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QUADRUPLED FIXED POINT IN G-METRIC SPACE WITH AN APPLICATION

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Abstract. In this paper, we prove some quadruple coincidence and quadruple common fixed point theorems for $F : X^4 \to X$ and $g : X \to X$ satisfying weak contractions in partially ordered G-metric spaces. We illustrate our results based on an example on the main theorems. We also give an application of obtained results of this paper.

Keywords: quadruple fixed point; ordered sets; generalized metric spaces; mixed g-monotone property.

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1. Introduction

In 1992, B.C. Dhage introduced a new class of generalized metric space called D-metric spaces (see [7]). In a subsequent series of papers, Dhage attempted to develop topological structures in such spaces (see [8],[9],[10]). In [11], Mustafa and Sims demonstrate the claims concerning the fundamental topological structure of D-metric space are incorrect, also introduce a valid

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generalized metric space structure, which we call G-metric spaces. Some other papers dealing with G-metric spaces are those in ([2, 3, 4, 5, 6],[14] - [25]). Recently, there has been growing interest in establishing fixed point theorems in partially ordered complete G-metric spaces with a contractive condition which holds for all points that are related by partial ordering ([26],[29] and [46]).

The aim of this paper is to prove some quadruple coincidence and quadruple common fixed point theorems for $F : X^4 \to X$ and $g : X \to X$ satisfying weak contractions in partially ordered G-metric spaces. We illustrate our results based on an example on the main theorems. We also give an application of obtained results of this paper.

Definition 1.1. ([12]) *Let X be a nonempty set, and let $G : X \times X \times X \to \boldsymbol{R}^+$, be a function satisfying the following properties:*

(G1) $G(x,y,z) = 0$ *if* $x = y = z$;

(G2) $0 < G(x,x,y)$; *for all* $x, y \in X$, *with* $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$, *for all* $x,y,z \in X$, *with* $z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, *(symmetry in all three variables); and*

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, *for all* $x,y,z,a \in X$, *(rectangle inequality).*

Then the function G is called a generalized metric*, or, more specifically a G-*metric *on X, and the pair (X,G) is called a G-*metric space.

Example 1.1. ([12]) *Let (X,d) be a usual metric space, and define G_s and G_m on $X \times X \times X$ to* $\mathbf{R}^+$ *by*

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z), and$$

$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$$

for all $x,y,z \in X$. Then (X,G_s) and (X,G_m) are G-metric spaces.

Definition 1.2. ([12]) *Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the* limit *of the sequence (x_n) if* $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, *and one say that the sequence (x_n) is G-*convergent *to x.*

Thus, that if $x_n \longrightarrow 0$ *in a G-metric space* (X,G)*, then for any* $\varepsilon > 0$*, there exists* $N \in \mathbf{N}$ *such that* $G(x,x_n,x_m) < \varepsilon$*, for all* $n,m \geq N$*, (we mean by* $\mathbf{N}$ *the Natural numbers).*

Proposition 1.1. ([12]) *Let* (X,G) *be G-metric space. Then the following are equivalent.*

(1) (x_n) *is G-convergent to x.*

(3) $G(x_n,x_n,x) \to 0$*, as* $n \to \infty$*.*

(4) $G(x_n,x,x) \to 0$*, as* $n \to \infty$*.*

(5) $G(x_m,x_n,x) \to 0$*, as* $m,n \to \infty$*.*

Definition 1.3. ([12]) *Let* (X,G) *be a G-metric space, a sequence* (x_n) *is called G-Cauchy if given* $\varepsilon > 0$*, there is* $N \in \mathbf{N}$ *such that* $G(x_n,x_m,x_l) < \varepsilon$*, for all* $n,m,l \geq N$*. That is* $G(x_n,x_m,x_l) \longrightarrow 0$ *as* $n,m,l \longrightarrow \infty$*.*

Proposition 1.2. ([12]) *In a G-metric space,* (X,G)*, the following are equivalent.*

(1) The sequence (x_n) *is G-Cauchy.*

(2) For every $\varepsilon > 0$*, there exists* $N \in \mathbf{N}$ *such that* $G(x_n,x_m,x_m) < \varepsilon$*, for all* $n,m \geq N$*.*

Proposition 1.3. ([12]) *Let* (X,G)*, and* $(X^{'},G^{'})$ *be two G-metric spaces. Then a function* $f : X \longrightarrow X^{'}$ *is G-continuous at a point* $x \in X$ *if and only if it is G-sequentially continuous at x; that is, whenever* (x_n) *is G-convergent to x we have* $(f(x_n))$ *is G-convergent to* $f(x)$*.*

Definition 1.4. ([12]) *A G-metric space* (X,G) *is called symmetric G-metric space if* $G(x,y,y) = G(y,x,x)$ *for all* $x,y \in X$*.*

It is clear that, any G-metric space where G derives from an underlying metric via G_s or G_m in Example 1.1 is symmetric.

Proposition 1.4. ([12]) *Let* (X,G) *be a G-metric space, then the function* $G(x,y,z)$ *is jointly continuous in all three of its variables.*

Proposition 1.5. ([12]) *Every G-metric space* (X,G) *induces a metric space* (X,d_G) *defined by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \forall x,y \in X.$$

Note that if (X,G) is symmetric, then

$$d_G(x,y) = 2G(x,y,y), \forall x,y \in X. \tag{1.1}$$

However, if (X,G) is not symmetric then it holds by the G-metric properties that

$$\frac{3}{2}G(x,y,y) \leq d_G(x,y) \leq 3G(x,y,y), \forall x,y \in X. \tag{1.2}$$

Definition 1.5. ([12]) A G-metric space (X,G) is said to be *G-complete* (or complete G-metric) if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

Definition 1.6. *Let (X,G) be a G-metric Space. A mapping $F : X \times X \times X \times X \to X$ is said to be continuous if for any G-convergent sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ converging to x,y,z and w respectively $\{F(x_n,y_n,z_n,w_n)\}$ is G-convergent to $F(x,y,z,w)$*

Proposition 1.6. ([12])*A G-metric space (X,G) is G-complete if and only if (X,d_G) is a complete metric space.*

Following Erdal [52] we introduced the following definitions.

Definition 1.7. [52] *Let X be a nonempty set and $F : X \times X \times X \times X \to X$ be a given mapping. An element $(x,y,z,w) \in X \times X \times X \times X$ is called a quadruple fixed point of F if*

$$F(x,y,z,w) = x,\ F(y,z,w,x) = y,\ F(z,w,x,y) = z \quad \text{and} \quad F(w,x,y,z) = w.$$

Definition 1.8. [52] *Let $(X,\leq)$ be a partially ordered set and $F : X \times X \times X \times X \to X$ be a mapping. We say that F has the mixed monotone property if $F(x,y,z,w)$ is monotone non-decreasing in x and z and is monotone non-increasing in y and w; that is, for any $x,y,z,w \in X$,*

$$x_1,x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies} \quad F(x_1,y,z,w) \leq F(x_2,y,z,w),$$

$$y_1,y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies} \quad F(x,y_2,z,w) \leq F(x,y_1,z,w),$$

$$z_1,z_2 \in X, \quad z_1 \leq z_2 \quad \text{implies} \quad F(x,y,z_1,w) \leq F(x,y,z_2,w),$$

and

$$w_1,w_2 \in X, \quad w_1 \leq w_2 \quad \text{implies} \quad F(x,y,z,w_2) \leq F(x,y,z,w_1).$$

Definition 1.9. [52] *Let X be a non-empty set. Then we say that the mappings $F : X^4 \to X$ and $g : X \to X$ are commutative if for all $x,y,z,w \in X$*

$$g(F(x,y,z,w)) = F(gx,gy,gz,gw).$$

Definition 1.10. [57] *Let $(X,\le)$ be a partially ordered set. Let $F : X^4 \to X$ and $g : X \to X$. The mapping F is said to has the mixed g-monotone property if for any $x,y,z,w \in X$*

$$\begin{aligned}
x_1,\, x_2 \in X, \quad gx_1 \le gx_2 \quad &\Longrightarrow \quad F(x_1,y,z,w) \le F(x_2,y,z,w),\\
y_1,\, y_2 \in X, \quad gy_1 \le gy_2 \quad &\Longrightarrow \quad F(x,y_1,z,w) \ge F(x,y_2,z,w),\\
z_1,\, z_2 \in X, \quad gz_1 \le gz_2 \quad &\Longrightarrow \quad F(x,y,z_1,w) \le F(x,y,z_2,w) \ \textit{and}\\
w_1,\, w_2 \in X, \quad gw_1 \le gw_2 \quad &\Longrightarrow \quad F(x,y,z,w_1) \ge F(x,y,z,w_2).
\end{aligned}$$

Definition 1.11. [57] *Let $F : X^4 \to X$ and $g : X \to X$. An element (x,y,z,w) is called a quadruple coincidence point of F and g if*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \ \textit{ and } \ F(w,x,y,z) = gw.$$

(gx,gy,gz,gw) is said a quadruple point of coincidence of F and g.

Definition 1.12. [57] *Let $F : X^4 \to X$ and $g : X \to X$. An element (x,y,z,w) is called a quadruple common fixed point of F and g if*

$$\begin{aligned}
F(x,y,z,w) = gx = x, &\qquad F(y,z,w,x) = gy = y,\\
F(z,w,x,y) = gz = z \quad \textit{and} &\qquad F(w,x,y,z) = gw = w.
\end{aligned}$$

2. Main result

Denote Φ be the set of functions ϕ such that $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions,

(i) ϕ is continuous and non decreasing,

(ii) $\phi(t) = 0$ if and only if $t = 0$,

(iii) $\phi(\alpha t) \le \alpha\phi(t)$ for $\alpha \in (0,\infty)$

(iv) $\phi(t+s) \le \phi(t) + \phi(s)$ for all $s,t \in [0,\infty)$.

Also, Ψ be the set of all functions ψ such that $\psi : [0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\to[0,\infty)$ satisfying condition $\lim_{(t_1,t_2,t_3,t_4)\to(r_1,r_2,r_3,r_4)}\psi(t_1,t_2,t_3,t_4) > 0$ for all $(r_1,r_2,r_3,r_4)\in[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)$ with $r_1+r_2+r_3+r_4>0$. For example

(a) $\psi(t_1,t_2,t_3,t_4) = k\max\{t_1,t_2,t_3,t_4\}$ for some $k\in[0,1)$,

(b) $\psi(t_1,t_2,t_3,t_4) = \alpha_1 t_1^{p_1}+\alpha_2 t_2^{p_2}+\alpha_3 t_3^{p_3}+\alpha_4 t_4^{p_4}$ for $\alpha_1,\alpha_2,\alpha_3,\alpha_4,p_1,p_2,p_3,p_4>0$

(c) $\psi(t_1,t_2,t_3,t_4) = \frac{1-k}{2}(t_1+t_2+t_3+t_4)$ for some $k\in[0,1)$.

Theorem 2.1. *Let $(X,\leq)$ be a partially ordered set and (X,G) be a G-metric space. Let $F : X\times X\times X\times X\to X$ and $g : X\to X$ such that F has the mixed g-monotone property. Assume that there exists a $\phi\in\Phi$ and $\psi\in\Psi$ such that*

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &= \alpha_1 G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \\ &\quad +\alpha_2 G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) \\ &\quad +\alpha_3 G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) \\ &\quad +\alpha_4 G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) \end{aligned}$$

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &\leq \phi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right) \\ &\quad -\psi(G(gx,gu,ga),G(gy,gv,gb),G(gz,gs,gc),G(gw,gt,gd)). \end{aligned} \tag{2.1}$$

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in(0,\infty)$, $x,y,z,w,u,v,s,t,a,b,c,d\in X$ with $gx\geq gu\geq ga$, $gy\leq gv\leq gb$, $gz\geq gs\geq gc$, and $gw\leq gt\leq gd$. Suppose $F(X^4)\subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0,y_0,z_0,w_0\in X$ such that

$$gx_0\leq F(x_0,y_0,z_0,w_0),\qquad gy_0\geq F(y_0,z_0,w_0,x_0),$$
$$gz_0\leq F(z_0,w_0,x_0,y_0)\quad \textit{and}\quad gw_0\geq F(w_0,x_0,y_0,z_0),$$

Suppose either

(a) (X,G) is a complete G-metric space and F is continuous or,

(b) $(g(X),G)$ is complete and $(X,G,\leq)$ has the following property:

(*i*) *if non-decreasing sequence* $x_n \to a$*, then* $x_n \leq x$ *for all* n,

(*ii*) *if non-increasing sequence* $y_n \to y$*, then* $y \leq y_n$ *for all* n.

then there exist $x,y,z,w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw$$

that is, F *and* g *have a quadruple coincidence point.*

Proof. Let $x_0,y_0,z_0,w_0 \in X$ such that

$$gx_0 \leq F(x_0,y_0,z_0,w_0), \qquad gy_0 \geq F(y_0,z_0,w_0,x_0),$$
$$gz_0 \leq F(z_0,w_0,x_0,y_0) \quad \text{and} \quad gw_0 \geq F(w_0,x_0,y_0,z_0).$$

Since $F(X^4) \subset g(X)$, then we can choose $x_1,y_1,z_1,w_1 \in X$ such that

$$\begin{aligned} gx_1 &= F(x_0,y_0,z_0,w_0), & gy_1 &= F(y_0,z_0,w_0,x_0), \\ gz_1 &= F(z_0,w_0,x_0,y_0) \quad \text{and} & gw_1 &= F(w_0,x_0,y_0,z_0). \end{aligned} \tag{2.2}$$

Taking into account $F(X^4) \subset g(X)$, by continuing this process, we can construct sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ in X such that

$$\begin{aligned} gx_{n+1} &= F(x_n,y_n,z_n,w_n), & gy_{n+1} &= F(y_n,z_n,w_n,x_n), \\ gz_{n+1} &= F(z_n,w_n,x_n,y_n) \quad \text{and} & gw_{n+1} &= F(w_n,x_n,y_n,z_n). \end{aligned} \tag{2.3}$$

We shall show that

$$gx_n \leq gx_{n+1}, \quad gy_{n+1} \leq gy_n, \quad gz_n \leq gz_{n+1} \text{ and } gw_{n+1} \leq gw_n \text{ for } n = 0,1,2,... \tag{2.4}$$

For this purpose, we use the mathematical induction. Since, $gx_0 \leq F(x_0,y_0,z_0,w_0)$, $gy_0 \geq F(y_0,z_0,w_0,x_0)$, $gz_0 \leq F(z_0,w_0,x_0,y_0)$ and $gw_0 \geq F(w_0,x_0,y_0,z_0)$, then by (2.2), we get

$$gx_0 \leq gx_1, \quad gy_1 \leq gy_0, \quad gz_0 \leq gz_1 \text{ and } gw_1 \leq gw_0$$

that is, (2.4) holds for $n = 0$.

We presume that (2.4) holds for some $n > 0$. As F has the mixed g-monotone property and

$gx_n \leq gx_{n+1}$, $gy_{n+1} \leq gy_n$, $gz_n \leq gz_{n+1}$ and $gw_{n+1} \leq gw_n$, we obtain

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n, w_n) \leq F(x_{n+1}, y_n, z_n, w_n) \\ &\leq F(x_{n+1}, y_n, z_{n+1}, w_n) \leq F(x_{n+1}, y_{n+1}, z_{n+1}, w_n) \\ &\leq F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}) = gx_{n+2}, \end{aligned}$$

$$\begin{aligned} gy_{n+2} &= F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1}) \leq F(y_{n+1}, z_n, w_{n+1}, x_{n+1}) \\ &\leq F(y_n, z_n, w_{n+1}, x_{n+1}) \leq F(y_n, z_n, w_n, x_{n+1}) \\ &\leq F(y_n, z_n, w_n, x_n) = gy_{n+1}, \end{aligned}$$

$$\begin{aligned} gz_{n+1} &= F(z_n, w_n, x_n, y_n) \leq F(z_{n+1}, w_n, x_n, y_n) \\ &\leq F(z_{n+1}, w_{n+1}, x_n, y_n) \leq F(z_{n+1}, w_{n+1}, x_{n+1}, y_n) \\ &\leq F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1}) = gz_{n+2} \end{aligned}$$

and

$$\begin{aligned} gw_{n+2} &= F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}) \leq F(w_{n+1}, x_n, y_{n+1}, z_{n+1}) \\ &\leq F(w_n, x_n, y_{n+1}, z_{n+1}) \leq F(w_n, x_n, y_n, z_{n+1}) \\ &\leq F(w_n, x_n, y_n, z_n) = gw_{n+1}. \end{aligned}$$

Thus, (2.4) holds for any $n \in \mathbb{N}$. Assume for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \quad gy_n = gy_{n+1}, \quad gz_n = gz_{n+1} \text{ and } gw_n = gw_{n+1}$$

then, by (2.3), we have $gx_n = F(x_n, y_n, z_n, w_n), gy_n = F(y_n, z_n, w_n, x_n)$, $gz_n = F(z_n, w_n, x_n, y_n)$ and $gw_n = F(w_n, x_n, y_n, z_n) \Rightarrow (x_n, y_n, z_n, w_n)$ is a quadruple coincidence point of F and g. From now on, assume for any $n \in \mathbb{N}$ that at least

$$gx_n \neq gx_{n+1} \quad \text{or} \quad gy_n \neq gy_{n+1} \quad \text{or} \quad gz_n \neq gz_{n+1} \text{ or } gw_n \neq gw_{n+1}. \tag{2.5}$$

Since $gx_n \leq gx_{n+1}$, $gy_{n+1} \leq gy_n$, $gz_n \leq gz_{n+1}$, and $gw_{n+1} \leq gw_n$ then from 2.1 and 2.3 we have

$$\begin{aligned}
& M(x_n,y_n,z_n,w_n,x_n,y_n,z_n,w_n,x_{n-1},y_{n-1},z_{n-1},w_{n-1}) \\
= \;& \alpha_1 G(F(x_n,y_n,z_n,w_n),F(x_n,y_n,z_n,w_n),F(x_{n-1},y_{n-1},z_{n-1},w_{n-1})) \\
& +\alpha_2 G(F(y_n,z_n,w_n,x_n),F(y_n,z_n,w_n,x_n),F(y_{n-1},z_{n-1},w_{n-1},x_{n-1})) \\
& +\alpha_3 G(F(z_n,w_n,x_n,y_n),F(z_n,w_n,x_n,y_n),F(z_{n-1},w_{n-1},x_{n-1},y_{n-1})) \\
& +\alpha_4 G(F(w_n,x_n,y_n,z_n),F(w_n,x_n,y_n,z_n),F(w_{n-1},x_{n-1},y_{n-1},z_{n-1})) \\
= \;& \alpha_1 G(gx_{n+1},gx_{n+1},gx_n)+\alpha_2 G(gy_{n+1},gy_{n+1},gy_n) \\
& +\alpha_3 G(gz_{n+1},gz_{n+1},gz_n)+\alpha_4 G(gw_{n+1},gw_{n+1},gw_n) \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
& M(x_n,y_n,z_n,w_n,x_n,y_n,z_n,w_n,x_{n-1},y_{n-1},z_{n-1},w_{n-1}) \\
\leq \;& \phi\left(\frac{G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})+G(gz_n,gz_n,gz_{n-1})+G(gw_n,gw_n,gw_{n-1})}{4}\right) \\
& -\psi(G(gx_n,gx_n,gx_{n-1}),G(gy_n,gy_n,gy_{n-1}),G(gz_n,gz_n,gz_{n-1}),G(gw_n,gw_n,gw_{n-1})). \qquad (2.7)
\end{aligned}$$

Similarly we have,

$$\begin{aligned}
& M(y_n,z_n,w_n,x_n,y_n,z_n,w_n,x_n,y_{n-1},z_{n-1},w_{n-1},x_{n-1}) \\
= \;& \alpha_1 G(F(y_n,z_n,w_n,x_n),F(y_n,z_n,w_n,x_n),F(y_{n-1},z_{n-1},w_{n-1},x_{n-1})) \\
& +\alpha_2 G(F(z_n,w_n,x_n,y_n),F(z_n,w_n,x_n,y_n),F(z_{n-1},w_{n-1},x_{n-1},y_{n-1})) \\
& +\alpha_3 G(F(w_n,x_n,y_n,z_n),F(w_n,x_n,y_n,z_n),F(w_{n-1},x_{n-1},y_{n-1},z_{n-1})) \\
& +\alpha_4 G(F(x_n,y_n,z_n,w_n),F(x_n,y_n,z_n,w_n),F(x_{n-1},y_{n-1},z_{n-1},w_{n-1})) \\
= \;& \alpha_1 G(gy_{n+1},gy_{n+1},gy_n)+\alpha_2 G(gz_{n+1},gz_{n+1},gz_n) \\
& +\alpha_3 G(gw_{n+1},gw_{n+1},gw_n)+\alpha_4 G(gx_{n+1},gx_{n+1},gx_n) \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
&M(y_n,z_n,w_n,x_n,y_n,z_n,w_n,x_n,y_{n-1},z_{n-1},w_{n-1},x_{n-1})\\
\leq\ &\phi\left(\frac{G(gy_n,gy_n,gy_{n-1})+G(gz_n,gz_n,gz_{n-1})+G(gw_n,gw_n,gw_{n-1})+G(gx_n,gx_n,gx_{n-1})}{4}\right)\\
&-\psi(G(gy_n,gy_n,gy_{n-1}),G(gz_n,gz_n,gz_{n-1}),G(gw_n,gw_n,gw_{n-1}),G(gx_n,gx_n,gx_{n-1})). \qquad (2.9)
\end{aligned}$$

$$\begin{aligned}
&M(z_n,w_n,x_n,y_n,z_n,w_n,x_n,y_n,z_{n-1},w_{n-1},x_{n-1},y_{n-1})\\
=\ &\alpha_1 G(F(z_n,w_n,x_n,y_n),F(z_n,w_n,x_n,y_n),F(z_{n-1},w_{n-1},x_{n-1},y_{n-1}))\\
&+\alpha_2 G(F(w_n,x_n,y_n,z_n),F(w_n,x_n,y_n,z_n),F(w_{n-1},x_{n-1},y_{n-1},z_{n-1}))\\
&+\alpha_3 G(F(x_n,y_n,z_n,w_n),F(x_n,y_n,z_n,w_n),F(x_{n-1},y_{n-1},z_{n-1},w_{n-1})) \qquad (2.10)\\
&+\alpha_4 G(F(y_n,z_n,w_n,x_n),F(y_n,z_n,w_n,x_n),F(y_{n-1},z_{n-1},w_{n-1},x_{n-1}))\\
=\ &\alpha_1 G(gz_{n+1},gz_{n+1},gz_n)+\alpha_2 G(gw_{n+1},gw_{n+1},gw_n)\\
&+\alpha_3 G(gx_{n+1},gx_{n+1},gx_n)+\alpha_4 G(gy_{n+1},gy_{n+1},gy_n)
\end{aligned}$$

$$\begin{aligned}
&M(z_n,w_n,x_n,y_n,z_n,w_n,x_n,y_n,z_{n-1},w_{n-1},x_{n-1},y_{n-1})\\
\leq\ &\phi\left(\frac{G(gz_n,gz_n,gz_{n-1})+G(gw_n,gw_n,gw_{n-1})+G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})}{4}\right)\\
&-\psi(G(gz_n,gz_n,gz_{n-1}),G(gw_n,gw_n,gw_{n-1}),G(gx_n,gx_n,gx_{n-1}),G(gy_n,gy_n,gy_{n-1})). \qquad (2.11)
\end{aligned}$$

$$\begin{aligned}
&M(w_n,x_n,y_n,z_n,w_n,x_n,y_n,z_n,w_{n-1},x_{n-1},y_{n-1},z_{n-1})\\
=\ &\alpha_1 G(F(w_n,x_n,y_n,z_n),F(w_n,x_n,y_n,z_n),F(w_{n-1},x_{n-1},y_{n-1},z_{n-1}))\\
&+\alpha_2 G(F(x_n,y_n,z_n,w_n),F(x_n,y_n,z_n,w_n),F(x_{n-1},y_{n-1},z_{n-1},w_{n-1}))\\
&+\alpha_3 G(F(y_n,z_n,w_n,x_n),F(y_n,z_n,w_n,x_n),F(y_{n-1},z_{n-1},w_{n-1},x_{n-1})) \qquad (2.12)
\end{aligned}$$

$$
\begin{aligned}
& +\alpha_4 G(F(z_n,w_n,x_n,y_n),F(z_n,w_n,x_n,y_n),F(z_{n-1},w_{n-1},x_{n-1},y_{n-1}))\\
= & \ \alpha_1 G(gw_{n+1},gw_{n+1},gw_n)+\alpha_2 G(gx_{n+1},gx_{n+1},gx_n)\\
& +\alpha_3 G(gy_{n+1},gy_{n+1},gy_n)+\alpha_4 G(gz_{n+1},gz_{n+1},gz_n)
\end{aligned}
$$

$$
\begin{aligned}
& M(w_n,x_n,y_n,z_n,w_n,x_n,y_n,z_n,w_{n-1},x_{n-1},y_{n-1},z_{n-1})\\
\leq & \ \phi\left(\frac{G(gw_n,gw_n,gw_{n-1})+G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})+G(gz_n,gz_n,gz_{n-1})}{4}\right)\\
& -\psi(G(gw_n,gw_n,gw_{n-1}),G(gx_n,gx_n,gx_{n-1}),G(gy_n,gy_n,gy_{n-1}),G(gz_n,gz_n,gz_{n-1})). \quad (2.13)
\end{aligned}
$$

We suppose that

$$
\begin{aligned}
\Omega^x_{n+1} &= G(gx_{n+1},gx_{n+1},gx_n),\ \ \Omega^y_{n+1}=G(gy_{n+1},gy_{n+1},gy_n)\\
\Omega^z_{n+1} &= G(gz_{n+1},gz_{n+1},gz_n),\ \ \Omega^w_{n+1}=G(gw_{n+1},gw_{n+1},gw_n). \quad (2.14)
\end{aligned}
$$

From 2.6, 2.8,2.10, 2.12,2.7, 2.9,2.11, 2.13 and 2.14 we have

$$
\begin{aligned}
(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1}) \leq & \ \phi(\Omega^x_n+\Omega^y_n+\Omega^z_n+\Omega^w_n)\\
& -4\psi\begin{pmatrix}\Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1},\\ \Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1},\\ \Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1},\\ \Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1}\end{pmatrix}.
\end{aligned}
$$

(2.15)

As $\psi(t_1,t_2,t_3,t_4)>0$ for all $(t_1,t_2,t_3,t_4)\in[0,\infty)^4$ and from the property of $\phi(kt)\leq kt$ for any $k>0$ (it should be noted that $(\alpha_1+\alpha_2+\alpha_3+\alpha_4)>0$) we have

$$
(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1}) \leq (\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_n+\Omega^y_n+\Omega^z_n+\Omega^w_n)
$$

$$(\Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1}) \quad < \quad (\Omega^x_n+\Omega^y_n+\Omega^z_n+\Omega^w_n)$$

for all $n \geq 0$.

Then the sequence $\{\Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1}\}$ is decreasing. Therefore, there exists $\eta \geq 0$ such that

$$\lim_{n\to\infty}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1}) = (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\eta. \tag{2.16}$$

Now, we show that $\eta = 0$. Suppose that $\eta > 0$. From 2.16, the sequences $\{G(gx_{n+1}, gx_{n+1}, gx_n)\}$, $\{G(gy_{n+1}, gy_{n+1}, gy_n)\}$, $\{G(gz_{n+1}, gz_{n+1}, gz_n)\}$ and $\{G(gw_{n+1}, gw_{n+1}, gw_n)\}$ have convergent subsequences $\{G(gx_{n(j)+1}, gx_{n(j)+1}, gx_{n(j)})\}$,$\{G(gy_{n(j)+1}, gy_{n(j)+1}, gy_{n(j)})\}$,$\{G(gz_{n(j)+1}, gz_{n(j)+1}, gz_{n(j)})\}$ and $\{G(gw_{n(j)+1}, gw_{n(j)+1}, gw_{n(j)})\}$, respectively. Assume that

$$\begin{aligned}\lim_{j\to\infty}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Omega^x_{n(j)} &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\lim_{j\to\infty}(G(gx_{n(j)}, gx_{n(j)}, gx_{n(j)-1}))\\ &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Omega^x_0\end{aligned}$$

$$\begin{aligned}\lim_{j\to\infty}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Omega^y_{n(j)} &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\lim_{j\to\infty}(G(gy_{n(j)}, gy_{n(j)}, gy_{n(j)-1}))\\ &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Omega^y_0\end{aligned}$$

$$\begin{aligned}\lim_{j\to\infty}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Omega^z_{n(j)} &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\lim_{j\to\infty}(G(gz_{n(j)}, gz_{n(j)}, gz_{n(j)-1}))\\ &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Omega^z_0\end{aligned}$$

and

$$\begin{aligned}\lim_{j\to\infty}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Omega^w_{n(j)} &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\lim_{j\to\infty}(G(gw_{n(j)}, gw_{n(j)}, gw_{n(j)-1}))\\ &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\Omega^w_0\end{aligned}$$

which gives that

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\lim_{j\to\infty}[\Omega^x_{n(j)}+\Omega^y_{n(j)}+\Omega^z_{n(j)}+\Omega^w_{n(j)}]=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\eta.$$

From 2.15, we have

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_{n(j)+1}+\Omega^y_{n(j)+1}+\Omega^z_{n(j)+1}+\Omega^w_{n(j)+1}) \leq \phi\left(\Omega^x_{n(j)}+\Omega^y_{n(j)}+\Omega^z_{n(j)}+\Omega^w_{n(j)}\right) -4\psi\begin{pmatrix}\Omega^x_{n(j)}+\Omega^y_{n(j)}+\Omega^z_{n(j)}+\Omega^w_{n(j)},\\ \Omega^x_{n(j)}+\Omega^y_{n(j)}+\Omega^z_{n(j)}+\Omega^w_{n(j)},\\ \Omega^x_{n(j)}+\Omega^y_{n(j)}+\Omega^z_{n(j)}+\Omega^w_{n(j)},\\ \Omega^x_{n(j)}+\Omega^y_{n(j)}+\Omega^z_{n(j)}+\Omega^w_{n(j)}\end{pmatrix}. \tag{2.17}$$

Then taking the limit as $j\to\infty$ in the above inequality, we obtain

$$\begin{aligned}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_0+\Omega^y_0+\Omega^z_0+\Omega^w_0) &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\eta\\ &\leq \phi(\eta)-4\psi(\eta,\eta,\eta,\eta)\\ &< (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\eta\end{aligned}$$

which is contradiction. Thus $\eta=0$, that is

$$\lim_{n\to\infty}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1})=0 \tag{2.18}$$

Next, we show that $\{g(x_n)\}$, $\{g(y_n)\}$, $\{g(z_n)\}$ and $\{g(w_n)\}$ are $G-$cauchy sequences. On the contrary, assume that at least one of $\{g(x_n)\}$ or $\{g(y_n)\}$ is not $G-$cauchy sequence. By Proposition 1.2 there is an $\varepsilon>0$ for which we can find subsequencs $\{g(x_{n(k)})\}$, $\{g(x_{m(k)})\}$ of $\{g(x_n)\}$, $\{g(y_{n(k)})\}$, $\{g(y_{m(k)})\}$ of $\{g(y_n)\}$, $\{g(z_{n(k)})\}$, $\{g(z_{m(k)})\}$ of $\{g(z_n)\}$ and $\{g(w_{n(k)})\}$, $\{g(w_{m(k)})\}$ of $\{g(w_n)\}$ with $n(k)>m(k)\geq k$ such that

$$\begin{pmatrix} G(g(x_{n(k)}),g(x_{n(k)}),g(x_{m(k)}))+G(g(y_{n(k)}),g(y_{n(k)}),g(y_{m(k)})) \\ G(g(z_{n(k)}),g(z_{n(k)}),g(z_{m(k)}))+G(g(w_{n(k)}),g(w_{n(k)}),g(w_{m(k)})) \end{pmatrix} \geq \varepsilon. \tag{2.19}$$

Further corresponding to $m(k)$ we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies 2.19. Then

$$\begin{pmatrix} G(g(x_{n(k)-1}),g(x_{n(k)-1}),g(x_{m(k)}))+G(g(y_{n(k)-1}),g(y_{n(k)-1}),g(y_{m(k)})) \\ G(g(z_{n(k)-1}),g(z_{n(k)-1}),g(z_{m(k)}))+G(g(w_{n(k)-1}),g(w_{n(k)-1}),g(w_{m(k)})) \end{pmatrix} < \varepsilon. \tag{2.20}$$

By Lemma 1.2, we have

$$\begin{aligned} G(g(x_{n(k)}),g(x_{n(k)}),g(x_{m(k)})) &\leq G(g(x_{n(k)}),g(x_{n(k)}),g(x_{n(k)-1})) \\ &\quad +G(g(x_{n(k)-1}),g(x_{n(k)-1}),g(x_{m(k)})) \\ G(g(y_{n(k)}),g(y_{n(k)}),g(y_{m(k)})) &\leq G(g(y_{n(k)}),g(y_{n(k)}),g(y_{n(k)-1})) \\ &\quad +G(g(y_{n(k)-1}),g(y_{n(k)-1}),g(y_{m(k)})) \\ G(g(z_{n(k)}),g(z_{n(k)}),g(z_{m(k)})) &\leq G(g(z_{n(k)}),g(z_{n(k)}),g(z_{n(k)-1})) \\ &\quad +G(g(z_{n(k)-1}),g(z_{n(k)-1}),g(z_{m(k)})) \\ G(g(w_{n(k)}),g(w_{n(k)}),g(w_{m(k)})) &\leq G(g(w_{n(k)}),g(w_{n(k)}),g(w_{n(k)-1})) \\ &\quad +G(g(w_{n(k)-1}),g(w_{n(k)-1}),g(w_{m(k)})). \end{aligned} \tag{2.21}$$

Form 2.19, 2.20 and 2.21 we have

$$\begin{aligned}
\varepsilon &\leq G(g(x_{n(k)}),g(x_{n(k)}),g(x_{m(k)}))+G(g(y_{n(k)}),g(y_{n(k)}),g(y_{m(k)})) \\
&\quad +G(g(z_{n(k)}),g(z_{n(k)}),g(z_{m(k)}))+G(g(w_{n(k)}),g(w_{n(k)}),g(w_{m(k)})) \\
&\leq G(g(x_{n(k)}),g(x_{n(k)}),g(x_{n(k)-1}))+G(g(x_{n(k)-1}),g(x_{n(k)-1}),g(x_{m(k)})) \\
&\quad +G(g(y_{n(k)}),g(y_{n(k)}),g(y_{m(k)}))+G(g(y_{n(k)}),g(y_{n(k)}),g(y_{n(k)-1})) \\
&\quad +G(g(z_{n(k)}),g(z_{n(k)}),g(z_{n(k)-1}))+G(g(z_{n(k)-1}),g(z_{n(k)-1}),g(z_{m(k)}))
\end{aligned}$$

$$\begin{aligned}
&\quad +G(g(w_{n(k)}),g(w_{n(k)}),g(w_{n(k)-1}))+G(g(w_{n(k)-1}),g(w_{n(k)-1}),g(w_{m(k)})) \\
&< G(g(x_{n(k)}),g(x_{n(k)}),g(x_{n(k)-1}))+G(g(y_{n(k)}),g(y_{n(k)}),g(y_{n(k)-1})) \\
&\quad G(g(z_{n(k)}),g(z_{n(k)}),g(z_{n(k)-1}))+G(g(w_{n(k)}),g(w_{n(k)}),g(w_{n(k)-1}))+\varepsilon.
\end{aligned}$$

Then letting $k\to\infty$ in the above inequality and using 2.18, we have

$$(2.22)\ \lim_{k\to\infty}\left[\begin{array}{l} G(g(x_{n(k)}),g(x_{n(k)}),g(x_{m(k)}))+G(g(y_{n(k)}),g(y_{n(k)}),g(y_{m(k)})) \\ +G(g(z_{n(k)}),g(z_{n(k)}),g(z_{m(k)}))+G(g(w_{n(k)}),g(w_{n(k)}),g(w_{m(k)})) \end{array}\right]=\varepsilon.$$

Again by rectangle inequality and using the fact that $G(x,y,y)\leq 2G(y,x,x)$, we have

$$\begin{aligned}
\varepsilon &\leq G(gx_{n(k)},gx_{n(k)},gx_{m(k)})+G(gy_{n(k)},gy_{n(k)},gy_{m(k)}) \\
&\quad +G(gz_{n(k)},gz_{n(k)},gz_{m(k)})+G(gw_{n(k)},gw_{n(k)},w_{m(k)}) \\
&\leq G(gx_{n(k)},gx_{n(k)},gx_{n(k)+1})+G(gx_{n(k)+1},gx_{n(k)+1},gx_{m(k)+1}) \\
&\quad +G(gx_{m(k)+1},gx_{m(k)+1},gx_{m(k)})+G(gy_{n(k)},gy_{n(k)},gy_{n(k)+1}) \\
&\quad +G(gy_{n(k)+1},gy_{n(k)+1},gy_{m(k)+1})+G(gy_{m(k)+1},gy_{m(k)+1},gy_{m(k)}) \\
&\quad +G(gz_{n(k)},gz_{n(k)},gz_{n(k)+1})+G(gz_{n(k)+1},gz_{n(k)+1},gz_{m(k)+1}) \\
&\quad +G(gz_{m(k)+1},gz_{m(k)+1},gz_{m(k)})+G(gw_{n(k)},gw_{n(k)},gw_{n(k)+1})
\end{aligned}$$

$$
\begin{aligned}
& +G(gw_{n(k)+1},gw_{n(k)+1},gw_{m(k)+1})+G(gw_{m(k)+1},gw_{m(k)+1},gw_{m(k)}) \\
\leq\ & 2[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_{n+1}+\Omega^y_{n+1}+\Omega^z_{n+1}+\Omega^w_{n+1})] \\
& +[(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega^x_{m+1}+\Omega^y_{m+1}+\Omega^z_{m+1}+\Omega^w_{m+1})] \\
& +G(gx_{n(k)+1},gx_{n(k)+1},gx_{m(k)+1})+G(gy_{n(k)+1},gy_{n(k)+1},gy_{m(k)+1}) \\
& +G(gz_{n(k)+1},gz_{n(k)+1},gz_{m(k)+1})+G(gw_{n(k)+1},gw_{n(k)+1},gw_{m(k)+1})
\end{aligned}
$$

Since $n(k)>m(k)$ then

$$
\begin{aligned}
gx_{n(k)} \geq gx_{m(k)}, &\qquad gy_{n(k)} \leq gy_{m(k)} \\
gz_{n(k)} \geq gz_{m(k)}, &\qquad gw_{n(k)} \leq gw_{m(k)}.
\end{aligned}
$$

Then from 2.1, we have

$$
\begin{aligned}
& M(x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{m(k)},y_{m(k)},z_{m(k)},w_{m(k)}) \\
=\ & \alpha_1 G(F(x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)}),F(x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)}),F(x_{m(k)},y_{m(k)},z_{m(k)},w_{m(k)}) \\
& +\alpha_2 G(F(y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)}),F(y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)}),F(y_{m(k)},z_{m(k)},w_{m(k)},x_{m(k)}) \\
& +\alpha_3 G(F(z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)}),F(z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)}),F(z_{m(k)},w_{m(k)},x_{m(k)},y_{m(k)}) \\
& +\alpha_4 G(F(w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)}),F(w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)}),F(w_{m(k)},x_{m(k)},y_{m(k)},z_{m(k)}) \\
=\ & \alpha_1 G(gx_{n(k)+1},gx_{n(k)+1},gx_{m(k)+1}))+\alpha_2 G(gy_{n(k)+1},gy_{n(k)+1},gy_{m(k)+1})) \\
& +\alpha_3 G(gz_{n(k)+1},gz_{n(k)+1},gz_{m(k)+1}))+\alpha_4 G(gw_{n(k)+1},gw_{n(k)+1},gw_{m(k)+1})).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& M(x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{m(k)},y_{m(k)},z_{m(k)},w_{m(k)}) \\
\leq\ & \phi\left(\frac{\begin{array}{c} G(gx_{n(k)},gx_{n(k)},gx_{m(k)})+G(gy_{n(k)},gy_{n(k)},gy_{m(k)}) \\ +G(gz_{n(k)},gz_{n(k)},gz_{m(k)})+G(gw_{n(k)},gw_{n(k)},gw_{m(k)}) \end{array}}{4}\right)
\end{aligned}
$$

(2.23)

$$-\psi\begin{pmatrix} G(gx_{n(k)},gx_{n(k)},gx_{m(k)}),G(gy_{n(k)},gy_{n(k)},gy_{m(k)}), \\ G(gz_{n(k)},gz_{n(k)},gz_{m(k)}),G(gw_{n(k)},gw_{n(k)},gw_{m(k)}) \end{pmatrix}$$

Similarly we can prove that

$$\begin{aligned} & M(y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{m(k)},z_{m(k)},w_{m(k)},x_{m(k)}) \\ = \ & \alpha_1 G(gy_{n(k)+1},gy_{n(k)+1},gy_{m(k)+1})) + \alpha_2 G(gz_{n(k)+1},gz_{n(k)+1},gz_{m(k)+1})) \\ & +\alpha_3 G(gw_{n(k)+1},gw_{n(k)+1},gw_{m(k)+1})) + \alpha_4 G(gx_{n(k)+1},gx_{n(k)+1},gx_{m(k)+1})). \end{aligned}$$

then,

$$\begin{aligned} & M(y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{m(k)},z_{m(k)},w_{m(k)},x_{m(k)}) \\ \leq \ & \phi\left(\frac{\begin{array}{c} G(gy_{n(k)},gy_{n(k)},gy_{m(k)}) + G(gz_{n(k)},gz_{n(k)},gz_{m(k)}) \\ +G(gw_{n(k)},gw_{n(k)},gw_{m(k)}) + G(gx_{n(k)},gx_{n(k)},gx_{m(k)}) \end{array}}{4}\right) \\ & -\psi\begin{pmatrix} G(gy_{n(k)},gy_{n(k)},gy_{m(k)}),G(gz_{n(k)},gz_{n(k)},gz_{m(k)}), \\ G(gw_{n(k)},gw_{n(k)},gw_{m(k)}),G(gx_{n(k)},gx_{n(k)},gx_{m(k)}) \end{pmatrix}, \end{aligned} \tag{2.24}$$

Also,

$$\begin{aligned} & M(z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{m(k)},w_{m(k)},x_{m(k)},y_{m(k)}) \\ = \ & \alpha_1 G(gz_{n(k)+1},gz_{n(k)+1},gz_{m(k)+1}) + \alpha_2 G(gw_{n(k)+1},gw_{n(k)+1},gw_{m(k)+1}) \\ & +\alpha_3 G(gx_{n(k)+1},gx_{n(k)+1},gx_{m(k)+1}) + \alpha_4 G(gy_{n(k)+1},gy_{n(k)+1},gy_{m(k)+1}). \end{aligned}$$

hence,

$$\begin{aligned}
& M(z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{m(k)},w_{m(k)},x_{m(k)},y_{m(k)}) \\
\leq\ & \phi\left(\frac{\begin{array}{c} G(gz_{n(k)},gz_{n(k)},gz_{m(k)})+G(gw_{n(k)},gw_{n(k)},gw_{m(k)}) \\ +G(gx_{n(k)},gx_{n(k)},gx_{m(k)})+G(gy_{n(k)},gy_{n(k)},gy_{m(k)}) \end{array}}{4}\right) \\
& -\psi\left(\begin{array}{c} G(gz_{n(k)},gz_{n(k)},gz_{m(k)}),G(gw_{n(k)},gw_{n(k)},gw_{m(k)}), \\ G(gx_{n(k)},gx_{n(k)},gx_{m(k)}),G(gy_{n(k)},gy_{n(k)},gy_{m(k)}) \end{array}\right)
\end{aligned} \tag{2.25}$$

and,

$$\begin{aligned}
& M(w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{m(k)},x_{m(k)},y_{m(k)},z_{m(k)}) \\
=\ & \alpha_1 G(gw_{n(k)+1},gw_{n(k)+1},gw_{m(k)+1})+\alpha_2 G(gx_{n(k)+1},gx_{n(k)+1},gx_{m(k)+1}) \\
& +\alpha_3 G(gy_{n(k)+1},gy_{n(k)+1},gy_{m(k)+1})+\alpha_4 G(gz_{n(k)+1},gz_{n(k)+1},gz_{m(k)+1}).
\end{aligned}$$

Thus,

$$\begin{aligned}
& M(w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{n(k)},x_{n(k)},y_{n(k)},z_{n(k)},w_{m(k)},x_{m(k)},y_{m(k)},z_{m(k)}) \\
\leq\ & \phi\left(\frac{\begin{array}{c} G(gw_{n(k)},gw_{n(k)},gw_{m(k)})+G(gx_{n(k)},gx_{n(k)},gx_{m(k)}) \\ +G(gy_{n(k)},gy_{n(k)},gy_{m(k)})+G(gz_{n(k)},gz_{n(k)},gz_{m(k)}) \end{array}}{4}\right) \\
& -\psi\left(\begin{array}{c} G(gw_{n(k)},gw_{n(k)},gw_{m(k)}),G(gx_{n(k)},gx_{n(k)},gx_{m(k)}), \\ G(gy_{n(k)},gy_{n(k)},gy_{m(k)}),G(gz_{n(k)},gz_{n(k)},gz_{m(k)}) \end{array}\right).
\end{aligned} \tag{2.26}$$

From 2.23, 2.24, 2.25 and 2.26 we have

$$(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\begin{pmatrix} G(gx_{n(k)},gx_{n(k)},gx_{m(k)})+G(gy_{n(k)},gy_{n(k)},gy_{m(k)}) \\ +G(gz_{n(k)},gz_{n(k)},gz_{m(k)})+G(gw_{n(k)},gw_{n(k)},gw_{m(k)}) \end{pmatrix}$$
$$\leq \phi\begin{pmatrix} G(gx_{n(k)},gx_{n(k)},gx_{m(k)})+G(gy_{n(k)},gy_{n(k)},gy_{m(k)}) \\ +G(gz_{n(k)},gz_{n(k)},gz_{m(k)})+G(gw_{n(k)},gw_{n(k)},gw_{m(k)}) \end{pmatrix}$$
$$-4\psi\begin{pmatrix} G(gx_{n(k)},gx_{n(k)},gx_{m(k)}),G(gy_{n(k)},gy_{n(k)},gy_{m(k)}), \\ G(gz_{n(k)},gz_{n(k)},gz_{m(k)}),G(gw_{n(k)},gw_{n(k)},gw_{m(k)}) \end{pmatrix} \tag{2.27}$$

Letting, $k \to \infty$ in above and using 2.18, then

$$\begin{aligned}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)[\Omega_0^x+\Omega_0^y+\Omega_0^z+\Omega_0^w] &\leq \phi(\Omega_0^x+\Omega_0^y+\Omega_0^z+\Omega_0^w)-4\psi(\Omega_0^x,\Omega_0^y,\Omega_0^z,\Omega_0^w) \\ &< (\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\Omega_0^x+\Omega_0^y+\Omega_0^z+\Omega_0^w)\end{aligned} \tag{2.28}$$

A contradiction, this implies that $(gx_n),(gy_n),(gz_n)$ and (gw_n) are G-cauchy sequences in (X,G).

Now suppose that assumption (a) holds.

Since X is G-complete metric space, there exists $x,y,z,w \in X$ such that

$$\begin{aligned}&\lim_{n\to\infty} g(x_n)=x, \quad \lim_{n\to\infty} g(y_n)=y \\ &\lim_{n\to\infty} g(z_n)=z, \quad \lim_{n\to\infty} g(w_n)=w\end{aligned} \tag{2.29}$$

From 2.29 and continuity of g we have

$\lim_{n\to\infty} g(g(x_n))=gx,\ \lim_{n\to\infty} g(g(y_n))=gy$

$\lim_{n\to\infty} g(g(z_n))=gz,$ and $\lim_{n\to\infty} g(g(w_n))=gw.$

From the commutativity of F and g we have,

$$g(gx_{n+1}) = g(F(x_n, y_n, z_n, w_n)) = F(gx_n, gy_n, gz_n, gw_n), \tag{2.30}$$

$$g(gy_{n+1}) = g(F(y_n, z_n, w_n, z_n)) = F(gy_n, gz_n, gw_n, gx_n), \tag{2.31}$$

$$g(gz_{n+1}) = g(F(z_n, w_n, x_n, y_n)) = F(gz_n, gw_n, gx_n, gy_n), \tag{2.32}$$

and

$$g(gw_{n+1}) = g(F(w_n, x_n, y_n, z_n)) = F(gw_n, gx_n, gy_n, gz_n). \tag{2.33}$$

We shall show that $gx = F(x,y,z,w),\ gy = F(y,z,w,x),\ gz = F(z,w,x,y)$ and $gw = F(w,x,y,z)$.

By Letting $n \to \infty$ in (2.30) $\to$ (2.33) and using the continuity of F we obtain

$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} F(gx_n, gy_n, gz_n, gw_n) =$
$F(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n, \lim_{n\to\infty} gz_n, \lim_{n\to\infty} gw_n) = F(x,y,z,w).$

Similarly, $gy = F(y,z,w,x),\ gz = F(z,w,x,y)$ and $gw = F(w,x,y,z)$.

Hence, (x,y,z,w) is coincidence point of F and g.

Now suppose that the assumption (b) holds.

Since $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ and $\{gw_n\}$ are G-Cauchy sequences in the complete G-metric space $(g(X),G)$. Then, there exist $x,y,z,w \in X$ such that

$$gx_n \to gx,\ \ gy_n \to gy,\ \ gz_n \to gz \ \text{ and } \ gw_n \to gw. \tag{2.34}$$

Since $\{gx_n\}, \{gz_n\}$ are non-decreasing and $\{gy_n\}, \{gw_n\}$ are non-increasing and since $(X,G,\leq)$ satisfy conditions (i) and (ii), we have

$$gx_n \leq gx,\ gy_n \geq gy,\ gz_n \leq gz,\ gw_n \geq gw \quad \text{for all } n.$$

If $gx_n = gx$, $gy_n = gy$, $gz_n = gz$ and $gw_n = gw$ for some $n \geq 0$, then $gx = gx_n \leq gx_{n+1} \leq gx = gx_n$, $gy \leq gy_{n+1} \leq gy_n = gy$, $gz = gz_n \leq gz_{n+1} \leq gz = gz_n$ and $gw \leq gw_{n+1} \leq gw_n = gw$, which implies that

$$gx_n = gx_{n+1} = F(x_n, y_n, z_n, w_n), \quad gy_n = gy_{n+1} = F(y_n, z_n, w_n, x_n),$$

and

$$gz_n = gz_{n+1} = F(z_n, w_n, x_n, y_n), \quad gw_n = gw_{n+1} = F(w_n, w_n, y_n, z_n),$$

that is, (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. Then, we suppose that $(gx_n, gy_n, gz_n, gw_n) \neq (gx, gy, gz, gw)$ for all $n \geq 0$. By (2.1), consider now

$$\begin{aligned}
&\begin{pmatrix} G(gx, F(x,y,z,w), F(x,y,z,w)) + G(gy, F(y,z,w,x), F(y,z,w,x)) \\ +G(gz, F(z,w,x,y), F(z,w,x,y)) + G(gw, F(w,x,y,z), F(w,x,y,z)) \end{pmatrix} \\
\leq &\begin{pmatrix} G(gx, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, F(x,y,z,w), F(x,y,z,w)) \\ G(gy, gy_{n+1}, gy_{n+1}) + G(gy_{n+1}, F(y,z,w,x), F(y,z,w,x)) \\ G(gz, gz_{n+1}, gz_{n+1}) + G(gz_{n+1}, F(z,w,x,y), F(z,w,x,y)) \\ G(gw, gw_{n+1}, gw_{n+1}) + G(gw_{n+1}, F(w,x,y,z), F(w,x,y,z)) \end{pmatrix} \\
= &\begin{pmatrix} G(gx, gx_{n+1}, gx_{n+1}) + G(F(x_n, y_n, z_n, w_n), F(x,y,z,w), F(x,y,z,w)) \\ G(gy, gy_{n+1}, gy_{n+1}) + G(F(y_n, z_n, w_n, x_n), F(y,z,w,x), F(y,z,w,x)) \\ G(gz, gz_{n+1}, gz_{n+1}) + G(F(z_n, w_n, x_n, y_n), F(z,w,x,y), F(z,w,x,y)) \\ G(gw, gw_{n+1}, gw_{n+1}) + G(F(w_n, x_n, y_n, z_n), F(w,x,y,z), F(w,x,y,z)) \end{pmatrix}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in above equation and using property of ϕ, ψ and fact that $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (0, \infty)$ we get that

$G(gx, F(x,y,z,w), F(x,y,z,w)) = 0$. Thus, $gx = F(x,y,z,w)$. Analogously, one finds

$$F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw.$$

Thus, we proved that F and g have a quadruple coincidence point. This completes the proof of Theorem 2.1.

□

Corollary 2.1. *Let $(X,\leq)$ be a partially ordered set and (X,G) be a G-metric space. Let $F: X\times X\times X\times X\to X$ such that F has the mixed monotone property. Assume that there exists a $\phi\in\Phi$ and $\psi\in\Psi$ such that*

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &= \alpha_1 G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \\ &+\alpha_2 G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) \\ &+\alpha_3 G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) \\ &+\alpha_4 G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) \end{aligned}$$

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &\leq \phi\left(\frac{G(x,u,a)+G(y,v,b)+G(z,s,c),G(w,t,d)}{4}\right) \\ &-\psi(G(x,u,a),G(y,v,b),G(z,s,c),G(w,t,d)). \end{aligned}$$

(2.35)

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in(0,\infty)$, $x,y,z,w,u,v,s,t,a,b,c,d\in X$ with $x\geq u\geq a$, $y\leq v\leq b$, $z\geq s\geq c$ and $w\leq t\leq d$. Suppose $F(X^4)\subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0,y_0,z_0,w_0\in X$ such that

$$\begin{aligned} x_0 \leq F(x_0,y_0,z_0,w_0), &\qquad g_0\geq F(y_0,z_0,w_0,x_0), \\ z_0\leq F(z_0,w_0,x_0,y_0) \quad \text{and} &\quad g_0\geq F(w_0,x_0,y_0,z_0), \end{aligned}$$

Suppose either

(a) (X,G) is a complete G-metric space and F is continuous or,

(b) F has the following property:

(i) if non-decreasing sequence $x_n\to a$, then $x_n\leq x$ for all n,

(ii) if non-increasing sequence $y_n\to y$, then $y\leq y_n$ for all n.

then there exist $x,y,z,w\in X$ such that

$$F(x,y,z,w)=x,\quad F(y,z,w,x)=y,\quad F(z,w,x,y)=z \text{ and } F(w,x,y,z)=w$$

that is, F have a quadruple fixed point.

Proof. Setting $g(x) = I_x$(Identity mapping) in Theorem 2.1, then the result follows. □

Corollary 2.2. *Let $(X,\leq)$ be a partially ordered set and (X,G) be a G-metric space. Let $F : X\times X\times X\times X \to X$ and $g : X \to X$ such that F has the mixed g-monotone property. Assume that there exists a $\psi \in \Psi$ such that*

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &= \alpha_1 G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \\ &\quad +\alpha_2 G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) \\ &\quad +\alpha_3 G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) \\ &\quad +\alpha_4 G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) \end{aligned}$$

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) \leq & \left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc),G(gw,gt,gd)}{4}\right) \\ & -\psi(G(gx,gu,ga),G(gy,gv,gb),G(gz,gs,gc),G(gw,gt,gd)). \end{aligned}$$

(2.36)

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in (0,\infty)$, $x,y,z,w,u,v,s,t,a,b,c,d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, and $gw \leq gt \leq gd$. Suppose $F(X^4) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0,y_0,z_0,w_0 \in X$ such that

$$gx_0 \leq F(x_0,y_0,z_0,w_0), \qquad gy_0 \geq F(y_0,z_0,w_0,x_0),$$
$$gz_0 \leq F(z_0,w_0,x_0,y_0) \quad \textit{and} \quad gw_0 \geq F(w_0,x_0,y_0,z_0),$$

Suppose either

(a) (X,G) is a complete G-metric space and F is continuous or,

(b) $(g(X),G)$ is complete and $(X,G,\leq)$ has the following property:

(i) if non-decreasing sequence $x_n \to a$, then $x_n \leq x$ for all n,

(ii) if non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n.

then there exist $x,y,z,w \in X$ such that

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \ \textit{ and } \ F(w,x,y,z) = gw$$

that is, F and g have a quadruple coincidence point.

Proof. It is sufficient if we take $\phi(t) = t$ in Theorem 2.1 then the result follows. □

Corollary 2.3. *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X \times X \times X \times X \to X$ and $g : X \to X$ such that F has the mixed g-monotone property. Assume that there exists a $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &= \alpha_1 G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \\ &+\alpha_2 G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) \\ &+\alpha_3 G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) \\ &+\alpha_4 G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) \end{aligned}$$

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) \leq \ & \phi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right) \\ & -\max\{G(gx,gu,ga),G(gy,gv,gb),G(gz,gs,gc),G(gw,gt,gd)\}. \end{aligned}$$

(2.37)

for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (0, \infty)$, $x,y,z,w,u,v,s,t,a,b,c,d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, and $gw \leq gt \leq gd$. Suppose $F(X^4) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$gx_0 \leq F(x_0,y_0,z_0,w_0), \qquad gy_0 \geq F(y_0,z_0,w_0,x_0),$$
$$gz_0 \leq F(z_0,w_0,x_0,y_0) \quad \text{and} \quad gw_0 \geq F(w_0,x_0,y_0,z_0),$$

Suppose either

(a) (X, G) is a complete G-metric space and F is continuous or,

(b) $(g(X), G)$ is complete and $(X, G, \leq)$ has the following property:

(i) *if non-decreasing sequence $x_n \to a$, then $x_n \leq x$ for all n,*

(ii) *if non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n.*

then there exist $x,y,z,w \in X$ such that

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw$$

that is, F and g have a quadruple coincidence point.

Proof. It is sufficient if we take $\psi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$ in Theorem 2.1, we get the above result. $\square$

Corollary 2.4. *Let $(X,\leq)$ be a partially ordered set and (X,G) be a G-metric space. Let $F : X\times X\times X\times X \to X$ and $g : X \to X$ such that F has the mixed g-monotone property. Assume that there exists a $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &= \alpha_1 G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \\ &+\alpha_2 G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) \\ &+\alpha_3 G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) \\ &+\alpha_4 G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) \end{aligned}$$

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) \leq &\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right) \\ &-\phi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right). \end{aligned} \tag{2.38}$$

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in (0,\infty)$, $x,y,z,w,u,v,s,t,a,b,c,d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, and $gw \leq gt \leq gd$. Suppose $F(X^4) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0,y_0,z_0,w_0 \in X$ such that

$$\begin{aligned} gx_0 \leq F(x_0,y_0,z_0,w_0), &\qquad gy_0 \geq F(y_0,z_0,w_0,x_0), \\ gz_0 \leq F(z_0,w_0,x_0,y_0) \quad \text{and} &\quad gw_0 \geq F(w_0,x_0,y_0,z_0), \end{aligned}$$

Suppose either

(a) (X,G) is a complete G-metric space and F is continuous or,

(b) $(g(X),G)$ is complete and $(X,G,\leq)$ has the following property:

(i) *if non-decreasing sequence* $x_n \to a$, *then* $x_n \leq x$ *for all* n,

(ii) *if non-increasing sequence* $y_n \to y$, *then* $y \leq y_n$ *for all* n.

then there exist $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw$$

that is, F *and* g *have a quadruple coincidence point.*

Proof. It is sufficient if we take $\phi(t) = t$, $\psi(t_1,t_2,t_3,t_4) = \phi\left(\frac{t_1+t_2+t_3+t_4}{4}\right)$ in Theorem 2.1, we get the above result. □

Corollary 2.5. *Let* $(X, \leq)$ *be a partially ordered set and* (X, G) *be a G-metric space. Let* $F : X \times X \times X \times X \to X$ *and* $g : X \to X$ *such that* F *has the mixed g-monotone property. Assume that there exists a* $\phi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &= \alpha_1 G(F(x,y,z,w), F(u,v,s,t), F(a,b,c,d)) \\ &\quad + \alpha_2 G(F(y,z,w,x), F(v,s,t,u), F(b,c,d,a)) \\ &\quad + \alpha_3 G(F(z,w,x,y), F(s,t,u,v), F(c,d,a,b)) \\ &\quad + \alpha_4 G(F(w,x,y,z), F(t,u,v,s), F(d,a,b,c)) \end{aligned}$$

$$M(x,y,z,w,u,v,s,t,a,b,c,d) \leq k\left(\frac{G(gx,gu,ga) + G(gy,gv,gb) + G(gz,gs,gc) + G(gw,gt,gd)}{4}\right)$$

for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (0, \infty)$, $k \in (0,1)$, $x,y,z,w,u,v,s,t,a,b,c,d \in X$ *with* $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, *and* $gw \leq gt \leq gd$. *Suppose* $F(X^4) \subseteq g(X)$, g *is continuous and commutes with* F. *If there exist* $x_0, y_0, z_0, w_0 \in X$ *such that*

$$\begin{aligned} gx_0 \leq F(x_0,y_0,z_0,w_0), &\qquad gy_0 \geq F(y_0,z_0,w_0,x_0), \\ gz_0 \leq F(z_0,w_0,x_0,y_0) \quad \text{and} &\quad gw_0 \geq F(w_0,x_0,y_0,z_0), \end{aligned}$$

Suppose either

(a) (X, G) *is a complete G-metric space and* F *is continuous or,*

(b) $(g(X), G)$ *is complete and* $(X, G, \leq)$ *has the following property:*

(i) *if non-decreasing sequence* $x_n \to a$*, then* $x_n \leq x$ *for all* n,

(ii) *if non-increasing sequence* $y_n \to y$*, then* $y \leq y_n$ *for all* n.

then there exist $x,y,z,w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw$$

that is, F *and* g *have a quadruple coincidence point.*

Proof. It is sufficient if we take $\phi(t) = kt$ and $\psi(t_1,t_2,t_3,t_4) = \left(\frac{1-k}{4}\right)(t_1+t_2+t_3+t_4)$ in Theorem 2.1, we get the above result. □

Corollary 2.6. *Let* $(X,\leq)$ *be a partially ordered set and* (X,G) *be a G-metric space. Let* $F: X\times X\times X\times X \to X$ *and* $g: X \to X$ *such that* F *has the mixed g-monotone property. Assume that there exists a* $\phi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &= G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \\ &\quad +G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) \\ &\quad +G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) \\ &\quad +G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) \end{aligned}$$

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) \leq\ & \phi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right) \\ & -\psi(G(gx,gu,ga),G(gy,gv,gb),G(gz,gs,gc),G(gw,gt,gd)). \end{aligned}$$

(2.39)

for all $x,y,z,w,u,v,s,t,a,b,c,d \in X$ *with* $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, *and* $gw \leq gt \leq gd$. *Suppose* $F(X^4) \subseteq g(X)$, g *is continuous and commutes with* F. *If there exist* $x_0,y_0,z_0,w_0 \in X$ *such that*

$$gx_0 \leq F(x_0,y_0,z_0,w_0), \qquad gy_0 \geq F(y_0,z_0,w_0,x_0),$$
$$gz_0 \leq F(z_0,w_0,x_0,y_0) \quad \text{and} \quad gw_0 \geq F(w_0,x_0,y_0,z_0),$$

Suppose either

(a) (X,G) *is a complete G-metric space and* F *is continuous or,*

(b) $(g(X), G)$ *is complete and* $(X, G, \leq)$ *has the following property:*

(i) *if non-decreasing sequence* $x_n \to a$*, then* $x_n \leq x$ *for all* n*,*

(ii) *if non-increasing sequence* $y_n \to y$*, then* $y \leq y_n$ *for all* n*.*

then there exist $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw$$

that is, F *and* g *have a quadruple coincidence point.*

Proof. If we take $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ in Theorem 2.1, we get the above result. □

Corollary 2.7. *Let* $(X, \leq)$ *be a partially ordered set and* (X, G) *be a G-metric space. Let* $F : X \times X \times X \times X \to X$ *and* $g : X \to X$ *such that* F *has the mixed g-monotone property. Assume that there exists a* $\phi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\begin{aligned} & G(F(x,y,z,w), F(u,v,s,t), F(a,b,c,d)) \\ \leq \ & \phi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right) \\ & -\psi(G(gx,gu,ga), G(gy,gv,gb), G(gz,gs,gc), G(gw,gt,gd)). \end{aligned}$$

(2.40)

for all $x,y,z,w,u,v,s,t,a,b,c,d \in X$ *with* $gx \geq gu \geq ga$*,* $gy \leq gv \leq gb$*,* $gz \geq gs \geq gc$*, and* $gw \leq gt \leq gd$*. Suppose* $F(X^4) \subseteq g(X)$*,* g *is continuous and commutes with* F*. If there exist* $x_0, y_0, z_0, w_0 \in X$ *such that*

$$\begin{aligned} gx_0 \leq F(x_0,y_0,z_0,w_0), \qquad & gy_0 \geq F(y_0,z_0,w_0,x_0), \\ gz_0 \leq F(z_0,w_0,x_0,y_0) \quad \text{and} \quad & gw_0 \geq F(w_0,x_0,y_0,z_0), \end{aligned}$$

Suppose either

(a) (X, G) *is a complete G-metric space and* F *is continuous or,*

(b) $(g(X), G)$ *is complete and* $(X, G, \leq)$ *has the following property:*

(i) *if non-decreasing sequence* $x_n \to a$*, then* $x_n \leq x$ *for all* n*,*

(ii) *if non-increasing sequence* $y_n \to y$, *then* $y \leq y_n$ *for all n.*

then there exist $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw$$

that is, F and g have a quadruple coincidence point.

Proof. If we take $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$ in Theorem 2.1, we get the above result. □

Corollary 2.8. *Let* $(X, \leq)$ *be a partially ordered set and* (X, G) *be a G-metric space. Let* $F : X \times X \times X \times X \to X$ *and* $g : X \to X$ *such that F has the mixed g-monotone property. Assume that there exists a* $\phi \in \Phi$ *such that*

$$\begin{aligned} & G(F(x,y,z,w), F(u,v,s,t), F(a,b,c,d)) \\ \leq \quad & \phi\left(\frac{G(gx,gu,ga) + G(gy,gv,gb) + G(gz,gs,gc) + G(gw,gt,gd)}{4}\right) \end{aligned} \tag{2.41}$$

for all $x,y,z,w,u,v,s,t,a,b,c,d \in X$ *with* $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, *and* $gw \leq gt \leq gd$. *Suppose* $F(X^4) \subseteq g(X)$, *g is continuous and commutes with F. If there exist* $x_0, y_0, z_0, w_0 \in X$ *such that*

$$\begin{aligned} gx_0 \leq F(x_0,y_0,z_0,w_0), \qquad & gy_0 \geq F(y_0,z_0,w_0,x_0), \\ gz_0 \leq F(z_0,w_0,x_0,y_0) \quad \text{and} \quad & gw_0 \geq F(w_0,x_0,y_0,z_0), \end{aligned}$$

Suppose either

(a) (X, G) *is a complete G-metric space and F is continuous or,*

(b) $(g(X), G)$ *is complete and* $(X, G, \leq)$ *has the following property:*

(i) *if non-decreasing sequence* $x_n \to a$, *then* $x_n \leq x$ *for all n,*

(ii) *if non-increasing sequence* $y_n \to y$, *then* $y \leq y_n$ *for all n.*

then there exist $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw$$

that is, F and g have a quadruple coincidence point.

Proof. If we take $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$ also $\psi(t_1,t_2,t_3,t_4) = 0$ in Theorem 2.1, we get the above result. □

Corollary 2.9. *Let $(X,\leq)$ be a partially ordered set and (X,G) be a G-metric space. Let $F : X\times X\times X\times X \to X$ and $g : X \to X$ such that F has the mixed g-monotone property. Assume that there exists a $\phi \in \Phi$ such that*

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) &= G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \\ &\quad +G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) \\ &\quad +G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) \\ &\quad +G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) \end{aligned}$$

$$\begin{aligned} &M(x,y,z,w,u,v,s,t,a,b,c,d) \\ \leq\ &\phi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right) \end{aligned}$$

(2.42)

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in (0,\infty)$, $x,y,z,w,u,v,s,t,a,b,c,d \in X$ with $gx \geq gu \geq ga$, $gy \leq gv \leq gb$, $gz \geq gs \geq gc$, and $gw \leq gt \leq gd$. Suppose $F(X^4) \subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0,y_0,z_0,w_0 \in X$ such that

$$\begin{aligned} gx_0 &\leq F(x_0,y_0,z_0,w_0), & gy_0 &\geq F(y_0,z_0,w_0,x_0), \\ gz_0 &\leq F(z_0,w_0,x_0,y_0) \quad \text{and} & gw_0 &\geq F(w_0,x_0,y_0,z_0), \end{aligned}$$

Suppose either

(a) (X,G) is a complete G-metric space and F is continuous or,

(b) $(g(X),G)$ is complete and $(X,G,\leq)$ has the following property:

(i) *if non-decreasing sequence $x_n \to a$, then $x_n \leq x$ for all n,*

(ii) *if non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n.*

then there exist $x,y,z,w \in X$ such that

$$F(x,y,z,w)=gx, \quad F(y,z,w,x)=gy, \quad F(z,w,x,y)=gz \text{ and } F(w,x,y,z)=gw$$

that is, F and g have a quadruple coincidence point.

Proof. If we take $\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$ also $\psi(t_1,t_2,t_3,t_4)=0$ in Theorem 2.1, we get the above result. □

Corollary 2.10. *Let $(X,\leq)$ be a partially ordered set and (X,G) be a G-metric space. Let $F: X\times X\times X\times X \to X$ and $g: X\to X$ such that F has the mixed g-monotone property. Assume that there exists a $\phi\in\Phi$ and $\psi\in\Psi$ such that*

$$M(x,y,z,w,u,v,s,t,a,b,c,d) = \alpha\begin{pmatrix} G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \\ +G(F(y,z,w,x),F(v,s,t,u),F(b,c,d,a)) \\ +G(F(z,w,x,y),F(s,t,u,v),F(c,d,a,b)) \\ +G(F(w,x,y,z),F(t,u,v,s),F(d,a,b,c)) \end{pmatrix}.$$

$$\begin{aligned} M(x,y,z,w,u,v,s,t,a,b,c,d) \leq\ & \phi\left(\frac{G(gx,gu,ga)+G(gy,gv,gb)+G(gz,gs,gc)+G(gw,gt,gd)}{4}\right) \\ & -\psi(G(gx,gu,ga),G(gy,gv,gb),G(gz,gs,gc),G(gw,gt,gd)). \end{aligned}$$

(2.43)

for all $\alpha\in(0,\infty)$, $x,y,z,w,u,v,s,t,a,b,c,d\in X$ with $gx\geq gu\geq ga$, $gy\leq gv\leq gb$, $gz\geq gs\geq gc$, and $gw\leq gt\leq gd$. Suppose $F(X^4)\subseteq g(X)$, g is continuous and commutes with F. If there exist $x_0,y_0,z_0,w_0\in X$ such that

$$gx_0\leq F(x_0,y_0,z_0,w_0), \qquad gy_0\geq F(y_0,z_0,w_0,x_0),$$
$$gz_0\leq F(z_0,w_0,x_0,y_0) \quad \text{and} \quad gw_0\geq F(w_0,x_0,y_0,z_0),$$

Suppose either

(a) (X,G) is a complete G-metric space and F is continuous or,

(b) $(g(X),G)$ is complete and $(X,G,\leq)$ has the following property:

(i) *if non-decreasing sequence $x_n\to a$, then $x_n\leq x$ for all n,*

(ii) *if non-increasing sequence* $y_n \to y$*, then* $y \leq y_n$ *for all* n*.*

then there exist $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \ \text{ and } \ F(w,x,y,z) = gw$$

that is, F *and* g *have a quadruple coincidence point.*

Proof. If we take $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$ in Theorem 2.1, we get the above result. □

Example 2.1. *Let* $X = \mathbb{R}$*. Define* $G : X \times X \times X \to [0,\infty)$ *by*

$$G(x,y,z) = |\, x - y \,| + |\, y - z \,| + |\, z - x \,|$$

$$F(x,y,z,w) = 2x - 3y + 2z - 3w, \quad g(x) = x$$

also $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$*,* $\phi(t) = 22t$ *and* $\psi(t_1,t_2,t_3,t_4) = \frac{t_1+t_2+t_3+t_4}{4}$*. Then we have from 2.1 we have a fixed point* $(0,0,0,0)$*.*

3. An Application

Theorem 3.1. *Let* $(X,\leq)$ *be a partially ordered set and* (X,G) *be a G-metric space. Let* $F : X \times X \times X \times X \to X$ *such that* F *has the mixed monotone property. Assume that there exists a* $\phi \in \Phi$ *such that*

$$G(F(x,y,z,w),F(u,v,s,t),F(a,b,c,d)) \ \leq \ \phi\left(\frac{G(x,u,s)+G(y,v,b)+G(z,s,c)+G(w,t,d)}{4}\right) \tag{3.1}$$

for all $x,y,z,w,u,v,s,t,a,b,c,d \in X$ *with* $x \geq u \geq a$*,* $y \leq v \leq b$*,* $z \geq s \geq c$*, and* $w \leq t \leq d$*. If there exist* $x_0, y_0, z_0, w_0 \in X$ *such that*

$$x_0 \leq F(x_0,y_0,z_0,w_0), \qquad y_0 \geq F(y_0,z_0,w_0,x_0),$$
$$z_0 \leq F(z_0,w_0,x_0,y_0) \quad \text{and} \quad w_0 \geq F(w_0,x_0,y_0,z_0),$$

Suppose either

(a) (X,G) *is a complete G-metric space and* F *is continuous or,*

(b) $(X,G,\leq)$ *has the following property:*

(i) *if non-decreasing sequence* $x_n \to a$*, then* $x_n \leq x$ *for all* n,

(ii) *if non-increasing sequence* $y_n \to y$*, then* $y \leq y_n$ *for all* n.

then there exist $x,y,z,w \in X$ *such that*

$$F(x,y,z,w) = x, \quad F(y,z,w,x) = y, \quad F(z,w,x,y) = z \ \text{and} \ F(w,x,y,z) = w$$

that is, F *has a quadruple coincidence point.*

Proof. If we take $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$, $\psi(t_1,t_2,t_3,t_4) = 0$ also $g(X) = I_X$ in Theorem 2.1, we get the above result. □

Finally by using the above results, we show the existence of solutions for the following integral equation:

(3.2)
$$(x(t),y(t),z(t),w(t)) = \begin{pmatrix} \int_0^T G(t,s)[f(s,x(s)+\lambda x(s)-(f(s,y(s))+\lambda y(s))]ds, \\ \int_0^T G(t,s)[f(s,y(s)+\lambda y(s)-(f(s,z(s))+\lambda z(s))]ds, \\ \int_0^T G(t,s)[f(s,z(s)+\lambda z(s)-(f(s,w(s))+\lambda w(s))]ds, \\ \int_0^T G(t,s)[f(s,w(s)+\lambda w(s)-(f(s,x(s))+\lambda x(s))]ds \end{pmatrix}$$

where $x,y,z,w \in C(I,R)$ where $C(I,R)$ is the set of continuous functions from I into R, $T > 0$, $f : I \times R \to R$ is continuous function and

(3.3)
$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T-1}} & if \ \ 0 \leq s \leq t \leq T \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T-1}} & if \ \ 0 \leq t < s \leq T \end{cases}$$

Definition 3.1. *A lower solution for the integral type equation 3.2 is an element* $(\alpha,\beta,\gamma,\eta) \in (C^1(I,R))^4$ *such that*

$$\begin{aligned}
\alpha'(t)+\lambda\beta(t)+\lambda\gamma(t)+\lambda\eta(t) &\leq f(t,\alpha(t))-f(t,\beta(t))-f(t,\gamma(t))-f(t,\eta(t)),\ \ \alpha(0)\leq\alpha(T),\\
\beta'(t)+\lambda\gamma(t)+\lambda\eta(t)+\lambda\alpha(t) &\leq f(t,\beta(t))-f(t,\gamma(t))-f(t,\eta(t))-f(t,\alpha(t)),\ \ \beta(0)\geq\beta(T),\\
\gamma'(t)+\lambda\eta(t)+\lambda\alpha(t)+\lambda\beta(t) &\leq f(t,\gamma(t))-f(t,\eta(t))-f(t,\alpha(t)-f(t,\beta(t)),\ \ \gamma(0)\leq\gamma(T),\\
\eta'(t)+\lambda\alpha(t)+\lambda\beta(t)+\lambda\gamma(t) &\leq f(t,\eta(t))-f(t,\alpha(t))-f(t,\beta(t))-f(t,\gamma(t)),\ \ \beta(0)\geq\beta(T),
\end{aligned}\tag{3.4}$$

where $C^1(I,R)$ denotes the set of differentiable functions from I to R.

Next we prove the existence of solution for the integral equation 3.2.

Theorem 3.2. *Let Φ be the class of the functions $\phi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:*

(a) ϕ is nondecreasing,

(b) for any $x\geq 0$, there exists $k\in[0,1)$ such that $\phi(x)\leq(k/4)x$.

In the integral equation 3.2 suppose that there exists $\lambda\succ 0$ such that for all $x,y\in R$ with $y\geq x$

$$[f(t,y)+\lambda y]-[f(t,x)+\lambda x]\leq\lambda\psi(y-x),\tag{3.5}$$

where $\phi\in\Phi$. If a lower solution of the integral equation 3.2 exists then the solution of integral equation 3.2 exists.

Proof. Define a mapping $F:(C(I,R))^4\to C(I,R)$ by

$$\begin{aligned}
F(x(t),y(t),z(t),w(t)) &= \int_0^T G(t,s)[f(s,x(s)+\lambda x(s))-(f(s,y(s))+\lambda y(s))\\
&\quad -(f(s,z(s))+\lambda z(s))-(f(s,w(s))+\lambda w(s))]ds,
\end{aligned}\tag{3.6}$$

Note that, if $(x(t),y(t),z(t),w(t))\in(C(I,R))^4$ is quadrupled fixed point of F, then $(x(t),y(t),z(t),w(t))$ is the solution of integral equation 3.2.

Now, we check the hypothesis in Theorem 3.1 as follows:

(1) $X^4 = (C(I,R))^4$ is a partially ordered set if we define the order relation in X^4 as follows;

$$(u(t),v(t),p(t),q(t)) \leq (x(t),y(t),z(t),w(t)) \tag{3.7}$$

iff

$$u(t) \leq x(t),\ \ v(t) \geq y(t),\ \ p(t) \leq z(t),\ \ q(t) \geq w(t),$$

for all

$$(u(t),v(t),p(t),q(t)),(x(t),y(t),z(t),w(t)) \in X^4$$

and $t \in I$.

(2) (X,G) is a complete G-metric space if we define a metric G as follows;

$$G(a(t),b(t),c(t)) = \sup_{t\in I}\{\mid a(t)-b(t)\mid, \mid b(t)-c(t)\mid, \mid c(t)-a(t)\mid : a(t),b(t),c(t) \in X\}. \tag{3.8}$$

(3) The mapping F has the mixed monotone property. In fact by hypothesis, if $x_2 \geq x_1$, then we have

$$f(t,x_2)+\lambda x_2 \geq f(t,x_1)+\lambda x_1 \tag{3.9}$$

which implies that for any $t \in I$,

$$\begin{aligned} F(x_2(t),y(t),z(t),w(t)) &= \int_0^T G(t,s)[f(s,x_2(s))+\lambda x_2(s)-(f(s,y(s))+\lambda y(s) \\ &\quad)-(f(s,z(s))+\lambda z(s))-(f(s,w(s))+\lambda w(s))]ds \end{aligned}$$

and

$$\begin{aligned} F(x_1(t),y(t),z(t),w(t)) &= \int_0^T G(t,s)[f(s,x_1(s))+\lambda x_1(s)-(f(s,y(s))+\lambda y(s)) \\ &\quad -(f(s,z(s))+\lambda z(s))-(f(s,w(s))+\lambda w(s))]ds, \end{aligned}$$

that is,

$$F(x_2(t),y(t),z(t),w(t)) \geq F(x_1(t),y(t),z(t),w(t)). \tag{3.10}$$

Similarly if $y_1 \geq y_2$, then we have

$$f(t,y_2)+\lambda y_2 \geq f(t,y_1)+\lambda y_1 \tag{3.11}$$

which implies that for any $t \in I$,

$$\begin{aligned} F(x(t),y_2(t),z(t),w(t)) &= \int_0^T G(t,s)[f(s,x(s))+\lambda x(s)-(f(s,y_2(s))+\lambda y_2(s)) \\ &\quad -(f(s,z(s))+\lambda z(s))-(f(s,w(s))+\lambda w(s))]ds \end{aligned}$$

and

$$\begin{aligned} F(x(t),y_1(t),z(t),w(t)) &= \int_0^T G(t,s)[f(s,x(s))+\lambda x(s)-(f(s,y_1(s))+\lambda y_1(s)) \\ &\quad -(f(s,z(s))+\lambda z(s))-(f(s,w(s))+\lambda w(s))]ds. \end{aligned}$$

that is

$$F(x(t),y_2(t),z(t),w(t)) \leq F(x(t),y_1(t),z(t),w(t)) \tag{3.12}$$

for any $t \in I$.

Also if $z_1 \leq z_2$, then we have

$$f(t,z_2)+\lambda z_2 \geq f(t,z_1)+\lambda z_1 \tag{3.13}$$

$$\begin{aligned} F(x(t),y(t),z_2(t),w(t)) &= \int_0^T G(t,s)[f(s,x(s))+\lambda x(s)-(f(s,y(s))+\lambda y(s)) \\ &\quad -(f(s,z_2(s))+\lambda z_2(s))-(f(s,w(s))+\lambda w(s))]ds \end{aligned}$$

and

$$\begin{aligned} F(x(t),y(t),z_1(t),w(t)) &= \int_0^T G(t,s)[f(s,x(s))+\lambda x(s)-(f(s,y(s))+\lambda y(s)) \\ &\quad -(f(s,z_1(s))+\lambda z_1(s))(f(s,w(s))+\lambda w(s))]ds \end{aligned}$$

that is

$$F(x(t),y(t),z_2(t),w(t)) \geq F(x(t),y(t),z_1(t),w(t)) \tag{3.14}$$

$$\begin{aligned} F(x(t),y(t),z(t),w_2(t)) &= \int_0^T G(t,s)[f(s,x(s))+\lambda x(s)-(f(s,y(s))+\lambda y(s)) \\ &\quad -(f(s,z(s))+\lambda z(s))-(f(s,w_2(s))+\lambda w_2(s))]ds \end{aligned}$$

and

$$\begin{aligned} F(x(t),y(t),z(t),w_1(t)) &= \int_0^T G(t,s)[f(s,x(s))+\lambda x(s)-(f(s,y(s))+\lambda y(s)) \\ &\quad -(f(s,z(s))+\lambda z(s))(f(s,w_1(s))+\lambda w_1(s))]ds \end{aligned}$$

that is

$$F(x(t),y(t),z(t),w_2(t)) \leq F(x(t),y(t),z(t),w_1(t)). \tag{3.15}$$

In fact, let $(x,y,z,w) \leq (u,v,p,q)$ and $t \in I$ then we have

$$\begin{aligned} &G(F(x(t),y(t),z(t),w(t)),F(u(t),v(t),p(t),q(t)),F(a(t),b(t),c(t),d(t))) \\ &= \sup\begin{pmatrix} \mid F(x(t),y(t),z(t),w(t))-F(u(t),v(t),p(t),q(t) \mid, \\ \mid F(u(t),v(t),p(t),q(t)-F(a(t),b(t),c(t),d(t)) \mid, \\ \mid F(a(t),b(t),c(t),d(t))-F(x(t),y(t),z(t),w(t)) \mid \end{pmatrix} (t \in I) \end{aligned}$$

$$
= \sup_{t\in I}\begin{pmatrix}
\mid \int_0^T G(t,s)[f(s,x(s))+\lambda x(s)-(f(s,y(s))+\lambda y(s)) \\
-(f(s,z(s))+\lambda z(s))-(f(s,w(s))+\lambda w(s))]ds \\
-\int_0^T G(t,s)[f(s,u(s))+\lambda u(s)-(f(s,v(s))+\lambda v(s)) \\
-(f(s,p(s))+\lambda p(s))(f(s,q(s))+\lambda q(s))]ds\mid, \\
\mid \int_0^T G(t,s)[f(s,u(s))+\lambda u(s)-(f(s,v(s))+\lambda v(s)) \\
-(f(s,p(s))+\lambda p(s))-(f(s,q(s))+\lambda q(s))]ds\mid \\
-\int_0^T G(t,s)[f(s,a(s))+\lambda a(s)-(f(s,b(s))+\lambda b(s)) \\
-(f(s,c(s))+\lambda c(s))-(f(s,d(s))+\lambda d(s))]ds\mid, \\
\int_0^T G(t,s)[f(s,a(s))+\lambda a(s)-(f(s,b(s))+\lambda b(s)) \\
-(f(s,c(s))+\lambda c(s))-(f(s,d(s))+\lambda d(s))]ds\mid, \\
-\int_0^T G(t,s)[f(s,x(s))+\lambda x(s)-(f(s,y(s))+\lambda y(s)) \\
-(f(s,z(s))+\lambda z(s))-(f(s,w(s))+\lambda w(s))]ds
\end{pmatrix}
$$

$$
\leq \sup_{t\in I}\begin{pmatrix}
\mid \int_0^T G(t,s)[(f(s,x(s))+\lambda x(s))-(f(s,u(s))+\lambda u(s))-[(f(s,y(s))+\lambda y(s))-(f(s,v(s))+\lambda v(s))] \\
-[(f(s,z(s))+\lambda z(s))+(f(s,p(s))+\lambda p(s))]-[(f(s,w(s))+\lambda w(s))-(f(s,q(s))+\lambda q(s))]]\mid ds, \\
\mid \int_0^T G(t,s)[[(f(s,u(s))+\lambda u(s))-(f(s,a(s))+\lambda a(s))]-[(f(s,v(s))+\lambda v(s))-(f(s,b(s))+\lambda b(s))] \\
-[(f(s,p(s))+\lambda p(s))-(f(s,c(s))+\lambda c(s))]-[(f(s,q(s))+\lambda q(s))-(f(s,d(s))+\lambda d(s))]]ds\mid, \\
\int_0^T G(t,s)[[(f(s,a(s))+\lambda a(s))-(f(s,x(s))+\lambda x(s))]-[(f(s,b(s))+\lambda b(s))-(f(s,y(s))+\lambda y(s))] \\
-[(f(s,c(s))+\lambda c(s))-f(s,z(s))+\lambda z(s))]-[(f(s,d(s))+\lambda d(s))-((f(s,w(s))+\lambda w(s))]]ds\mid
\end{pmatrix}
$$

Since the function $\phi(x)$ is nondecreasing and $(x,y,z,w)\leq(u,v,p,q)$, we have

$$\begin{aligned}
\phi(\max\{|x(s)-u(s)|,|u(s)-a(s)|,|a(s)-x(s)|\}) &\leq \phi(G(x(s),u(s),a(s)) \\
\phi(\max\{|y(s)-v(s)|,|v(s)-b(s)|,|b(s)-y(s)|\}) &\leq \phi(G(y(s),v(s),b(s)) \\
\phi(\max\{|z(s)-p(s)|,|p(s)-c(s)|,|c(s)-z(s)|\}) &\leq \phi(G(z(s),p(s),c(s)) \\
\phi(\max\{|w(s)-q(s)|,|q(s)-d(s)|,|d(s)-w(s)|\}) &\leq \phi(G(w(s),q(s),d(s)). \quad (3.16)
\end{aligned}$$

By using property of ϕ, 3.2, 3.3, 3.16,3.16,3.16 we get $(\alpha(t),\beta(t),\gamma(t),\eta(t)) \in (C^1(I,R))^4$ be a lower solution for the integral equation 3.2 then we show that

$$\alpha \leq F(\alpha,\beta,\gamma,\eta),\ \ \beta \geq F(\beta,\gamma,\eta,\alpha),\ \ \gamma \leq F(\gamma,\eta,\alpha,\beta),\ \ \eta \geq F(\eta,\alpha,\beta,\gamma). \quad (3.17)$$

Indeed, we have

$$\alpha'(t)+\lambda\beta(t)+\lambda\gamma(t)+\lambda\eta(t) \leq f(t,\alpha(t))-f(t,\beta(t))-f(t,\gamma(t))-f(t,\eta(t))$$

for any $t \in I$ and so

(3.18)

$$\alpha'(t)+\lambda\alpha(t) \leq f(t,\alpha(t))-f(t,\beta(t))-f(t,\gamma(t)))-f(t,\eta(t))+\lambda\alpha(t)-\lambda\beta(t)-\lambda\gamma(t)-\lambda\eta(t)$$

for any $t \in I$.

Multiplying 3.18 by $e^{\lambda t}$, we get the following:

$$\begin{aligned}
\left((\alpha(t)e^{\lambda t}\right)' \leq\ & [(f(t,\alpha(t))+\lambda\alpha(t))-(f(t,\beta(t))-\lambda\beta(t)) \\
& -(f(t,\gamma(t))-\lambda\gamma(t))-(f(t,\eta(t))-\lambda\eta(t))]e^{\lambda t} \quad (3.19)
\end{aligned}$$

for any $t \in I$, which implies that

$$\begin{aligned}
\alpha(t)e^{\lambda t} \preceq\ & \alpha(0)+\int_0^t [(f(s,\alpha(s))+\lambda\alpha(s))-(f(s,\beta(s))-\lambda\beta(s)) \\
& -(f(s,\gamma(s))-\lambda\gamma(s))-(f(s,\eta(s))-\lambda\eta(s))]e^{\lambda s}ds \quad (3.20)
\end{aligned}$$

for any $t \in I$, this implies that

$$\begin{aligned}\alpha(0)e^{\lambda t} &\prec \alpha(T)e^{\lambda T} \\ &\preceq \alpha(0)+\int_0^T[(f(s,\alpha(s))+\lambda\alpha(s))-(f(s,\beta(s))-\lambda\beta(s)) \\ &\quad -(f(s,\gamma(s))-\lambda\gamma(s))-(f(s,\eta(s))-\lambda\eta(s))]e^{\lambda s}ds \qquad (3.21)\end{aligned}$$

and so

$$\begin{aligned}\alpha(0) &\prec \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ &\quad -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))-(f(s,\eta(s))-\lambda\eta(s))]ds \qquad (3.22)\end{aligned}$$

Thus it follows from 3.20 and 3.22 that

$$\begin{aligned}\alpha(t)e^{\lambda t} &\prec \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ &\quad -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))-(f(s,\eta(s))-\lambda\eta(s))]ds \\ &\quad +\int_0^t \frac{e^{\lambda(T-s)}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ &\quad -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))-(f(s,\eta(s))-\lambda\eta(s))]ds \qquad (3.23)\end{aligned}$$

and so

$$\begin{aligned}\alpha(t) &\leq \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ &\quad -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))]ds \\ &\quad +\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ &\quad -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))-(f(s,\eta(s))-\lambda\eta(s))]ds \qquad (3.24)\end{aligned}$$

then,

$$\begin{aligned}\alpha(t) &\leq \int_0^T G(t,s)[f(s,\alpha(s)+\lambda\alpha(s) \\ &\quad -(f(s,\beta(s))+\lambda\beta(s))-(f(s,\gamma(s))+\lambda\gamma(s))-(f(s,\eta(s))+\lambda\eta(s))]ds \\ &= F(\alpha(t),\beta(t),\gamma(t),\eta(t)) \qquad (3.25)\end{aligned}$$

for any $t \in I$.

Similarly, we have

$$\beta(t) \geq F(\beta(t), \gamma(t), \eta(t), \alpha(t)),$$

$$\gamma(t) \leq F(\gamma(t), \eta(t), \alpha(t), \beta(t))$$

and

$$\eta(t) \geq F(\eta(t), \alpha(t), \beta(t), \gamma(t)).$$

Therefore by Theorem 3.1, F has a quadrupled fixed point. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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