Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 2, 167-174
ISSN: 1927-6303

COMMON FIXED POINT THEOREMS IN METRIC SPACES SATISFYING AN IMPLICIT RELATION

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Abstract. In this paper, we establish common fixed point theorems in metric spaces using the (CLRg) property based on the implicit functions due to Popa. An example is provided to support our main results, which generalize and improve the corresponding results announced recently.

Keywords: weakly compatible mappings; (CLRg) property; property (E. A); common fixed points; implicit functions.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

Metric fixed point theory plays an important role in mathematics because of its wide range of applicability in applied mathematics and sciences. Banach contraction principle is one of the fundamental results in fixed point theory and is generalized in various directions. Jungck [4] gave an interesting generalization of Banach contraction principle and established common

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Received January 12, 2016

fixed point result for a pair of commuting mappings. Afterward, study of common fixed points of mappings satisfying some contractive type condition has been a center of vigorous research activity and a number of interesting results have been obtained using commutativity and its weaker forms such as weak commutativity [12], compatibility [5], R-weak commutativity [8], semi-compatibility [2], compatibility of type (A) [6], compatibility of type (B) [9], compatible mappings of type (T) [10] and weak compatibilty [7] etc.

Amari and Moutawakil [1] defined the notion of property (E. A) which contains the class of non-compatible mappings. Recently, Imdad and Ali [3] proved common fixed point theorems using (E. A) property. Most recently, Sintunavarat and Kumam [13] defined the notion of (CLRg) property. It has been noticed that (CLRg) property never requires completeness (or closedness) of subspaces (also see [14]). On the otherhand Popa[11] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions. In this paper, we prove results of Imdad and Ali [3] using (CLRg) property. In proving existence of common fixed point completeness (or closedness) of subspace is not required in our results. Many of the common fixed point theorems in existing literature can be proved by using modifications suggested in this paper.

2. Preliminaries

Sessa [12] introduced the notion of weak commutativity:

Definition 1.1. Two self-mappings f and g of a metric space (X,d) are said to be weakly commuting if

$$d(fgx,gfx) \leq d(fx,gx), \quad \forall x \in X,$$

It is clear that two commuting mappings are weakly commuting but the converse is not true, for more details; see [12] and the references therein.

Definition 1.2. Two self-mappings f and g of a metric space (X,d) are said to be compatible if

$$\lim_{n\to\infty} d(fgx_n,gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$.

Obviously, two weakly commuting mappings are compatible, but the converse is not true; see [5] and the references therein..

Definition 1.3. Two self-mappings f and g of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence points, i.e. if $fu = gu$ for some $u \in X$, than $fgu = gfu$.

It is easy to see that two compatible mappings are weakly compatible.

Definition 1.4. Two self-mappings f and g of a metric space (X,d) are said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$.

Definition 1.5. Two self-mappings f and g of a metric space (X,d) are said to satisfy the common limit in the range of g property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$.

In what follows, the common limit in the range of g property will be denoted by the (CLRg) property.

Now, we give examples of mappings f and g which satisfy the (CLRg) property.

Example 1.6. *Let $X = [0,\infty)$ with the usual metric on X. Define $f,g : X \to X$ by $fx = x/2$ and $gx = 2x$ for all $x \in X$. Consider the sequence $\{x_n\} = \{1/n\}$. Since* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0$, *therefore f and g satisfy the (CLRg) property.*

Example 1.7. *Let $X = [0,\infty)$ with the usual metric on X. Define $f,g : X \to X$ by $fx = x+2$ and $gx = 3x$ for all $x \in X$. Consider the sequence $\{x_n\} = \{1+1/n\}$. Since* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3 = g1$, *therefore f and g satisfy the (CLRg) property.*

Remark 1.8. It is clear from the Jungck's definition [5] that two self-mappings f and g of a metric space (X,d) will be non-compatible if there exists atleast one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$, but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either non-zero or non-existent. Therefore, two non-compatible self-mappings of a metric space (X,d) satisfy the property $(E.A)$.

3. Main results

The idea of implicit relations was introduced by Popa [11].

Example 3.1. *Define* $F(t_1,t_2,\cdots,t_6) : R_6^+ \to R$ *as*

$F(t_1,t_2,\cdots,t_6) = t_1 - k\ max\{t_2,t_3,t_4,(t_5+t_6)/2\}$, *where* $k \in (0,1)$.

Example 3.2. *Define* $F(t_1,t_2,\cdots,t_6) : R_6^+ \to R$ *as*

$F(t_1,t_2,\cdots,t_6) = t_1^2 - c_1\ max\{t_2^2,t_3^2,t_4^2\} - c_2\ max\{t_3t_5,t_4t_6\} - c_3t_5t_6$,

where $c_1 > 0, c_2, c_3 \geq 0, c_1 + 2c_2 < 1$ *and* $c_1 + c_3 < 1$.

Example 3.3. *Define* $F(t_1,t_2,\cdots,t_6) : R_6^+ \to R$ *as*

$F(t_1,t_2,\cdots,t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$,

where $a > 0, b, c, d \geq 0,\ a+b+c < 1$ *and* $a+d < 1$.

Example 3.4. *Define* $F(t_1,t_2,\cdots,t_6) : R_6^+ \to R$ *as*

$F(t_1,t_2,\cdots,t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2$,

where $a > 0, b, c, d \geq 0,\ a+b+d < 1$ *and* $a+b < 1$.

Example 3.5. *Define* $F(t_1,t_2,\cdots,t_6) : R_6^+ \to R$ *as*

$F(t_1,t_2,\cdots,t_6) = t_1^3 - c\frac{t_3^2t_4^2 + t_5^2t_6^2}{1+t_2+t_3+t_4}$, *where* $c \in (0,1)$.

Example 3.6. *Define* $F(t_1,t_2,\cdots,t_6) : R_6^+ \to R$ *as*

$F(t_1,t_2,\cdots,t_6) = t_1^3 - at_2^2 - \frac{bt_5t_6}{1+t_3^2+t_4^2}$, *where* $a > 0, b \geq 0$ *and* $a+b < 1$.

The details and verifications of all above implicit relations can be found in [11]. Implicit functions are quite fruitful in deducing many known contraction conditions.

Theorem 3.7. *Let f and g be two weakly compatible self-mappings of a metric space (X,d) such that*

(i) *f and g satisfy the (CLRg) property,*

(ii) $F(d(fx,fy),d(gx,gy),d(fx,gx),d(fy,gy),d(fy,gx),d(fx,gy)) \leq 0$

for all $x,y \in X$ and $F \in \Psi$, than f and g have a unique common fixed point in X.

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu, \ for\ some\ u\ in\ X.$$

First, we show that $fu = gu$. Suppose that $fu \neq gu$, than $d(fu,gu) > 0$. Using condition (ii) with $x = u$ *and* $y = x_n$, we get

$$F(d(fu,fx_n),d(gu,gx_n),d(fu,gu),d(fx_n,gx_n),d(fx_n,gu),d(fu,gx_n)) \leq 0.$$

Making $n \to \infty$ yields

$$F(d(fu,gu),d(gu,gu),d(fu,gu),d(gu,gu),d(gu,gu),d(fu,gu)) \leq 0,$$

or

$$F(d(fu,gu),0,d(fu,gu),0,0,d(fu,gu)) \leq 0,$$

which implies (due to F_{2b}), $d(fu,gu) \leq 0$. Hence $fu = gu$. Since f and g are weakly compatible, $fu = gu$ implies $fgu = gfu$ and therefore $ffu = fgu = gfu = ggu$.

Finally, we show that fu is a common fixed point of f and g. Suppose that $fu \neq ffu$, than $d(ffu,fu) > 0$. Using condition (ii) with $x = fu$ and $y = u$, we get

$$F(d(ffu,fu),d(gfu,gu),d(ffu,gfu),d(fu,gu),d(fu,gfu),d(ffu,gu)) \leq 0,$$

or

$$F(d(ffu,fu),d(ffu,fu),0,0,d(fu,ffu),d(ffu,fu)) \leq 0,$$

which contradicts F_3. Hence $fu = ffu$ and $gfu = ffu = fu$. Thus fu is a common fixed point of mappings f and g.

Uniqueness of the common fixed point is a direct consequence of condition (ii).

Next, we prove a common fixed point theorem for two finite families of mappings.

Theorem 3.8. *Let $\{f_1, f_2, \cdots, f_m\}$ and $\{g_1, g_2, \cdots, g_p\}$ be two families of self-mappings of a metric space (X,d) with $f = f_1 f_2 \cdots f_m$ and $g = g_1 g_2 \cdots g_p$ satisfying the (CLRg) property and* *condition* (ii) *of Theorem 3.7. If* $f_if_j = f_jf_i$; $g_kg_l = g_lg_k$ *for all* $i, j \in I_1 = \{1,2,\cdots,m\}$ *and* $k,l \in I_2 = \{1,2,\cdots,p\}$, *than (for all* $i \in I_1$ *and* $k \in I_2$*)* f_i *and* g_k *have a common fixed point.*

Proof. Using componentwise commutativity of various pairs, we can easily prove that $fg = gf$, thus the mappings f and g are weakly compatible. Since all the conditions of Theorem 3.7 are satisfied (for the mappings f and g), hence the mappings f and g have a unique common fixed point, say t.

Now we show that t is fixed point of all the component mappings. For this, consider

$$f(f_it) = ((f_1f_2\cdots f_m)f_i)t = (f_1f_2\cdots f_{m-1})((f_mf_i)t) = (f_1f_2\cdots f_{m-1})(f_if_m)t$$

$$= \cdots = f_1f_i(f_2f_3\cdots f_m)t = f_i(f_1f_2\cdots f_m)t = f_i(ft) = f_it.$$

Similarly, we can show that

$$f(g_kt) = g_k(ft) = g_kt,\ g(g_kt) = g_k(gt) = g_kt,$$

and

$$g(f_it) = f_i(gt) = f_it,$$

which shows that (for all i and k) f_it and g_kt are other fixed points of the mappings f and g. Now appealing to the uniqueness of common fixed point of the mappings f and g separately, we get

$$t = f_it = g_kt,$$

which shows that t is common fixed point of f_i and g_k for all i and k.

We are now in a position to give an example to illustrate the Theorems.

Example 3.9. *Let* $X = (0,1]$ *with the usual metric on* X*. Define* $f,g : X \to X$ *as follows:*

$$f(x) = \begin{cases} \frac{2}{3}, & 0 < x \leq \frac{2}{3}, \\ \frac{1}{3}, & \frac{2}{3} < x \leq 1. \end{cases}$$

$$g(x) = \begin{cases} 1-\frac{x}{2}, & 0 < x \leq \frac{2}{3}, \\ \frac{4}{5}, & \frac{2}{3} < x \leq 1. \end{cases}$$

It is clear that f *and* g *satisfy (CLRg) property. To see this let us consider the sequence* $\{x_n\}$ *given by* $x_n = \frac{2}{3} - \frac{1}{n}$*. Than* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \frac{2}{3} = g\frac{2}{3}$*. Also* $f\frac{2}{3} = g\frac{2}{3} \Rightarrow fg\frac{2}{3} = gf\frac{2}{3}$, *which shows that the f and g are weakly compatible. Define a continuous function $F : R_6^+ \to R$ as $F(t_1,t_2,\cdots,t_6) = t_1 - k\, max\{t_2,t_3,t_4,(t_5+t_6)/2\}$, where $k \in (0,1)$; than one can verify that F satisfies F_1, F_2 and F_3. By a routine calculation one can also show that condition (ii) is satisfied for $k = \frac{5}{7}$. Thus all the conditions of Theorem 3.7 are satisfied and $x = \frac{2}{3}$ is the unique common fixed point of f and g.*

Here one needs to note that neither $f(X)$ is contained in $g(X)$ nor $g(X)$ is contained in $f(X)$. Also the mappings f and g are discontinuous and $g(X)$ is not complete.

Remark 3.10. Our results improve several known results including the results of Jungck [4] and Imdad and Ali [3] for a pair of mappings in the following ways:

(i) the completeness of space is not required,

(ii) the completeness of subspace is not required even closedness of subspace is not required,

(iii) containment of ranges of involved mappings is droped,

(iv) continuity of mappings is not required.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

The authors were grateful to the referee, who made a number of valuable comments and suggestions which have improved the manuscript greatly.

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