclusion immediately.

Theorem 3.2. *Let N and H be commuting self-mappings on A. Suppose that either N or H is continuous and $\mu(NA) < \Phi(\mu(HA))$, then N and H have at least one common fixed point in A and the set of common fixed points is compact.*

Proof. For a given $x_0 \in X$, the set $\mathscr{S} = \{S \cup \{x_0\} \diagup S$ is closed and convex subset of $X\}$. Denote $A = \bigcap S$. Suppose that A is not compact, that is, $\mu(A) \neq 0$. It is clear that A is non empty (since $x_0 \in A$).

Now, define $R = Conv(NA) \cup \{x_0\}$. Obviously, R is closed and convex subset of X that contains x_0. Then $R \in \mathscr{S}$. Thus, $A \subseteq R$.

Moreover, $R \subseteq A \cup \{x_0\} \subseteq A$ (since N is a self-mapping, then $Conv(NA) \subseteq A$). Consequently, $R = A$. Let $\mu(A) = \mu(R) = \mu(Conv(NA) \cup \{x_0\})$, using the properties of the measure of noncompactness, we get

$$\mu(A) = \mu(NA) < \Phi(\mu(HA)) < \mu(HA) \leqslant \mu(A).$$

Then, A should be compact. If N (or H) is continuous, then by Schaulder Theorem we conclude that N (Or H) has at least one fixed points on A. Suppose that x^* is a fixed point of N, that is, $Nx^* = x^*$, then $NHx^* = HNx^* = Hx^*$, hence Nx^* $(= x^*)$ is a fixed point of H. (Similarly if x^* is a fixed point of H).

Finally, N and H have at least one common fixed point on A and as previously we can easily show that the set of common fixed points is compact.

Corollary 3.4. *Let N and H given as in Theorem 3.2. such that instead of Condition 3. we have the following inequality $\mu(NA) < \mu(HA)$, then N and H have at least one common fixed point in A and the set of common fixed points is compact.*

Proof. Let for an arbitrary $\varepsilon > 0$, $\Phi(t) = t - \varepsilon$, then Condition 3. become

$$\mu(NA) < \mu(HA) - \varepsilon.$$

Since ε is arbitrary, then by taking the limit as ε goes to 0 we get $\mu(NA) < \mu(HA)$.

Corollary 3.5. (Sadovski Theorem [19]) *Let A be a nonempty closed, bounded and convex subset of X. If $N : A \to A$ is a continuous mapping such that*

$$\mu(NA) < \mu(A),$$

then H has a fixed point in A and the set of common fixed points is compact.

Proof. Taking $H = I$ in Corollary 3.4, we obtain the desired conclusion immediately.

Corollary 3.6. *Let N and H given as in Theorem 3.2 such that instead of Condition 3 we have the following inequality*

$$\mu(NA) < k\mu(HA) + (1-k)\mu(A), \quad k \in [0,1).$$

Then, N and H have a common fixed point and the set of common fixed points is compact.

Proof. Since H is a self-mapping then $HA \subseteq A$ which implies that $\mu(HA) \leqslant \mu(A)$. Then,

$$\begin{aligned} \mu(NA) &\leqslant k\mu(HA) + (1-k)\mu(A) \\ &< k\mu(A) + (1-k)\mu(A) \\ &= \mu(A). \end{aligned}$$

Then by Corollary 3.5, N has a fixed point in A.

Moreover, by assuming that x is the fixed point of N, that is, $Nx = x$, we get $NHx = HNx = Hx$, hence $Nx\ (= x)$ is a fixed point of H. Thus, N and H have common fixed point.

Definition 3.1. Let N and H two self mappings on X, N is an (ε, δ) H-set-contraction if and only if for any $\varepsilon > 0$ there exists $\delta > 0$, such that $\varepsilon \leqslant \mu(HA) < \varepsilon + \delta$ implies that $\mu(NA) < \varepsilon$.

Definition 3.2. Let N and H two self mappings on X, N is said to be a generalized (ε, δ) H-set-contraction if and only if for any $\varepsilon > 0$ there exists $\delta > 0$, such that $\varepsilon \leqslant \Phi(\mu(HA)) < \varepsilon + \delta$ implies that $\mu(NA) < \varepsilon$.

Remark 3.1. The case $H = I$, was introduced by Aghajani *et al.* in [4] under the name of N Meir-Keeler condensing operator.

Theorem 3.3. *If N is a continuous self-mapping of X and H is a generalized (ε, δ) H-set-contraction which commutes with N. Then, N and H have at least one common fixed point in X and the set of fixed point is compact.*

Proof. Interestingly, although the class of generalized (ε, δ) H-set-contraction mapping is larger than that of generalized set-contraction mappings, one may prove Theorem 3.3. by means of Theorem 3.1. Indeed, since

$$\varepsilon \leqslant \Phi(\mu(HA)) < \varepsilon + \delta \text{ implies that } \mu(NA) < \varepsilon.$$

Obviously $\mu(NA) < \Phi(\mu(HA))$ .Hence by Theorem 3.2. N and H have at least one fixed point and the set of fixed points is compact.

Corollary 3.7. *If N is a continuous self-mapping of X and H is an (ε, δ) H-set-contraction which commutes with N. Then, N and H have at least one common fixed point in X.*

Proof. We can easily notice that, $\mu(NA) < \mu(HA)$. Then by Corollary 3.4, N and H have at least one fixed point and the set of fixed points is compact.

Corollary 3.8. [4] *If N is a Meir-Keeler continuous condensing self mapping, then N have at least one fixed point and the set of fixed point is compact.*

Proof. Taking $H = I$ in Theorem 3.3, we obtain the desired conclusion immediately.

4. Applications

In order to extend the well-known Banach-Caccioppoli theorem, Branciari [11] was the first to introduce contractive mappings of integral type. Afterwards, many authors continued Branciari's study and obtained many fixed point theorems for several classes of contractive mapping of integral type (see, e.g. [18], [21], [22] and the references therein).

In this section, we establish integral versions of the fixed point theorems presented in the previous section. The proofs of this section are given only in sketches since they are based on arguments similar to the ones used previously.

Theorem 4.1. *Let N , $H : A \to A$ be commuting mappings such that either N or H is continuous. If there exists $\Phi : R_+ \to R_+$ nondecreasing and upper semicontinuous function such that $\Phi(t) < t$ for all $t > 0$ and for which we have*

$$\int_0^{\mu(NA)} \varphi(t)\,dt \leqslant \Phi\left(\int_0^{\mu(HA)} \varphi(t)\,dt\right).$$

Then $F = \{x \in A/\ Sx = Tx = x\}$ *is nonempty and compact.*

Proof. First, let $(A_n)_{n=0}^{\infty}$ be a closed and convex sequence of subset of X, such that $A_{n+1} = Conv(N(A_n))$. We notice that $A_1 = Conv(N(A_0)) \subseteq A_0$ and $A_2 = Conv(N(A_1)) \subseteq A_1$. By induction, we get

$$...A_{n+1} \subseteq A_n \subseteq ... \subseteq A_0.$$

Obviously,

$$\int_0^{\mu(A_{n+1})} \varphi(r)\,dr \leqslant \Phi\left(\int_0^{\mu((A_n))} \varphi(r)\,dr\right).$$

Repeating this process n-time, we get

$$\int_0^{\mu(A_{n+1})} \varphi(r)\,dr \leqslant \Phi^n\left(\int_0^{\mu(A_0)(p)} \varphi(r)\,dr\right).$$

As previously, since $\lim_{n\to\infty} \Phi^n\left(\int_0^{\mu(A_0)(p)} \varphi(r)\,dr\right) = 0$, we get $\lim_{n\to\infty} \mu(A_{n+1}) = 0$. Consequently, if N is continuous, then by Schaulder Theorem N has at least one fixed point and as previously by a simple calculation we can show that H and N have at least one common fixed point and the set of fixed points is compact (Similarly if H is continuous).

Corollary 4.1. *Let N , $H : A \to A$ be commuting mappings such that either N or H is continuous and*

$$\int_0^{\mu(NA)} \varphi(r)\,dr \leqslant k\int_0^{\mu(HA)} \varphi(r)\,dr, for\ k \in [0,1).$$

Then, $F = \{x \in A/\ Sx = Tx = x\}$ is nonempty and compact.

Proof. Take $\Phi(t) = kt$ in Theorem 4.1, we obtained the desired conclusion immediately.

Corollary 4.2. [2] *Let A be a nonempty closed, bounded and convex subset of X. If $N : A \to A$ is a continuous mapping such that*

$$\int_0^{\mu(NA)} \varphi(r)\,dr \leqslant \Phi\left(\int_0^{\mu(A)} \varphi(r)\,dr\right),$$

where $\Phi : R_+ \to R_+$ *is a nondecreasing and upper semicontinuous function such that* $\Phi(t) < t$ *for all* $t > 0$. *Then,* H *has at least one fixed point in* A.

Proof. Taking $H = I$ in Theorem 4.1, we obtain the desired conclusion immediately.

Corollary 4.3. *Let* A *be a nonempty closed, bounded and convex subset of* X. *If* $N : A \to A$ *is a continuous mapping such that*

$$\int_0^{\mu(NA)} \varphi(r)\, dr \leqslant k \int_0^{\mu(A)} \varphi(r)\, dr,, \quad k \in [0,1),$$

then N *has a fixed point in* A.

Proof. Taking $\Phi(t) = kt$ in Corollary 4.2, we obtain the desired conclusion immediately.

Theorem 4.2. *Let* N *and* H *be commuting self-mappings from* A *to* A *such that* N *commutes with* H. *Suppose that either* N *or* H *is continuous and*

$$\int_0^{\mu(NA)} \varphi(r)\, dr < \Phi\left(\int_0^{\mu(A)} \varphi(r)\, dr\right),$$

then N *and* H *have at least one common fixed point in* A *and the set of common fixed points is compact.*

Proof. The proos is similar to the proof of Theorem 3.2.

Corollary 4.4. *Let* N *and* H *given as in Theorem 4.2. such that*

$$\int_0^{\mu(NA)} \varphi(r)\, dr < \int_0^{\mu(A)} \varphi(r)\, dr,$$

then N *and* H *have at least one common fixed point in* A *and the set of common fixed points is compact.*

Proof. Since H is a self-mapping then $HA \subseteq A$ which implies that $\mu(HA) \leqslant \mu(A)$. Then,

$$\begin{aligned}
\int_0^{\mu(NA)} \varphi(r)\, dr &< k \int_0^{\mu(HA)} \varphi(r)\, dr + (1-k) \int_0^{\mu(A)} \varphi(r)\, dr \\
&\leqslant k \int_0^{\mu(A)} \varphi(r)\, dr + (1-k) \int_0^{\mu(A)} \varphi(r)\, dr \\
&= \int_0^{\mu(A)} \varphi(r)\, dr.
\end{aligned}$$

Then by Corollary 4.3. N has a fixed point in A and by easy calculation we show that N and H have common fixed point.

Definition 4.1. Let N and H two self mappings on X, N is an (ε,δ) H-set-contraction of integral type if and only if for any $\varepsilon>0$ there exists $\delta>0$, such that $\varepsilon\leqslant\int_0^{\mu(HA)}\varphi(r)\,dr<\varepsilon+\delta$ implies that $\int_0^{\mu(NA)}\varphi(r)\,dr<\varepsilon$.

Definition 4.2. Let N and H two self mappings on X, N is said to be a generalized (ε,δ) H-set-contraction of integral type if and only if for any $\varepsilon>0$ there exists $\delta>0$, such that $\varepsilon\leqslant\Phi\left(\int_0^{\mu(HA)}\varphi(r)\,dr\right)<\varepsilon+\delta$ implies that $\int_0^{\mu(NA)}\varphi(r)\,dr<\varepsilon$.

Remark 4.1. If $H=I$ in Definition 4.2, then N is a Meir Keeler condensing operator type (see [4]).

Theorem 4.3. Let H be a continuous self mapping on A and N is a generalized $(\varepsilon,\delta)-H-$contraction of integral type which commutes with H, then H and N have at least one common fixed point in X and th set of fixed points is compact.

Proof. Obviously,
$$\int_0^{\mu(NA)}\varphi(r)\,dr<\Phi\left(\int_0^{\mu(HA)}\varphi(r)\,dr\right).$$
Then by Theorem 4.2. N and H have at least one fixed point and the set of fixed points is compact.

Corollary 4.5. Let H be a continuous self mapping on A and N is a $(\varepsilon,\delta)-H-$integral type set contraction which commutes with H, then H and N have at least one common fixed point in X and th set of fixed points is compact.

Proof. The proof is obvious.

Corollary 4.6. If N is a Meir-Keeler continuous condensing integral type self-mapping, then N have at least one fixed point and the set of fixed point is compact.

Proof. Taking $H=I$ in Theorem 4.3, we get the desired conclusion immediately.

5. Fredholm integral equations of integral type

Let the following Volterra integral equation

$$\int_0^{x(t)}\varphi(r)\,dr=\int_0^{f\left(t,x(t),\int_0^1 K(s,t)x(s)ds\right)}\varphi(r)\,dr. \tag{5.1}$$

In this section, we characterize solutions of Integral Equation 5.1. defined on the Banach space $C([0,1])$ consisting of all defined, bounded and continuous functions on the interval $C([0,1])$.

We endow the space $C([0,1])$ with the usual norm

$$\|x\| = \sup\{|x(t)| : t > 0\}.$$

Let X be a nonempty and bounded subset of the space $C([0,1])$. Fix $x \in C[0,1]$, $T \geqslant 0$ and $\varepsilon > 0$. Let us denote

$$\omega_0(X) = \lim_{\varepsilon \to 0} \sup\{\omega(x,\varepsilon) : x \in X\},$$

where $X \in \mathscr{M}_{C([0,1])}$ $\omega(x,\varepsilon) = \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in [0,1], |t_2 - t_1| \leqslant \varepsilon\}$.

According to [10] the above defined $\omega_0(X)$ is a regular measure of noncompactness in the space $C[0,1]$.

Then under the following hypotheses:

(i) The function $K : [0,1]^2 \to \mathbb{R}$ is continuous and there exists $M = \sup\{|K(s,t)|, \ \ s,t \in [0,1]\} < 1$.

(ii) The function $f : [0,1]^3 \to \mathbb{R}$ is continuous and there exist bounded functions $a(t)$, $b(t)$ defined on $[0,1]$ such that

$$\begin{aligned} |f(t,x_1,y) - f(t,x_2,y)| &\leqslant a(t)|x_1 - x_2|, \\ |f(t,x,y_1) - f(t,x,y_2)| &\leqslant b(t)|y_1 - y_2|, \end{aligned}$$

and there exists a positive α such that $f(t,x(t),0) \leqslant \alpha$.

Then, we have the following theorem.

Theorem 5.1. *Assuming the hypotheses* $(i)-(ii)$ *hold. Then, Equation* 5.1. *has at least one fixed point in* $C[0,1]$.

Proof. In order to study the existence of solution for Equation 5.1. we study the existence of fixed points of the following operator

$$Fx(t) = f\left(t, x(t), \int_0^1 K(s,t)x(s)\,ds\right).$$

Firstly, we verify that F is a self-mappings. To do this, we fix $x \in B_r$, then

$$\begin{aligned} |Fx(t)| &= \left| f\left(t, x(t), \int_0^1 K(s,t)x(s)\,ds\right) \right| \\ &\leqslant \left| f\left(t, x(t), \int_0^1 K(s,t)x(s)\,ds\right) - f(t, x(t), 0) \right| + \left| f\left(t, x(t), \int_0^1 K(s,t)x(s)\,ds\right) \right| \\ &\leqslant b(t)M\|x\| + \alpha. \end{aligned}$$

Since $b(t)M < 1$, we can found r_0 that satifies the inequality

$$r \leqslant \frac{\alpha}{1 - b(t)M}.$$

Thus, $FB_{r_0} \subset B_{r_0}$ and F is a self mapping.

Next, we verify that F is continuous on B_{r_0}. To do this, we fix $\delta > 0$ and take arbitrary x, $y \in B_{r_0}$ such that $\|x - y\| \leqslant \delta$. Then for $t \geqslant 0$,

$$\begin{aligned} |Fx(t) - Fy(t)| &= \left| f\left(t, x(t), \int_0^1 K(s,t)x(s)\,ds\right) - f\left(t, y(t), \int_0^1 K(s,t)y(s)\,ds\right) \right| \\ &\leqslant \left| f\left(t, x(t), \int_0^1 K(s,t)x(s)\,ds\right) - f\left(t, x(t), \int_0^1 K(s,t)y(s)\,ds\right) \right| \\ &\quad + \left| f\left(t, x(t), \int_0^1 K(s,t)y(s)\,ds\right) - f\left(t, y(t), \int_0^1 K(s,t)y(s)\,ds\right) \right| \\ &\leqslant a(t)|x(t) - y(t)| + b(t)\int_0^1 |K(s,t)||x(s) - y(s)|\,ds \\ &\leqslant (a(t) + b(t)M)\|x - y\|. \end{aligned}$$

Now for $x \in B_{r_0}$, let

$$\begin{aligned} &\int^{|Fx(t_1) - Fx(t_2)|} \varphi(r)\,dr \\ &= \int^{\left| f\left(t_1, x(t_1), \int_0^1 K(s,t_1)x(s)ds\right) - f\left(t_2, x(t_2), \int_0^1 K(s,t_2)x(s)ds\right) \right|} \varphi(r)\,dr \\ &\leqslant \int^{\substack{\left| f\left(t_1, x(t_1), \int_0^1 K(s,t_1)x(s)ds\right) - f\left(t_2, x(t_1), \int_0^1 K(s,t_1)x(s)ds\right) \right| \\ + \left| f\left(t_2, x(t_1), \int_0^1 K(s,t_1)x(s)ds\right) - f\left(t_2, x(t_2), \int_0^1 K(s,t_1)x(s)ds\right) \right| \\ + \left| f\left(t_2, x(t_2), \int_0^1 K(s,t_1)x(s)ds\right) - f\left(t_2, x(t_2), \int_0^1 K(s,t_2)x(s)ds\right) \right|}} \varphi(r)\,dr \\ &\leqslant \int^{\omega(f,\varepsilon) + a(t)\omega(x,\varepsilon) + b(t)\omega(K,\varepsilon)\int_0^1 |x(s)|ds} \varphi(r)\,dr, \end{aligned}$$

where

$$\omega(f,\varepsilon) = \sup\{|f(t_1,.,.) - f(t_2,.,.)| : t_i \in [0,1],\ |t_2 - t_1| \leqslant \varepsilon\},$$

and

$$\omega(K,\varepsilon) = \sup\{|K(.,t_1) - K(.,t_2)| : t_i \in [0,1],\ |t_2 - t_1| \leqslant \varepsilon\}.$$

Since f and K are continuous on $[0,1]\times B_{r_0}\times B_{r_0}$, $[0,1]\times[0,1]$ (Resp.), then they are uniformly continuous on $[0,1]\times B_{r_0}\times B_{r_0}$, $[0,1]\times[0,1]$ (Resp.). Hence,

$$\lim_{\varepsilon\to 0}\omega(f,\varepsilon) = \lim_{\varepsilon\to 0}\omega(K,\varepsilon) = 0.$$

Consequently,

$$\int^{\omega(Fx,\varepsilon)} \varphi(r)\,dr \leqslant \int^{\sup_{t\in[0,1]}|a(t)|\omega(x,\varepsilon)} \varphi(r)\,dr.$$

By making a change of variable, we get

$$\int^{\omega(Fx,\varepsilon)} \varphi(r)\,dr \leqslant \sup_{t\in[0,1]}|a(t)| \int^{\omega(x,\varepsilon)} \varphi\left(r \sup_{t\in[0,1]}|a(t)|\right) dr.$$

Since $\sup_{t\in[0,1]}|a(t)| \leqslant 1$ and φ is increasing then it is obviously that $\varphi\left(r \sup_{t\in[0,1]}|a(t)|\right) \leqslant \varphi(r)$.

Hence,

$$\int^{\omega(Fx,\varepsilon)} \varphi(r)\,dr \leqslant \sup_{t\in[0,1]}|a(t)| \int^{\omega(x,\varepsilon)} \varphi(r)\,dr.$$

By taking limits, we get

$$\int^{\omega_0(FX)} \varphi(r)\,dr \leqslant \sup_{t\in[0,1]}|a(t)| \int^{\omega_0(X)} \varphi(r)\,dr.$$

Then, using Corollary 4.4. Equation 5.1. has at least one solution in $C[0,1]$.

Example 5.1. Let solve the following integral equation on $C[0,1]$

$$\int_0^{x(t)} \sqrt{r}dr = \int_0^{-\frac{1}{8}e^{-t}+x(t)+\frac{1}{8}\int_0^1 e^{s-t}x(s)ds} \sqrt{r}dr, \tag{5.2}$$

where $t\in[0,1]$.

We notice that by taking $\varphi(r) = \sqrt{r}$, $K(s,t) = \frac{1}{4}e^{s-t}$ and

$$f(t,x(t),y(t)) = -\frac{1}{8}e^{-t} + x(t) + \frac{1}{2}y(t),$$

we get the integral equation,

$$\int_0^{x(t)} \varphi(r)\,dr = \int_0^{f\left(t,x(t),\int_0^1 K(s,t)x(s)ds\right)} \varphi(r)\,dr.$$

Since, for any $t, s \in [0,1]$

$$|K(s,t)| = \frac{1}{4}e^{s-t} \leqslant \frac{e^s}{4} \leqslant \frac{e}{4} \simeq 0.68.$$

In further, we know that $x(t) \in C[0,1]$,then

$$f(t, x(t), 0) = -\frac{1}{8}e^{-t} + x(t) \leqslant \frac{7}{8} = \alpha.$$

It is easy to see that $a(t) = 1$ and $b(t) = \frac{1}{2}$, since

$$|f(t, x(t), y(t)) - f(t, u(t), y(t))| = |x(t) - u(t)|,$$

and

$$|f(t, x(t), y(t)) - f(t, x(t), v(t))| = \frac{1}{2}|y(t) - v(t)|.$$

Then, by Theorem 5.1. Equation 5.2. has at least one solution in $C[0,1]$.

We notice that, $\int_0^{e^{-t}} \sqrt{r} dr = \frac{2}{3}e^{-\frac{3}{2}t}$ is one of the solutions of Equation 5.2. and has the following graphic,

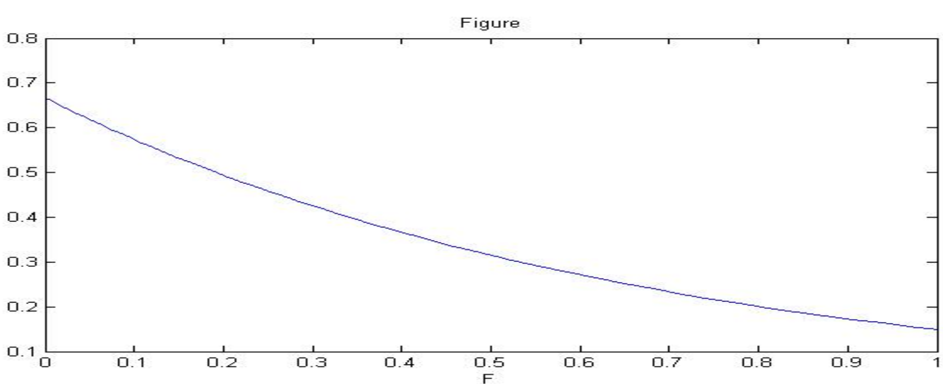

FIGURE 1

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] R. Agarwal, M. Meehan, D. O'Regan, Fixed point theory and applications, Cambridge University Press (2004).

[2] A. Aghajani, M. Aliaskari, Generalization of Darbo's fixed point theorem and application, Int. J. Nonlinear Anal. Appl. 2 (2011), 86-95.

[3] A. Aghajani, R. Allahyari, M. Mursaleen, A generalization of Darbo's theorem with application to the solvability of systems of integral equations, Comput. Math. Appl. 260 (2014), 68–77.

[4] A. Aghajani, M. Mursaleen, A. Shole Haghighi , Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness, Acta. Math. Sci. 2015 (2015), 552–566.

[5] A. Aghajani, J. Banas, N. Sabzali, Some generalizations of Darbo xed point theorem and applictions, Bull. Belg. Math. Soc. 20 (2013), 345-358.

[6] R. R Akmerov, M. I. Kamenski, A. S. Potapov, A. E. Rodkina, B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhauser-Verlag,Basel (1992).

[7] J. Apell, Measure of noncompactness, condensing operator and fixed points : An application-oriented survey, Fixed Point Theory 6 (2005), 157-229.

[8] J. Banas, Measures of noncompactness in the study of solutions of nonlinear differential and integral equations, Cent. Eur. J. Math. 10 (2012), 2003–2011.

[9] J. Banas, On measures of noncompactness in Banach spaces, Comment. Math. Univ. Carolin. 21 (1980), 131–143.

[10] J. Banas, M. Mursaleen, Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations, Springer (2014).

[11] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 29 (2002), 531–536.

[12] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag (1985).

[13] G. Jungck, Commuting mappings and fixed points, Amer. Math. 83 (1976), 261-263.

[14] V. Karakaya, N. H. Bouzara, K. Dogan, Y. Atalan, Existence of tripled fixed points for a class of condensing operators in Banach spaces, Sci. World J. doi:10.1155/2014/541862

[15] K. Kuratowski, Sur les espaces complets, Fund. Math. 5 (1930), 301–309.

[16] A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326–329.

[17] S. Park, J. S. Bae, Extensions of a fixed point theorem of Meir and Keeler, Ark. Mat. 19 (1981), 223-228.

[18] B. E. Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 2003 (2003), 4007–4013.

[19] B. N. Sadovski, On a fixed point principle, Functional Anal. Appl. 1 (1967), 74-76.

[20] A. Samadi, M. B. Ghaemi, An extension of Darbo's theorem and its Application, Abst. Appl. Anal. 2014 (2014). Article ID 852324.

[21] W. Sintunavarat, P. Kumam, Gregus type fixed points for a tangential multivalued mappings satisfying contractive conditions of integral type, J. Inequal. Appl. doi:10.1186/1029-242X-2011-3.

[22] T. Suzuki, Meir-Keeler contractions of integral type are still Meir-Keeler contractions, Int. J. Appl. Math. Sci. 2007 (2007), Article ID 39281.