Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 3, 262-276
ISSN: 1927-6303

ITERATIVE METHOD FOR CONVEX OPTIMIZATION PROBLEMS IN REAL LEBESGUE SPACES

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Abstract. We introduce a new iterative scheme for finding common fixed points of a finite family of nonextensive mapping and zeros of strongly monotone mappings in L_p spaces, which yields a solution to a convex optimization problem. This provides a partial extension of a theorem of Yamada and some other authors from Hilbert spaces to the more general Banach spaces.

Keywords: Convex optimization, monotone mappings, uniformly smooth spaces.

2010 AMS Subject Classification: 47H04, 47H09, 47H10.

1. Introduction

Let X be a real Banach space, X^* its dual, $T : X \to X$ be a mapping and $f : X \to \mathbb{R}$ be a convex functional. We denote the collection of all elements x of X satisfying $Tx = x$ by $Fix(T)$.

Definition 1.1 A mapping $M : X \to X^*$ is called

- $\eta-$strongly monotone if $\langle x-y, Mx-My\rangle \geq \eta\|x-y\|^2,\ \forall x,y \in X,$

and $A : X \to X$ is called

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Received February 5, 2016

- $\eta-$strongly accretive if $\langle Ax-Ay, j(x-y)\rangle \geq \eta\|x-y\|^2,\ \forall x,y \in X,$

where $\langle .,.\rangle$ is the duality pairing between X and X^*. If X is Hilbert space, these two notion agree and it is simply refered to as monotone.

Definition 1.2 A mapping $T: X \to X$ is called $L-$Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|,\ \forall x,y,\in X. \tag{1}$$

Remark

If $L = 1$ in the inequality (1), the mapping is called nonexpansive. It is well known that $Fix(T)$ is closed and convex whenever T is nonexpansive.

Given a nonexpansive self mapping T on a Hilbert space H, and a (possibly nonlinear) monotone mapping $A : T(H) \to H$, the variational inequality problem VIP$(A, Fix(T))$ over $Fix(T)$ is stated as:

$$\text{Find } \; x^* \in Fix(T) \text{ such that } \langle y - x^*, Ax^* \rangle \geq 0, \;\; \forall y \in Fix(T). \tag{2}$$

It is known that an element x^*, of a closed and convex set K, solves $VIP(A, Fix(P_K))$ if and only if $x^* = P_K(x^* - \lambda Ax^*)$ for some positive number λ.

This result is very important because it gives a basis for constructing iterative methods of approximating solutions of variational inequalities in Hilbert spaces.

The method previously used in solving the variational inequality problem in the late 1960's and later was the gradient projection method

$$x_{n+1} := P_K(x_n - \lambda_{n+1} \nabla f(x_n)), n \geq 1, \tag{3}$$

where λ_n is a suitably defined sequence of real numbers. This algorithm has been employed widely in applications because it has a good rate of convergence.

Under suitable conditions, the sequence generated from this algorithm converges to a solution of the smooth convex optimization problem posed in the Hilbert space H as:

$$(SCOP) \begin{cases} \text{Minimize } f : H \to \mathbb{R}, \text{(G-differentiable convex functional)} \\ \text{subject to } x \in K(\subseteq H) \text{(closed convex set).} \end{cases} \tag{4}$$

It is well know that x^* in K solves problem $(SCOP)$ if and only if it satisfies $\langle y - x^*, \nabla f(x^*) \rangle \geq 0, \forall y \in K$. The gradient projection method relies on the fact that for any closed convex subset K of a Hilbert space, $Fix(P_K) = K$ and $P_K : H \to K \subset H$ is a nonexpansive mapping with a nonempty fixed point set. However, the computation of the projection mapping P_K is difficult(except when the convex set K has simple structures) in application.

Based on the fact above, replacing the projection mapping P_K by an arbitrary nonexpansive mapping T, Yamada [16] introduced the steepest descent method given by

$$x_{n+1} := Tx_n - \lambda_{n+1}A(Tx_n), n \geq 1 \ (A := \nabla f). \tag{5}$$

This choice is because $y_n := Tx_n$ is generated by $y_{n+1} := T(y_n - \lambda_{n+1}\nabla f(y_n))$ (the gradient projection method) and for $x^* \in Fix(T)$, if $x^* = \lim x_n$, then $x^* = \lim y_n$. Thus the method can solve the problem $(SCOP)$ over $K = Fix(T)$ where T is a nonexpansive self map of H and $\{\lambda_n\}_{n=1}^{\infty}$ is suitably defined as stated below.

Theorem 1.4 [Hybrid steepest descent method for VIP(A,Fix(T)[16] Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Suppose that a mapping $A : H \to H$ is $L-$Lipschitzian and $\eta-$ strongly monotone over $T(H)$. Then for any $x_0 \in H$, and $\mu \in (0, \frac{2\eta}{L^2})$, and any sequence satisfying

$$(A1) \ \lim \lambda_n = 0, \ (A2) \ \sum_{n=1}^{\infty} \lambda_n = \infty, \ and \ (A3) \ \lim(\lambda_n - \lambda_{n+1})\lambda_{n+1}^{-2} = 0,$$

the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (5) converges strongly to the uniquely existing solution of the problem (2).

If $K = \cap_{n=1}^{r} Fix(T_i) \neq \emptyset$, where $\{T_i\}_{i=1}^{r}$ is a finite family of nonexpansive mappings, Yamada [16] studied the following algorithm

$$x_{n+1} = T_{[n]}x_n - \lambda_n \mu A(T_{[n]}x_n), \ n \geq 1, \tag{6}$$

where $T_{[k]} = T_{k \bmod r}$, for $k \geq 1$ and the sequence $\{\lambda_n\}$ satisfies condition $(A1)$, $(A2)$, and $(A4)$: $\sum |\lambda_n - \lambda_{n+N}| < \infty$, and proved the strong convergence of $\{x_n\}$ to the unique solution of problem (2).

In the case where $A := \nabla f$, he obtained $x_n \to x^* \in \arg \inf_{x \in Fix(T)} f(x)$, where $\nabla f : H \to H^*(= H)$ is the gradient of the convex functional f.

However, most problems of practical significance are not posed in Hilbert spaces. Obviously, for an arbitary real Banach space $X^* \neq X$. Besides, the exact expression of the duality mapping $J_q : X \to 2^{X^*}$ defined by $J_q(x) = \{x^* \in X^* : \|x\|^q = \|x^*\|^q = \langle x, x^* \rangle\}$ is known only in L_p spaces,

$1 < p < \infty$. Therefore, it makes sense if we limit our study to L_p spaces, $1 < p < \infty$ where it is practically possible to compute the duality mapping.

Based on these assertions, an ideal extension of the problem of Yamada (5) to a Banach spaces and which would solve the problem (SCOP) in Banach spaces ought to be:

$$VIP^*(A, Fix(T)) \begin{cases} \text{Given a } \textit{nonexpansive} \text{ mapping } T : X \to X \text{ and a strongly} \\ \text{monotone } L\text{−Lipschitzian mapping } A : X \to X^*, \\ \text{find } x^* \in Fix(T) \ : \ \langle y - x^*, Ax^* \rangle \geq 0, \forall y \in Fix(T). \end{cases}$$

Considerable research efforts have been devoted to this problem in Hilbert spaces. For example, Xu and Kim [15], replaced the condition $(A3)$ by the less restrictive condition $\lim\limits_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0$ and the condition $(A4)$ replaced by $\lim\limits_{n\to\infty} \frac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0$. The theorems of Xu and Kim [15] are improvements of the results of Yamada because the canonical choice sequence $\lambda_n = \frac{1}{n+1}$ is applicable there but it is not applicable in the result of Yamada [16] with condition $(C3)$. Other significant extensions of the theorems in Hilbert spaces can be found in Wang [18] , Zeng and Yao [19], and Yamada et al. [17].

Some of the extensions of the theorem to the more general Banach spaces include Chidume *et al.* [5, 6], Sahu et al. [20],

Most of the extensions of the theorem of Yamada [16] to more general Banach spaces have focused on the problem

$$VIP(A, Fix(T)) \begin{cases} \text{Given a } \textit{nonexpansive} \text{ mapping } T : X \to X \text{ and a strongly} \\ \text{accretive } L\text{−Lipschitzian mapping } A : X \to X, \\ \text{find } x^* \in Fix(T) \ : \ \langle y - x^*, j_q(Ax^*) \rangle \geq 0, \forall y \in Fix(T). \end{cases}$$

This problem certainly has a lot of applications in evolution equation and other area of interest , but it does not neccesarily solve the original optimization problem (SCOP). The problem (SCOP) arise in diverse disciplines as differential equations, convex optimization problems, time-optimal control, mathematical programming, demand problems, transport and network problems and so on. Details about these problems can be found, for example, in Kindelehrer and Stampacchia [9], Nagurney [11], and Noor [12].

Though there has been significant progress in solving problem VIP(A, Fix(T)), the successes achieved so far in using many geometric properties of spaces, developed in the last two centuries or so, in approximating zeros of accretive-type operators in Banach spaces have not been acchieved in approximating zeros of monotone mappings. The major difficulty in any attempt in this direction is that A goes from E to E^* and most iterative algorithm involving x_n and Ax_n are not suitably defined.

In some case, attempts are made to construct the algorithm by introducing the duality mapping. However the exact values of the duality mapping is unknown outside L_p spaces, for $1 < p < \infty$. Thus, the sequence obtained thereby are usually not possible to implement for practical uses.

In this paper, motivated by Chidume et al.[7], we propose an algorithm for the problem VIP$^*(A, Fix(T))$ in L_p, spaces for $1 < p < \infty$. Our theorems complements the results of Chidume *et al.*[5,6], Tan and Xu [14], extends to L_p spaces the result of Yamada [16], and generalize the results of Chidume *et al.* [7].

2. Preliminaries

In this section we recall some definitions and characterizations of the L_p spaces by the duality mappings. The normalized duality mapping is $J : X \to 2^{X^*}$ given by

$$J(x) = \{ j(x) \in X^* : \langle j(x), x \rangle = \|x\|^2 = \|j(x)\|^2 \}. \tag{7}$$

Some of its very useful properties are:

(a) For any $x \in X$, $J(x) \neq \emptyset$ (due to Hahn Banach theorem).

(b) For any real number, say α, $J(\alpha x) = \alpha J(x)$, for all $x \in X$.

(c) If X is a reflexive and smooth Banach space, then J is single-valued and onto.

(d) If X is strictly convex, then J is $1-1$.

(e) If X is reflexive and strictly convex and X^* is strictly convex, then $J^* : X^* \rightrightarrows X^{**}(=X)$ is a duality mapping on X^* satisfying $J^{-1} = J^*$.

The normalized duality mapping is in most cases nonlinear and it is not symmetric unless X is a Hilbert space. Thus, in the conjectural formula

$$\langle x, J(y) \rangle = \langle y, J(x) \rangle \quad (?),$$

the left hand side is linear in x, but the right hand side is not, unless J is a linear map.

Example: Let $X = \ell^4$. Then, the duality map $J : \ell^4 \to \ell^{4/3}$ is

$$J(x) = (x_1^3, x_2^3, x_3^3, \dots)$$

Therefore,

$$\langle x, J(y) \rangle = \sum_i x_i y_i^3,$$

which is not the same as

$$\langle y, J(x) \rangle = \sum_i x_i^3 y_i.$$

Lemma 2.1[see e.g. Chidume [4]] Let $E = L_p,\ 1 < p \leq 2$. Then the following inequalities hold.

(i) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, j(x) \rangle + c_p \|y\|^2,$ for some $c_p > 0$

(ii) $\langle x-y, j(x)-j(y)\rangle \geq (p-1)\|x-y\|^2$,

where j is the normalized duality mapping.

For the case of L_p spaces, $p \geq 2$, the following lemma is applicable.

Lemma 2.2[Alber and Ryanzantseva [3], p.48] Let $X = L_p,\ p \geq 2.$ Then, the inverse of the normalized duality mapping $j^{-1} : X^* \to X$ is Holder continuous on balls. i.e. $\forall u, v \in X^*$ such that $\|u\| \leq R,\ \ \|v\| \leq R$, then

$$\|j^{-1}(u) - j^{-1}(v)\| \leq m_p\|u-v\|^{\frac{1}{p-1}},$$

where $m_p := (2^{p+1}L_p c_2^p)^{\frac{1}{p-1}} > 0$ for some $c_2 > 0$.

Definition 2.3 Let E be a smooth real Banach space. The Lyapunov's function is a distance function $\phi : E \times E \to \mathbb{R}$ given by

$$\phi(x,y) := \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2.$$

In recent times, this type of functional has been studied extensively by many authors including Alber [1], Alber and Guerre-Delabriere [2], Kamimura and Takahashi [10], Reich [13]. It has proved to be a very useful tool for the study of nonlinear mappings in the general Banach spaces. It is known that on a Hilbert space H, there holds $\phi(x,y) = \|x-y\|^2$. Moreover, by the fact that the normalized duality mapping is the subdifferential of the functional defined by $f(x) = \frac{1}{2}\|x\|^2$, we have that $\phi(x,y) \geq 0$ for all x, y in E.

We define a parallel function $V : E \times E^* \to \mathbb{R}$ by

$$V(x,x^*) = \phi(x, j^{-1}(x^*)), \quad \forall x \in X, x^* \in X^*.$$

The functional is characterized by the following

Lemma2.4[Alber [1]] Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,

$$V(x,x^*) \leq V(x,x^*+y^*) - 2\langle j^{-1}x^* - x, y^*\rangle,\ \forall\, x \in X, x^*, y^* \in X^*. \tag{8}$$

Following the terminology of Alber and Guerre-Delabriere [2], as can be found also in Chidume *et al.*,[8], we present the following definitions.

Definition 2.5 Let K be a nonempty subset of a Banach space E. A map $T: K \to E$ is called:

- *strongly suppresive* on K if there exist $0 < q < 1$ such that

$$\phi(Tx, Ty) \leq q\phi(x,y) \ \forall \ x, y \in K, and \tag{9}$$

- nonextensive if

$$\phi(Tx, Ty) \leq \phi(x,y) \ \forall \ x, y \in K. \tag{10}$$

It follow from inequalities (9) and (10) above that in Hilbert spaces, nonextensive mappings are precisely the nonexpansive mapping and the strongly suppresive mappings are the strict contractions. For this reason, in this paper, we will weaken the nonexpansive assumption in theorem of Yamada to Nonextensive.

Lemma 2.6[Xu and Kim [15]] Assume that $\{x_n\}$ is a sequence of nonnegative reals numbers satisfying the conditions

$$x_{n+1} \leq (1-\alpha_n)x_n + \alpha_n\beta_n, \forall n \geq 1 \tag{11}$$

(i) $\{\alpha_n\} \subseteq [0,1]$, (ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and (iii) $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n < \infty$. Then, $\lim\limits_{n\to\infty} x_n = 0$.

3. Convergence Theorems in L_p spaces $1 < p \leq 2$

Theorem 3.1 Let $E = L_p, 1 < p \leq 2$, and $E^* = L_q$, $\frac{1}{p} + \frac{1}{q} = 1$. For $k = 1, 2, ..., N$, let $T_k : E \to E$ be a finite family of nonextensive mappings and $A : E \to E^*$ be an $\eta-$strongly monotone mapping which is also $L-$Lipschitzian. Assume that $S := A^{-1}(0) \cap \cap_{k=1}^{N} Fix(T_k) \neq \emptyset$. Then for arbitrary $x_1 \in E$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = j^{-1}\Big(j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)\Big), n \geq 1 \tag{12}$$

converges to the common solution of the problem VIP$^*(A, Fix(T_{[n]}))$, where $T_{[n]} := T_{n \, mod \, N}$, and $\lambda \in (0, \frac{\eta}{2L_1^2 L_2})$, L_1, L_2 the Lipschitz constants for the mappings A and j^{-1}, respectively.

Proof. Let $x^* \in S$. Then the sequence $\{x_n\}$ satisfies

$$\begin{aligned}
\phi(x^*,x_{n+1}) &= V(x^*, j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)) \\
&\leq V(x^*, j(T_{[n]}x_n)) - 2\lambda \left\langle j^{-1}\Big(j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)\Big) - x^*, AT_{[n]}x_n - Ax^* \right\rangle \\
&= \phi(x^*, T_{[n]}x_n) - 2\lambda \left\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^* \right\rangle \\
&+ 2\lambda \left\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^* \right\rangle \\
&- 2\lambda \left\langle j^{-1}((j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)) - x^*, A(T_{[n]}x_n) - Ax^* \right\rangle \\
&= \phi(x^*, T_{[n]}x_n) - 2\lambda \left\langle Tx_n - x^*, AT_{[n]}x_n - Ax^* \right\rangle \\
&- 2\lambda \left\langle j^{-1}((j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)) - j^{-1}(j(T_{[n]}x_n)), AT_{[n]}x_n - Ax^* \right\rangle. \\
&\leq \phi(x^*, T_{[n]}x_n) - 2\lambda\eta \|T_{[n]}x_n - x^*\|^2 \\
&+ 2\lambda \|j^{-1}\Big((j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)\Big) - j^{-1}\Big(j(T_{[n]}x_n)\Big)\| \|AT_{[n]}x_n - Ax^*\|
\end{aligned}$$

By the $\eta-$strong monotonicity of A, we obtain that

$$\langle T_{[n]}x_n - x^*, AT_{[n]}x_n - Ax^* \rangle \geq \eta \|T_{[n]}x_n - x^*\|^2.$$

On the other hand, using the fact that each of the mappings T_k are nonextensive, we have that

$$\phi(x^*, T_{[n]}x_n) = (T_{[n]}x^*, T_{[n]}x_n) \leq \phi(x^*, x_n)$$

Therefore, substituting these relations into the chain of inequalities above, and using the fact that $\lambda \in (0, \frac{\eta}{2L_1^2 L_2})$, we obtain:

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \phi(x^*, T_{[n]}x_n) - 2\lambda\eta \|T_{[n]}x_n - x^*\|^2 \\
&\quad + 2\lambda^2 L_1^2 L_2 \|T_{[n]}x_n - x^*\|^2 \\
&\leq \phi(x^*, T_{[n]}x_n) - \lambda\eta \|T_{[n]}x_n - x^*\|^2 \\
&\leq \phi(x^*, x_n) - \lambda\eta \|T_{[n]}x_n - x^*\|^2.
\end{aligned}$$

Thus $\phi(x^*, x_n)$ is a monotone non-increasing sequence of real numbers that is bounded below, and therefore converges. On the otherhand the same inequality yields

$$\lambda\eta \|T_{[n]}x_n - x^*\|^2 \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}). \tag{13}$$

Taking limits on both sides of the inequality (13), we have that $\lim_{n\to\infty} T_{[n]}x_n = x^*$. But we have that

$$\begin{aligned}\|x_{n+1} - T_{[n]}x_n\| &= \|j^{-1}(j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)) - j^{-1}(j(T_{[n]}x_n)\| \\ &\leq \lambda L_2 \|A(T_{[n]}x_n - A(x^*)\| \\ &\leq \lambda L_2 L_1^2 \|T_{[n]}x_n - x^*\| \to 0, \text{ as } n \to \infty.\end{aligned}$$

□

Therefore, we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \|x_{n+1} - T_{[n]}x_n\| + \|T_{[n]}x_n - x^*\| \\ &\leq (1 + \lambda L_2 L_1^2)\|T_{[n]}x_n - x^*\|\end{aligned}$$

and thus $\lim_{n\to\infty} x_n = x^*$. The uniqueness of x^* follows from the strong monotonicity of the mapping A.

Iin the special case when $T_k = I$ the identity mapping for each k, we have the following result of Chidume *et al.* [7]:

Corollary 3.2 Let $E = L_p, 1 < p \leq 2$, and $E^* = L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, and $A : E \to E^*$ be an $\eta-$strongly monotone mapping which is also $L-$Lipschitzian. Assume that $A^{-1}(0) \neq \emptyset$. Then for arbitrary $x_1 \in E$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = j^{-1}\Big(j(x_n) - \lambda A(x_n)\Big), n \geq 1 \tag{14}$$

converges to the uniquely existing $x^* \in A^{-1}(0)$, where $\lambda \in (0, \frac{\eta}{2L_1^2 L_2})$, L_1, L_2 the Lipschitz constants for the mappings A and j^{-1}, respectively.

4. Convergence Theorems in L_p spaces, $2 \leq p < \infty$.

Theorem 4.1 Let $E = L_p,\ 2 \leq p < \infty$ and $A : L_p \to L_q,\ \frac{1}{p} + \frac{1}{q} = 1$, be an η-strongly monotone mapping which is also Lipschitzian. For $k = 1, 2, ..., N$, let $T_k : L_p \to L_p$ be a finite family of nonextensive mappings. Assume that $S := A^{-1}(0) \cap \cap_{k=1}^N Fix(T_k) \neq \emptyset$. Then for arbitrary $x_1 \in E$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = j^{-1}\Big(j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big), n \geq 1 \tag{15}$$

converges strongly to the unique common solution of the problem VIP$^*(A, Fix(T_k))$, where $T_{[n]} := T_{n\,mod\,N}$, and $\lambda_n \in \Big(0, \frac{\eta}{2L_1 L_2^{\frac{p}{p-1}}}\Big)$ satisfies $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} \lambda_n^{\frac{p}{p-1}} < \infty$, L_1, L_2 are the Lipschitz constants for the mappings A and j^{-1}, respectively.

Proof. Let $x^* \in S$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated satisfies

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= V(x^*, j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)) \\
&\leq V(x^*, j(T_{[n]}x_n)) - 2\lambda_n \Big\langle j^{-1}\Big(j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big) - x^*, Ax_n - Ax^*\Big\rangle \\
&= \phi(x^*, T_{[n]}x_n) - 2\lambda_n \Big\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^*\Big\rangle \\
&+ 2\lambda_n \Big\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^*\Big\rangle \\
&- 2\lambda_n \Big\langle j^{-1}\Big((j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big) - x^*, A(T_{[n]}x_n) - Ax^*\Big\rangle \\
&= \phi(x^*, T_{[n]}x_n) - 2\lambda_n \Big\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^*\Big\rangle \\
&+ 2\lambda_n \Big\langle j^{-1}(jT_{[n]}x_n) - j^{-1}\Big((j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big), A(T_{[n]}x_n) - Ax^*\Big\rangle \\
&\leq \phi(x^*, T_{[n]}x_n) - 2\lambda_n \Big\langle T_{[n]}x_n - x^*, AT_{[n]}x_n - Ax^*\Big\rangle \\
&+ 2\lambda_n \|j^{-1}\Big((j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big) - j^{-1}\Big(j(T_{[n]}x_n)\Big)\|\|AT_{[n]}x_n - Ax^*\|
\end{aligned}$$

By the strong monotonicity of A, and the Holder continuity of j^{-1}, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \phi(x^*, T_{[n]}x_n) - 2\lambda_n \eta \|T_{[n]}x_n - x^*\| \\
&+ 2\lambda_n^{\frac{p}{p-1}} m_p \|AT_{[n]}x_n - Ax^*\|^{\frac{p}{p-1}}, \\
&\leq \phi(x^*, T_{[n]}x_n) - 2\lambda_n \eta \|T_{[n]}x_n - x^*\| \\
&+ 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}} \|T_{[n]}x_n - x^*\|^{\frac{p}{p-1}}.
\end{aligned}$$

Now, for $p \geq 2$, if $\|T_{[n]}x_n - x^*\| \geq 1$, then, $\|T_{[n]}x_n - x^*\|^{\frac{p}{p-1}} \leq \|T_{[n]}x_n - x^*\|^2$. So $2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}} \|T_{[n]}x_n - x^*\|^{\frac{p}{p-1}} \leq \lambda \eta \|T_{[n]}x_n - x^*\|^2$ Therefore, we have for this case

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - \lambda_n \eta \|T_{[n]}x_n - x^*\|^2.$$

Otherwise $\|T_{[n]}x_n - x^*\| < 1$ and thus $2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}} \|T_{[n]}x_n - x^*\|^{\frac{p}{p-1}} \leq 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}$. Thus, in any case,

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \phi(x^*, T_{[n]}x_n) - \lambda_n \eta \|T_{[n]}x_n - x^*\|^2 \\ &\quad + 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}. \\ &\leq \phi(x^*, T_{[n]}x_n) - \lambda_n \eta \phi(T_{[n]}x_n, x^*) \\ &\quad + 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}.\end{aligned}$$

□

Using the fact that the mapping T_k are nonextensive we conclude that

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq (1 - \lambda_n \eta)\phi(x^*, T_{[n]}x_n) + 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}} \\ &\leq (1 - \lambda_n \eta)\phi(x^*, x_n) + 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}\end{aligned}$$

.

Therefore we may conclude by Lemma () that $x_n \to x^*$.

Remark The canonical choice for the sequence λ_n is $\lambda_n := \frac{1}{n}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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