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FIXED POINTS OF $(\psi,\phi)-$ ALMOST WEAKLY CONTRACTIVE MAPS IN FUZZY METRIC SPACES

MANTHENA PRAPOORNA*, MANCHALA RANGAMMA

Department of Mathematics, University College of Science, Osmania University, Hyd., Telangana, 500007 India

Abstract. In this paper,we come out with the approach of $(\psi,\phi)-$ almost weakly contractive maps in the context of fuzzy metric spaces by extending the results given by G. V. R. Babu, Dasari Ratna babu, Kanuri Nageswara Rao, Bendi Venkata Siva Kumar [1] in the notion of G - Metric Spaces. We prove a theorem to show the existence of a fixed point and also provide an example in support to our result.

Keywords: $(\psi,\phi)-$ almost weakly contractive maps; continuous t - norm; fuzzy metric space.

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1. Introduction

The advancement of fixed point theory is rooted on the generalization of contractive conditions and also on the generalization of circumambient spaces of the operator under consideration. Banach Contraction Principle portrays a key role in the theory of fixed point. In the view of generalizations of contractive conditions, Alber and Guerre- Delabriere[2] in the year 1997 came out with the concept of weakly contractive mappings as an extension of contractive maps in Hilbert spaces which were later outstretched by Rhoades[3] in the perception of arbitrary

*Corresponding author

E-mail address: m_prapoorna@yahoo.com

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banach spaces. Weak contractions were presented by Berinde[4] for a complete metric space as a generalization of contraction maps which were later renamed by him as almost contractions. In 2008, Dutta and Choudary[5] introduced (ψ,ϕ) - weakly contractive maps and showed the existence of fixed points in complete metric spaces. In 2009, Doric[6] unfolded it to a pair of maps by broadening the result that was proposed by Q. Zhang and Y. Song[7]. J.Harjani and K.Sadarangani[8], presented some fixed point results in a complete metric space bestowed with a partial order for weakly C- contractive mappings. P. Saha[9] established a weakened version of contraction mapping principle in fuzzy metric space with a partial ordering. In the present work, we insinuate the concept of (ψ,ϕ) - almost weakly contractive maps in the panorama of fuzzy metric spaces and observe few results.

2. Preliminaries

Let us recall some prefaces here

Let $\Psi = \{\psi \,/\, \psi : [0, \infty) \to [0, \infty)$ is continuous on $[0, \infty)$, ψ is nondecreasing,
$\psi(t) > 0$ for $t > 0$, $\psi(0) = 0\}$

Definition 2.1.[5] Let (X, d) be a metric space. Let T: X $\to$ X be a self map. If $\exists$ ψ, $\phi \in \Psi$ such that

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y)) \tag{1}$$

for all x, y $\in$ X, then T is said to be a $(\psi, \phi) -$ weakly contractive map.

Definition 2.2.[5] Let (X, d) be a complete metric space. and T: X $\to$ X be a self map satisfying the inequality

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$$

where ψ, ϕ : $[0, \infty) \to [0, \infty)$ are both continuous and monotone non - decreasing functions with $\psi(t) = \phi(t) = 0$ iff $t = 0$. Then, T has a unique fixed point.

Theorem 2.3.[7] Let (X, d) be a complete metric space and T, S: X $\to$ X be two maps such that for all x, y $\in$ X

$$d(Tx,Ty) \leq M(x,y)) - \phi(M(x,y)) \tag{2}$$

where ϕ : [0, ∞) → [0, ∞) is a lower semi- continuous function with ϕ(t) > 0 for t ∈ (0, ∞) and $\phi(0) = 0$

$$M(x,y) = max\left\{d(x,y), d(Tx,x), d(Sy,y), \frac{1}{2}[d(y,Tx)+d(x,Sy)]\right\} \tag{3}$$

then ∃ a unique fixed point u ∈ X such that $u = Tu = Su$

Theorem 2.4.[6] Let (X, d) be a complete metric space and let T, S: X → X be two self-mappings such that for all x, y ∈ X

$$\psi(d(Tx,Sy)) \leq \psi(M(x,y)) - \phi(M(x,y)) \tag{4}$$

where

(a). ψ : [0, ∞) → [0, ∞) is a continuous monotone nondecreasing function with ψ(t) = 0 if and only if t = 0,

(b). ϕ : [0, ∞) → [0, ∞) is a lower semi-continuous function with ϕ(t) = 0 if and only if t = 0,

(c). M is defined by (3).

Then there exists a unique point u ∈ X such that u = Tu = Su.

On the other hand, fuzzy set introduced by Zadeh[10] in the year 1965 had led to enormous research in expanding the theory by defining different concepts and blending various properties. Here, in our work we consider the notion of fuzzy metric space that was stated by George and Veeramani.

Definition 2.5.[11] A binary operation ∗ : [0,1] × [0,1] → [0,1] is said to a continuous t - norm if ([0, 1], ∗) is an abelian topological monoid with unit 1 ∋ p ∗ q ≤ r ∗ s whenever p ≤ r and q ≤ s (p, q, r, s ∈ [0,1]).

Definition 2.6.[12] Let X be any non - empty set, ∗ is a continuous t - norm and M is a fuzzy set on X × X × (0,∞) satisfying

- M(x, y, t) > 0
- M(x, y, t) = 1 ⟺ x = y
- M(x, y, t) = M(y, x, t)
- M(x, y, t) ∗ M(y, z, s) ≤ M(x, z, t + s)
- M(x, y, .) : (0, ∞) → (0,1] is continuous where x, y, z ∈ X, s, t > 0

Here, M(x, y, t) denotes the degree of nearness between x , y with respect to t. Then, the 3- tuple (X, M, t) is called a fuzzy metric space.

M(x, y, .) is monotonic in third variable $\forall$ x, y $\in$ X.

Definition 2.7.[12] If $\{x_n\}$ is a sequence in a fuzzy metric space such that $M(x_n, x, t) \to 1$ whenever $n \to \infty$, then $\{x_n\}$ is said to converge to x $\in$ X.

Definition 2.8.[12] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is said to be a Cauchy sequence if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N \ni M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

Definition 2.9.[12] If every cauchy sequence in a fuzzy metric space X is convergent, then X is said to be complete.

Definition 2.10.[9] Let (X ,M,$*$) be a complete fuzzy metric space. Let C be a subset of X. Let T: C $\to$ C be a self mapping which satisfies the following inequality :

$$\psi(M(Tx,Ty,t)) \leq \psi(M(x,y,t)) - \phi(M(x,y,t) \tag{5}$$

where x, y $\in$ X , $t > 0$, ψ and ϕ : (0,1] $\to$ [0,∞) are two functions such that,

I. ψ is continuous and monotone decreasing with $\psi(t) = 0 \iff t = 1$

II. ϕ is continuous with $\phi(s) = 0 \iff s = 1$

Then T is said to be a weak contraction on C.

Definition 2.11.[1] Let (X, G) be a G-metric space and let T be a selfmap of X. If there exist ϕ, ψ in Ψ and $L \geq 0$ such that

$$\psi(G(Tx,Ty,Tz)) \leq \psi(G(x,y,z)) - \phi(G(x,y,z)) + Lm(x,y,z) \tag{6}$$

for all x, y, z $\in$ X, where

$$m(x,y,z) = min\{G(Tx,x,x), G(Tx,y,y), G(Tx,z,z), G(Tx,y,z)\} \tag{7}$$

and $\Psi = \{\psi \,/\, \psi : [0, \infty) \to [0, \infty)$ is continuous on $[0, \infty)$, ψ is nondecreasing,

$\psi(t) > 0$ for $t > 0$, $\psi(0) = 0\}$

then T is said to be a $(\psi, \phi) -$ almost weakly contractive map on X.

3. Main Results

Inspired by the (ψ, ϕ) − almost weakly contractive maps developed by G. V. R. Babu et al. in [1] , in this paper we bestow the (ψ, ϕ) − almost weakly contractivity condition in the context of fuzzy metric spaces and define it as follows:

3.1. **Definition:** Let (X, M, $*$) be a fuzzy metric space. Let there exists ψ, ϕ : (0, 1] $\to$ [0, ∞) such that

(3.1.1). ψ is continuous and monotonically decreasing,

(3.1.2). ψ (t) = 0 $\iff$ t = 1,

(3.1.3). ϕ is continuous and ϕ(s) = 0 $\iff$ s = 1

let T: X $\to$ X be a self map satisfying the inequality:

$$\psi(M(Tx,Ty,t)) \leq \psi(M(x,y,t)) - \phi(M(x,y,t)) + L\{m(x,y) - 1\} \tag{8}$$

for all x, y, z $\in$ X, where

$$m(x,y) = max\{M(Tx,x,t), M(Tx,y,t), M(Ty,x,t), M(Ty,y,t)\}$$

then T is said to be a (ψ, ϕ) − almost weakly contractive map on X.

3.2. **Theorem.** Let (X, M, $*$) be a complete fuzzy metric space and T: X $\to$ X be a (ψ, ϕ) − almost weakly contractive map. Then, T has a fixed point in X which is unique.

Proof. Let $\{x_n\}$ be a sequence in X $\ni$ $Tx_n = x_{n+1}$

If $x_n = x_{n+1}$, then the theorem is obvious.

let $x_n \neq x_{n+1}$

consider

$$\begin{aligned} \psi(M(x_n,x_{n+1},t)) &= \psi(M(Tx_{n-1},Tx_n,t)) \\ &\leq \psi(M(x_{n-1},x_n,t)) - \phi(M(x_{n-1},x_n,t)) \\ &\quad + L\{m(x_{n-1},x_n) - 1\} \end{aligned} \tag{3.2.1}$$

$$\begin{aligned} m(x_{n-1},x_n) &= max\{M(Tx_{n-1},x_{n-1},t),M(Tx_{n-1},x_n,t),M(Tx_n,x_{n-1},t), \\ &\qquad M(Tx_n,x_n,t)\} \\ &= max\{M(x_n,x_{n-1},t),M(x_n,x_n,t),M(x_{n+1},x_{n-1},t), \\ &\qquad M(x_{n+1},x_n,t)\} \\ &= max\{M(x_n,x_{n-1},t),1,M(x_{n+1},x_{n-1},t),M(x_{n+1},x_n,t)\} \\ &= 1 \end{aligned}$$

(3.2.2) $$m(x_{n-1},x_n) = 1$$

from (3.2.1) and (3.2.2), we get that

(3.2.3) $$\psi(M(x_n,x_{n+1},t)) \leq \psi(M(x_{n-1},x_n,t)) - \phi(M(x_{n-1},x_n,t))$$

(3.2.4) $$\psi(M(x_n,x_{n+1},t)) \leq \psi(M(x_{n-1},x_n,t))$$

we know ψ is monotonically decreasing

(3.2.5) $$\implies M(x_n,x_{n+1},t) > M(x_{n-1},x_n,t)$$

$\therefore \{M(x_n,x_{n+1},t)\}$ is an increasing sequence of non- negative real numbers.

Let $\lim_{n\to\infty} M(x_n,x_{n+1},t) = r$ then (3.2.3) $\implies \psi(r) \leq \psi(r) - \phi(r)$

$$\implies \phi(r) \leq 0 \implies \phi(r) = 0$$

$$\iff r = 1 \ (\because \text{from def } (3.1))$$

(3.2.6) $$\therefore \lim_{n\to\infty} M(x_n,x_{n+1},t) = 1$$

- To prove that $\{x_n\}$ is a Cauchy sequence:

Let $\{x_n\}$ is not cauchy. then, for any given $\varepsilon > 0$, we can find subsequences $\{x_{nk}\},\{x_{mk}\}$ of $\{x_n\}$ with $n_k > m_k \ni$

(3.2.7) $$M(x_{n_k},x_{m_k},t) \leq 1-\varepsilon$$

then, we have

$$M(x_{n_k-1},x_{m_k},t) > 1-\varepsilon, M(x_{n_k-1},x_{m_k-1},t) > 1-\varepsilon \tag{3.2.8}$$

Consider

$$\begin{aligned} &1-\varepsilon \geq M(x_{n_k},x_{m_k},t) \\ &1-\varepsilon \geq \lim_{k\to\infty} sup M(x_{n_k},x_{m_k},t) \end{aligned} \tag{3.2.9}$$

$$\begin{aligned} M(x_{n_k},x_{m_k},t) &\geq M(x_{n_k},x_{n_k-1},\frac{t}{2}) * M(x_{n_k-1},x_{m_k},\frac{t}{2}) \\ &> M(x_{n_k},x_{n_k-1},\frac{t}{2}) * 1-\varepsilon \quad (\because from(3.2.8)) \\ &> 1 * 1-\varepsilon \quad as \quad k\to\infty \quad (\because from(3.2.6)) \\ &\implies \lim_{k\to\infty} M(x_{n_k},x_{m_k},t) > 1-\varepsilon \end{aligned} \tag{3.2.10}$$

$$\therefore \lim_{k\to\infty} inf M(x_{n_k},x_{m_k},t) > 1-\varepsilon \tag{3.2.11}$$

from (3.2.9) and (3.2.11) we see that

$$1-\varepsilon < \lim_{k\to\infty} inf M(x_{n_k},x_{m_k},t) \leq \lim_{k\to\infty} sup M(x_{n_k},x_{m_k},t) < 1-\varepsilon$$

$\therefore \lim_{k\to\infty} M(x_{n_k},x_{m_k},t)$ exists and is equal to $1-\varepsilon$

$$i.e., \lim_{k\to\infty} M(x_{n_k},x_{m_k},t) = 1-\varepsilon \tag{3.2.12}$$

Consider

$$\begin{aligned} \psi(M(x_{n_k},x_{m_k},t)) &= \psi(M(Tx_{n_k-1},Tx_{m_k-1},t)) \\ &\leq \psi(M(x_{n_k-1},x_{m_k-1},t) - \phi(M(x_{n_k-1},x_{m_k-1},t) \\ &\quad + L\{m(x_{n_k-1},x_{m_k-1}) - 1\} \end{aligned} \tag{3.2.13}$$

from def. 3.1, (3.2.8), and since we know that ψ is a decreasing function, we have

$$M(x_{n_k-1},x_{m_k-1},t) > 1-\varepsilon \implies \psi(M(x_{n_k-1},x_{m_k-1},t)) \leq \psi(1-\varepsilon) \tag{3.2.14}$$

Since ϕ is continuous, we have

$$M(x_{n_k-1},x_{m_k-1},t) > 1-\varepsilon \implies \phi(M(x_{n_k-1},x_{m_k-1},t)) \geq \phi(1-\varepsilon) \tag{3.2.15}$$

also,

$$\begin{aligned} m(x_{n_k-1},x_{m_k-1}) &= max\{M(Tx_{n_k-1},x_{n_k-1},t),M(Tx_{n_k-1},x_{m_k-1},t),\\ &\qquad M(Tx_{m_k-1},x_{n_k-1},t),M(Tx_{m_k-1},x_{m_k-1},t)\}\\ &= max\{M(x_{n_k},x_{n_k-1},t),M(x_{n_k},x_{m_k-1},t),\\ &\qquad M(x_{m_k},x_{n_k-1},t),M(x_{m_k},x_{m_k-1},t)\} \end{aligned} \tag{3.2.16}$$

as k $\to \infty$, (3.2.16) $\implies$

$$\begin{aligned} m(x_{n_k-1},x_{m_k-1}) &= max\{1,M(x_{n_k},x_{m_k-1},t),M(x_{m_k},x_{n_k-1},t),1\}(\because eq.3.2.6)\\ &= 1 \end{aligned}$$

$$\therefore m(x_{n_k-1},x_{m_k-1}) = 1 \quad as \quad k \to \infty \tag{3.2.17}$$

Using (3.2.12), (3.2.14), (3.2.15)and (3.2.17), equation (3.2.13) becomes

$$\begin{aligned} \psi(M(x_{n_k},x_{m_k},t)) &\leq \psi(1-\varepsilon) - \phi(1-\varepsilon) + L\{m(x_{n_k-1},x_{m_k-1}) - 1\}\\ \psi(1-\varepsilon) &\leq \psi(1-\varepsilon) - \phi(1-\varepsilon) + L\{1-1\} \quad as \quad k \to \infty\\ \psi(1-\varepsilon) &\leq \psi(1-\varepsilon) - \phi(1-\varepsilon)\\ \implies \phi(1-\varepsilon) &\leq 0\\ \implies 1-\varepsilon &= 1 \implies \varepsilon = 0 \end{aligned}$$

which is a contradiction.Hence, our assumption is wrong and thus, $\{x_n\}$ is a Cauchy sequence in X.

Since, X is complete, we can find a z $\in$ X $\ni$ the sequence $\{x_n\}$ is convergent to z as n $\to \infty$

- To prove z is a fixed point of T in X:

Consider

$$\begin{aligned} \psi(M(x_n,Tz,t)) &= \psi(M(Tx_{n-1},Tz,t))\\ &\leq \psi(M(x_{n-1},z,t)) - \phi(M(x_{n-1},z,t))\\ &\qquad + L\{m(x_{n-1},z) - 1\} \end{aligned} \tag{3.2.18}$$

as n $\to \infty$ (3.2.18) becomes

$$\psi(M(z,Tz,t)) \leq \psi(M(z,z,t)) - \phi(M(z,z,t)) + L\{m(z,z) - 1\}$$

$$\psi(M(z,Tz,t)) \leq \psi(1) - \phi(1) + L\{M(Tz,z,t) - 1\}$$

$$\implies \psi(M(z,Tz,t)) \leq 0 + L\{M(Tz,z,t) - 1\} \quad (\because from \quad def.3.1)$$

$$\implies \psi(M(z,Tz,t)) \leq L\{M(Tz,z,t) - 1\}$$

which is possible $\iff$ M(z, Tz, t) = 1

$\therefore$ Tz = z $\implies$ z is a fixed point of T in X.

- To prove z is unique:

If possible, let z, w be two fixed points of T in X, then

$$\psi(M(z,w,t)) = \psi(M(Tz,Tw,t))$$
$$\leq \psi(M(z,w,t)) - \phi(M(z,w,t)) + L\{m(z,w) - 1\}$$
$$\leq \psi(M(z,w,t)) - \phi(M(z,w,t)) + L(0) \quad (\because m(z,w) = 1)$$
$$\psi(M(z,w,t)) \leq \psi(M(z,w,t)) - \phi(M(z,w,t))$$
$$\implies \phi(M(z,w,t)) \leq 0 \implies \phi(M(z,w,t)) = 0 \iff M(z,w,t) = 1$$

$\therefore$ M(z, w, t) = 1 which implies z = w i.e., fixed point is unique. Hence the theorem.

In support to our result let us see the following example:

3.3. **Example:** Let X= [0, 1] and $M(x,y,t) = \begin{cases} 1, & \text{if either x = 0 or y = 0} \\ \text{Max}\{\text{x, y}\} & \text{if x} \neq 0 \text{ and y} \neq 0 \end{cases}$ and $*$ be the continuous t-norm defined by a $*$ b = min$\{a,b\}$. Then, Clearly (X, M, $*$) is a complete fuzzy metric space. Let $T: X \to X$ be a self map in X defined by

$$Tx = \begin{cases} \frac{1}{2} & \text{if x = 0} \\ \text{2x} & \text{if } 0 < \text{x} < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Let ψ and ϕ on (0, 1] be defined by ψ(s) = 1 - s^2 and ϕ(s) = 1 - s Here, T satisfies the inequality (8) with any L $\geq$ 0.

$\therefore$ T is a (ψ, ϕ) $-$ almost weakly contractive map on X.

Thus, T satisfies the hypothesis of theorem 3. 2 and so, have a unique fixed point in X i.e., at x = 1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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