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AN ABSTRACT PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this work, we investigate the existence and regularity of solutions for some partial functional integrodifferential equations with finite delay. The continuous dependence upen initial values and asymptotic stability are also studied. Firstly, we show the existence of the mild solutions. Secondly, we give sufficient conditions ensuring the existence of the strict solutions. The method used treats the equations in the domain of A with the graph norm employing results from linear semigroup theory. To illustrate our abstract result, we conclude this work with an application.

Keywords: mild and strict solutions; partial functional integrodifferential equations; C_0-semigroup; infinitesimal generator; finite delay; phase space.

2010 AMS Subject Classification: 45K05.

1. Introduction

Integrodifferential equations with delay are important for investigating some problems raised from natural phenomena. They have been studied in many different aspects. The purpose of this work is to study global existence and regularity of the following integrodifferential equation

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with finite delay in a Banach space $(X, |.|)$, namely

$$(1.1)\qquad \begin{cases} u'(t) &= Au(t) + \int_0^t g\,(t-s, u(s))\,ds + f(t, u_t), \quad \text{for} \quad t \geq 0, \\ \\ u_0 &= \varphi \in \mathscr{C} = \mathscr{C}\,([-r, 0]; \mathscr{D}(A))\,, \end{cases}$$

where $A : \mathscr{D}(A) \subset \mathscr{X} \to \mathscr{X}$ is the infinitesimal generator of a linear semigroup $(T(t))_{t\geq 0}$ a Banach space $\mathscr{X}$, g is in general a nonlinear operator from $\mathbb{R}^+ \times \mathscr{D}(A)$ to $\mathscr{X}$, $f : \mathbb{R}^+ \times \mathscr{C} \to \mathscr{X}$ is a continuous function and the phase space $\mathscr{C}$ is a linear space of functions mapping $[-r, 0]$ into $\mathscr{D}(A)$ endowed with the graph norm namely for $x \in \mathscr{D}(A)$, $|x|_{\mathscr{D}(A)} = |x|_{\mathscr{X}} + |Ax|_{\mathscr{X}}$ then $\left(\mathscr{D}(A), |.|_{\mathscr{D}(A)}\right)$ is a Banach space, for every $t \geq 0$, the history function $u_t \in \mathscr{C}$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for} \quad \theta \in [-r, 0].$$

As in [33], we consider a nonlinear Volterra integrodifferential equation of parabolic type

$$(1.2)\qquad \begin{cases} \dfrac{\partial}{\partial t} w(t,x) = \dfrac{\partial^2}{\partial x^2} w(t,x) + \displaystyle\int_0^t k\left(t-s, \frac{\partial^2}{\partial x^2} w(s,x)\right) ds + h(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t > 0 \quad \text{and} \quad 0 < x < 1, \\ \\ w(t,0) = w(t,1) = 0, \quad \text{for} \quad t > 0, \\ \\ w(0,x) = w_0(x), \quad \text{for} \quad 0 < x < 1. \end{cases}$$

The abstract version of the initial boundary value problem (1.2) is given by

$$(1.3)\qquad \begin{cases} u'(t) = Au(t) + \int_0^t g\,(t-s, u(s))\,ds + F(t), \quad \text{for} \quad t \geq 0, \\ \\ u(0) = x \in \mathscr{X}. \end{cases}$$

Some results are proved concerning local existence, global existence, continuous dependence upon initial values and asymptotic stability for $Eq.(1.3)$ under some suitable assumptions. A vast literature has investigated this equation in various aspects. $Eq.(1.3)$ has many physical applications and arises in such problems as heat flow in materials with memory [7], [8]. As a model see $Eq.(1.2)$. For his study, we also refer the reader to [[3], [9], [23], [27]].

Partial functional differential equations arise in a variety of areas of biological, physical and engineering applications, see, for example, the books and the papers in the following references [[20], [21], [25], [30], [34]], [19, 28] and the references therein. Equations with delay appear in many mathematical models of natural phenomena. Recently, the following differential equations with delay have been studies by many authors ([32], and references therein):

$$\begin{cases} u'(t) = Au(t) + F(t, u_t), \quad \text{for} \quad t \geq 0, \\ u_0 = \varphi \in \mathscr{C}([-r, 0]; \mathscr{X}). \end{cases} \tag{1.4}$$

There has been a great deal of work contributed to the study of partial differential equations with delay by using different methods under different conditions. The most classical work is due to Travis and Webb [32].

In, the recent years, many authors have attracted much attention to the study of existence problems for differential and integrodiffern(tial equations. We refer to [1, 2, 6, 10, 11, 12, 13, 14, 15, 22, 26, 29, 31] where numerous approaches that are commonly used: the contraction mapping principle, Leray-Schauder alternative, Schauder and Sadovskii fixed point theorems.

$Eq.(1.1)$ is the mixed type of $Eq.(1.3)$ and $Eq.(1.4)$. It well enable us to study the nonlinear Volterra integrodifferential equation with delay. On the basis of the results in $Eq.(1.4)$ we generalize the method used in [33] to derive global existence and regularity of $Eq.(1.1)$. The result obtained is a generalization and a continuation of [33]. The method used treats the equation in the domain of A with the graph norm employing results from linear semigroup theory concerning abstract inhomogeneous linear differential equations.

In Section 2, we recall some preliminary results about $Eq.(1.3)$ and $Eq.(1.4)$. Some basic notations and assumptions are also given in this section. In Section 3, we prove global existence and regularity of solution to $Eq.(1.1)$ which are the main results of this paper. Moreover, some properties of solutions are also studied. In Section 4, we give an example of application to show that our results valuable.

2. **Preliminary results**

In this section, we recall some fundamental results needed to establish our results. Throughout the paper, $\mathscr{X}$ is a Banach space, A is closed linear operator on $\mathscr{X}$. $\mathscr{Y}$ represent the Banach space $\mathscr{D}(A)$ equipped with the graph norm defined by $|y|_{\mathscr{Y}} = |y|_{\mathscr{X}} + |Ay|_{\mathscr{X}}$ for $y \in \mathscr{Y}$. $\mathscr{C}([-r,0];\mathscr{Y})$ is the space of continuous function from $[-r,0]$ to $\mathscr{Y}$. It is well know by the Hille-Yosida theorem that A is the infinitesimal generator of a C_0-semigroup of bounded linear operators in $\mathscr{X}$ if and only if

(i) $\overline{\mathscr{D}(A)} = \mathscr{X}$,

(ii) there exist $M \geq 1$, $w \in \mathbb{R}$ such that for $\lambda > w$, $(\lambda I - A)^{-1} \in \mathscr{B}(\mathscr{X})$ and

$$\left|(\lambda I - A)^{-n}\right| \leq \frac{M}{(\lambda - w)^n} \quad \text{for} \quad \lambda > w \quad \text{and} \quad n \in \mathbb{N},$$

where $\mathscr{B}(\mathscr{X})$ is the space of bounded linear operators on $\mathscr{X}$.

Definition 2.1. A continuous function $u : [0,+\infty[\to \mathscr{D}(A)$ is said to be strict solution of $Eq.(1.3)$ if

(i) $u \in \mathscr{C}^1([0,+\infty[;\mathscr{X}) \cap \mathscr{C}([0,+\infty[;\mathscr{Y})$

(ii) u satisfies $Eq.(1.3)$ for all $t \geq 0$.

Remark 2.2. From this definition, we deduce that $u(t) \in \mathscr{D}(A)$, the function $t \mapsto g(t-s,u(s))$ is integrable for all $t \geq 0$ and $s \in [0,t]$.

Theorem 2.3. [33]. If u is a strict solution of $Eq.(1.3)$ then u satisfies

$$u(t) = T(t)x + \int_0^t T(t-s)\int_0^s g(s-r,u(r))drds + \int_0^t T(t-s)F(s)ds. \tag{2.1}$$

Remark 2.4. If u satisfies the formula (2.1) u is not in general a strict solution. That is why we give the definition of the mild solution.

Definition 2.5. A continuous function $u : [0,+\infty[\to \mathscr{D}(A)$ is called a mild solution of $Eq.(1.3)$ if u satisfies the formula (2.1).

3. **Existence and regularity of the solutions for Eq.(1.1)**

In this section, we prove global existence and regularity of solution to $Eq.(1.1)$, which are the main results of this work. Moreover, the continuous dependence upen initial values and asymptotic stability are also studied. Firstly, we show the existence of the mild solutions. Secondly, we give sufficient conditions ensuring the existence of the strict solutions.

3.1 Global existence of the mild solutions

Definition 3.1. We say that a continuous function $u : [-r,+\infty[\to \mathscr{D}(A)$ is a strict solution of $Eq.(1.1)$ if the following conditions hold

(i) $u \in \mathscr{C}^1([0,+\infty[;\mathscr{X}) \cap \mathscr{C}([0,+\infty[;\mathscr{Y})$,

(ii) u satisfies $Eq.(1.1)$ on $[0,+\infty[$,

(iii) $u(\theta) = \varphi(\theta)$ for $-r \leq \theta \leq 0$.

Proposition 3.2. If u is a strict solution of $Eq.(1.1)$, then u is given by

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)\int_0^s g(s-r,u(r))drds + \int_0^t T(t-s)f(s,u_s)ds. \tag{3.1}$$

Proof. It is just a consequence of Theorem.(2.3). In fact, let us suppose $F(t) = f(t,u_t)$ for $t \geq 0$. Then we get the desired result.

Definition 3.3. We say that a continuous function $u : [-r,+\infty[\to \mathscr{D}(A)$ is a mild solution of $Eq.(1.1)$ if u satisfies the formula (3.1) and $u_0 = \varphi$.

To establish existence of mild solutions, we assume the following assumptions.

$(\mathbf{H_1})$ $f : \mathbb{R}^+ \times \mathscr{C}([-r,0];\mathscr{D}(A)) \to \mathscr{D}(A)$ is continuous and Lipschitzian with respect to the second argument. Let $L_f > 0$ be such that

$$|f(t,\varphi) - f(t,\hat{\varphi})| \leq L_f |\varphi - \hat{\varphi}|_{\mathscr{C}} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \varphi, \hat{\varphi} \in \mathscr{C}([-r,0];\mathscr{D}(A)).$$

$(\mathbf{H_2})$ The derivative $\frac{\partial g}{\partial t}(t,u)$ exists and is continuous from $\mathbb{R}^+ \times \mathscr{D}(A)$ into $\mathscr{X}$, moreover there exist two nondecreasing continuous functions $b : \mathbb{R}^+ \to \mathbb{R}^+$ and $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|g(s,u_1)-g(s,u_2)| \leq b(s)\,|u_1-u_2|_{\mathscr{D}(A)}$$

and

$$\left|\frac{\partial g}{\partial s}(s,u_1)-\frac{\partial g}{\partial s}(s,u_2)\right| \leq c(s)\,|u_1-u_2|_{\mathscr{D}(A)}$$

for all $s\in\mathbb{R}+$ and $u_1,u_2\in\mathscr{Y}$.

Theorem 3.4. Assume that $(\mathbf{H_1})$ and $(\mathbf{H_2})$ hold. If $\varphi\in\mathscr{C}([-r,0];\mathscr{Y})$, then there exist a unique continuous function $u:[-r,+\infty[\to\mathscr{Y}$ which solves (3.1).

Proof. Let $t_1>0$. Define the set $M_{t_1}(\varphi):=\{u\in\mathscr{C}([0,t_1];\mathscr{Y}):u(0)=\varphi(0)\}$.

$M_{t_1}(\varphi)$ is a closed subset of $\mathscr{C}([0,t_1];\mathscr{Y})$, where $\mathscr{C}([0,t_1];\mathscr{Y})$ is the space of continuous functions from $[0,t_1]$ to $\mathscr{Y}$ equipped with the uniform norm topology. Next, for each $u\in M_{t_1}(\varphi)$, we define its extension $\tilde{u}:[-r,t_1]\to\mathscr{X}$ by

$$\tilde{u}(t)=\begin{cases}\varphi(t) & \text{for} \quad t\in[-r,0],\\ u(t) & \text{for} \quad t\in[0,t_1].\end{cases}$$

Define the operator $\Gamma:M_{t_1}(\varphi)\to\mathscr{C}([-r,0];\mathscr{X})$ by

$$(\Gamma u)(t)=T(t)\varphi(0)+\int_0^t T(t-s)\left[\int_0^s g(s-r,\tilde{u}(r))\,dr+f(s,\tilde{u}_s)\right]ds. \tag{3.2}$$

The first step is to show that $\Gamma(M_{t_1}(\varphi))\subset M_{t_1}(\varphi)$. In fact, we have

$$(\Gamma u)(t)=T(t)\varphi(0)+\int_0^t T(t-s)\int_0^s g(s-r,\tilde{u}(r))drds+\int_0^t T(t-s)f(s,\tilde{u}_s)ds \quad 0\leq t\leq t_1,$$

and

$$(A\Gamma u)(t)=AT(t)\varphi(0)+A\int_0^t T(t-s)\int_0^s g(s-r,\tilde{u}(r))drds+A\int_0^t T(t-s)f(s,\tilde{u}_s)ds \quad 0\leq t\leq t_1.$$

Since A is closed, then

$$(A\Gamma u)(t)=AT(t)\varphi(0)+A\int_0^t T(t-s)\int_0^s g(s-r,\tilde{u}(r))\,drds+\int_0^t T(t-s)Af(s,\tilde{u}_s)ds \quad 0\leq t\leq t_1.$$

For the next, we need the following Lemmas.

Lemma 3.5. Let $u:[0,t_1]\to\mathscr{X}$ be continuously differentiable. Assume that $(\mathbf{H_2})$ hold. Then, $k(t)=\int_0^t g(t-s,u(s))ds$ is continuously differentiable from $[0,t_1]$ to $\mathscr{X}$.

Proof. Let $k(t)=\int_0^t g(t-s,u(s))ds$ for all $t\in[0,t_1]$. Let $h>0$.

$$
\begin{aligned}
\frac{k(t+h)-k(t)}{h} &= \frac{1}{h}\left[\int_0^{t+h} g(t+h-s,u(s))ds)-\int_0^t g(t-s,u(s))ds\right]\\
&= \frac{1}{h}\int_0^t (g(t+h-s,u(s))-g(t-s,u(s)))ds+\frac{1}{h}\int_t^{t+h} g(t+h-s,u(s))ds
\end{aligned}
$$

passing to the limit we obtain $k'(t)=\dfrac{k(t+h)-k(t)}{h}\longrightarrow\int_0^t\dfrac{\partial}{\partial t}g(t-s,u(s))ds+g(0,u(t))$ when $h\longrightarrow 0^+$.

By virtue of the hypothesis we have placed on g, we see that $k(t)$ is continuously differentiable from $[0,t_1]$ to $\mathscr{X}$.

We require the following Lemma, which is proved in $[24,p.488]$.

Lemma 3.6. [24]. Let $k:[0,t_1]\to\mathscr{X}$ be continuously differentiable and q be defined by

$$q(t)=\int_0^t T(t-s)k(s)ds \quad \text{for} \quad t\in[0,t_1].$$

Then $q(t)\in\mathscr{D}(A)$ for $t\in[0,t_1]$, q is continuously differentiable and

$$Aq(t)=q'(t)-k(t)=\int_0^t T(t-s)k'(s)ds+T(t)k(0)-k(t).$$

By virtue of the hypothesis $(\mathbf{H_2})$, then, by Lemmas 3.5 and 3.6, we deduce that,

$$
\begin{aligned}
(A\Gamma u)(t) &= AT(t)\varphi(0)+\int_0^t T(t-s)g\left(0,\tilde{u}(s)\right)ds\\
&+\int_0^t T(t-s)\int_0^s\frac{\partial g}{\partial s}\left(s-r,\tilde{u}(r)\right)drds-\int_0^t g\left(t-s,\tilde{u}(s)\right)ds \qquad (3.3)\\
&+\int_0^t T(t-s)Af(s,\tilde{u}_s)ds \quad 0\le t\le t_1.
\end{aligned}
$$

Thus, for $u\in M_{t_1}(\varphi)$, Γu and $A\Gamma u$ are both continuous from $[0,t_1]$ to $\mathscr{X}$, Γ maps $M_{t_1}(\varphi)$ into $M_{t_1}(\varphi)$.

Let $u,v\in M_{t_1}(\varphi)$. Then

$$\begin{aligned}
|(\Gamma u)(t)-(\Gamma v)(t)|_{\mathscr{X}} &\leq \left|\int_0^t T(t-s)\int_0^s (g(s-r,\tilde{u}(r))-g(s-r,\tilde{v}(r))\,drds\right|_{\mathscr{X}} \\
&+\left|\int_0^t T(t-s)\,(f(s,\tilde{u}_s)-f(s,\tilde{v}_s))\,ds\right|_{\mathscr{X}} \\
&\leq M\int_0^t e^{w(t-s)}\int_0^s |g(s-r,\tilde{u}(r))-g(s-r,\tilde{v}(r))|_{\mathscr{X}}\,drds \\
&+M\int_0^t e^{w(t-s)}\,|f(s,\tilde{u}_s)-f(s,\tilde{v}_s)|_{\mathscr{X}}\,ds \\
&\leq M\int_0^t e^{w(t-s)}\int_0^s |g(s-r,\tilde{u}(r))-g(s-r,\tilde{v}(r))|_{\mathscr{X}}\,drds \\
&+M\int_0^t e^{w(t-s)}\,|f(s,\tilde{u}_s)-f(s,\tilde{v}_s)|_{\mathscr{D}(A)}\,ds.
\end{aligned}$$

Without loss of generality, we assume that $w>0$. By $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we obtain that

$$|(\Gamma u)(t)-(\Gamma v)(t)|_{\mathscr{X}} \quad \leq Me^{wt_1}\int_0^t\int_0^s b(s-r)\,|\tilde{u}(r)-\tilde{v}(r)|_{\mathscr{D}(A)}\,drds+ML_f e^{wt_1}\int_0^t |\tilde{u}_s-\tilde{v}_s|_{\mathscr{D}(A)}\,ds.$$

On the other hand, we have

$$\begin{aligned}
&|(A\Gamma u)(t)-(A\Gamma v)(t)|_{\mathscr{X}} \\
&\leq M\textstyle\int_0^t e^{w(t-s)}\,|g(0,\tilde{u}(s))-g(0,\tilde{v}(s))|_{\mathscr{X}}\,ds+\int_0^t |g(t-s,\tilde{u}(s))-g(t-s,\tilde{v}(s))|_{\mathscr{X}}\,ds \\
&+M\int_0^t e^{w(t-s)}\int_0^s \left|\frac{\partial g}{\partial s}(s-r,\tilde{u}(r))-\frac{\partial g}{\partial s}(s-r,\tilde{v}(r))\right|_{\mathscr{X}} drds \\
&+M\int_0^t e^{w(t-s)}\,|Af(s,\tilde{u}_s)-Af(s,\tilde{v}_s)|_{\mathscr{Y}}\,ds \\
&\leq Mb(0)e^{wt_1}\int_0^t |\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{D}(A)}\,ds+Me^{wt_1}\int_0^t\int_0^s c(s-r)\,|\tilde{u}(r)-\tilde{v}(r)|_{\mathscr{D}(A)}\,drds \\
&+\int_0^t b(t-s)\,|\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{D}(A)}\,ds+ML_f e^{wt_1}\int_0^t |\tilde{u}_s-\tilde{v}_s|_{\mathscr{D}(A)}\,ds.
\end{aligned}$$

Which implies that

$$\begin{aligned}
|(\Gamma u)(t)-(\Gamma v)(t)|_{\mathscr{D}(A)} &\leq Mb(0)e^{wt_1}\int_0^t |\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{D}(A)}\,ds \\
&+Me^{wt_1}\int_0^t\int_0^s [b(s-r)+c(s-r)]\,|\tilde{u}(r)-\tilde{v}(r)|_{\mathscr{D}(A)}\,drds \\
&+\int_0^t b(t-s)\,|\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{D}(A)}\,ds+2ML_f e^{wt_1}\int_0^t |\tilde{u}_s-\tilde{v}_s|_{\mathscr{D}(A)}\,ds.
\end{aligned}$$

Define $\alpha(t)=\int_0^t e^{-ws}(b(s)+c(s))ds$ and $\beta(t)=\max_{0\le s\le t} e^{-ws}b(s)$ for $t>0$.

$$
\begin{aligned}
|(\Gamma u)(t)-(\Gamma v)(t)|_{\mathscr{D}(A)} &\le Mb(0)e^{wt_1}\int_0^{t_1}|\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{D}(A)}ds+Me^{wt_1}\alpha(t)\int_0^{t_1}|\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{D}(A)}ds\\
&+Me^{wt_1}\beta(t)\int_0^{t_1}|\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{D}(A)}ds+2ML_f e^{wt_1}\int_0^{t_1}|\tilde{u}_s-\tilde{v}_s|_{\mathscr{D}(A)}ds
\end{aligned}
$$

$$
|(\Gamma u)(t)-(\Gamma v)(t)|_{\mathscr{D}(A)} \le Mt_1e^{wt_1}\left[b(0)+\alpha(t)+\beta(t)+2L_f\right]|\tilde{u}-\tilde{v}|_{\mathscr{D}(A)}.
$$

If we choose t_1 such that $Mt_1e^{wt_1}\left[b(0)+\alpha(t)+\beta(t)+2L_f\right]<1$, then Γ is a strict contraction in $M_{t_1}(\varphi)$, then by applying the Banach fixed point Theorem, we deduce that there exists a unique fixed point $u=u(.,\varphi)$ for Γ in $M_{t_1}(\varphi)$, which implies that $Eq.(1.1)$ has a unique mild solution on $[-r,t_1]$. A similar argument can be used for $[t_1,2t_1],...,[nt_1,(n+1)t_1]$, for all $n\ge 0$, which implies that the mild solution exists uniquely in $[-r,+\infty[$. This completes the proof. □

Proposition 3.7. (Dependence continuous with respect to the initial data)
Suppose that $(\mathbf{H_1})$ and $(\mathbf{H_2})$ hold. Then there exist continuous functions $\alpha:\mathbb{R}^+\to\mathbb{R}^+$ and $\beta:\mathbb{R}^+\to\mathbb{R}^+$ such that if u and v satisfy $Eq.(1.1)$ for $0\le t\le t_1$ with $u_0=\varphi_1, v_0=\varphi_2$. Then

$$
\begin{cases}
|u_t-v_t|_{\mathscr{D}(A)}\le M|\varphi_1-\varphi_2|e^{[w+M(b(0)+\alpha(t)+\beta(t)+k)]t} & \text{if} \quad w\ge 0\\[2ex]
|u_t-v_t|_{\mathscr{D}(A)}\le Me^{-wr}|\varphi_1-\varphi_2|e^{\left[w+M(b(0)+\alpha(t)+\beta(t)+k)e^{-wr}\right]t} & \text{if} \quad w<0,
\end{cases}
$$

where k is the Lipschitz constant of f.

Proof. Define

$$
\alpha(t)=\int_0^t e^{-ws}(b(s)+c(s))ds \quad \text{and} \quad \beta(t)=\max_{0\le s\le t} b(s)e^{-ws} \quad \text{for} \quad t>0.
$$

Using (3.2) and (3.3), we obtain that

$$
\begin{aligned}
|u(t)-v(t)|_{\mathscr{X}} &\le Me^{wt}|\varphi_1-\varphi_2|+M\int_0^t e^{w(t-s)}\int_0^s|g(s-r,u(r))-g(s-r,v(r))|_{\mathscr{X}}drds\\
&+M\int_0^t e^{w(t-s)}|f(s,u_s)-f(s,v_s)|_{\mathscr{X}}ds
\end{aligned}
$$

$$|u(t)-v(t)|_{\mathscr{X}} \leq Me^{wt}|\varphi_1-\varphi_2|+M\int_0^t e^{w(t-s)}\int_0^s |g(s-r,u(r))-g(s-r,v(r))|_{\mathscr{X}}\,drds$$

$$+M\int_0^t e^{w(t-s)}|f(s,u_s)-f(s,v_s)|_{\mathscr{D}(A)}\,ds$$

$$|u(t)-v(t)|_{\mathscr{X}} \leq Me^{wt}|\varphi_1-\varphi_2|+Me^{wt}\int_0^t e^{-ws}\int_0^s b(s-r)|u(r)-v(r)|_{\mathscr{D}(A)}\,drds$$

$$+ML_f e^{wt}\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds.$$

On the other hand, we have

$$\begin{aligned}&|(Au)(t)-(Av)(t)|_{\mathscr{X}} \leq Me^{wt}|A(\varphi_1-\varphi_2)|\\&+M\int_0^t e^{w(t-s)}\left[|g(0,u(s))-g(0,v(s))|_{\mathscr{X}}+\int_0^s\left|\frac{\partial g}{\partial s}(s-r,u(r))-\frac{\partial g}{\partial s}(s-r,v(r))\right|_{\mathscr{X}}dr\right]ds\\&+\int_0^t|g(t-s,u(s))-g(t-s,v(s))|_{\mathscr{X}}\,ds+M\int_0^t e^{w(t-s)}|Af(s,u_s)-Af(s,v_s)|_{\mathscr{X}}\,ds\end{aligned}$$

$$\begin{aligned}|(Au)(t)-(Av)(t)|_{\mathscr{X}} &\leq Me^{wt}|A(\varphi_1-\varphi_2)|\\&+M\int_0^t e^{w(t-s)}\left[b(0)|u(s)-v(s)|_{\mathscr{D}(A)}+\int_0^s c(s-r)|u(r)-v(r)|_{\mathscr{D}(A)}\,dr\right]ds\\&+\int_0^t b(t-s)|u(s)-v(s)|_{\mathscr{D}(A)}\,ds+ML_f\int_0^t e^{w(t-s)}|u_s-v_s|_{\mathscr{D}(A)}\,ds.\end{aligned}$$

$$|u(t)-v(t)|_{\mathscr{D}(A)} \leq Me^{wt}|\varphi_1-\varphi_2|+Me^{wt}\int_0^t e^{-ws}\int_0^s (b(s-r)+c(s-r))|u(r)-v(r)|_{\mathscr{D}(A)}\,drds$$

$$+Mb(0)e^{wt}\int_0^t e^{-ws}|u(s)-v(s)|_{\mathscr{D}(A)}\,ds+Me^{-wt}\int_0^t b(t-s)|u(s)-v(s)|_{\mathscr{D}(A)}\,ds$$

$$+2ML_f e^{wt}\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds$$

$$|u(t)-v(t)|_{\mathscr{D}(A)} \leq Me^{wt}|\varphi_1-\varphi_2|+Me^{wt}\alpha(t)\int_0^t |u(s)-v(s)|_{\mathscr{D}(A)}\,ds$$

$$+Mb(0)e^{wt}\int_0^t e^{-ws}|u(s)-v(s)|_{\mathscr{D}(A)}\,ds+M\beta(t)\int_0^t|u(s)-v(s)|_{\mathscr{D}(A)}\,ds$$

$$+2ML_f e^{wt}\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds.$$

$$|u(t+\theta)-v(t+\theta)|_{\mathscr{D}(A)} \leq \begin{cases} |\varphi_1-\varphi_2| \quad \text{if} \quad t+\theta \leq 0, \\ \\ Me^{w(t+\theta)}|\varphi_1-\varphi_2| + Me^{w(t+\theta)}\alpha(t+\theta)\displaystyle\int_0^{t+\theta}|u(s)-v(s)|_{\mathscr{D}(A)}\,ds \\ \\ +Mb(0)e^{w(t+\theta)}\displaystyle\int_0^{t+\theta} e^{-ws}|u(s)-v(s)|_{\mathscr{D}(A)}\,ds \\ \\ +M\beta(t+\theta)\displaystyle\int_0^{t+\theta}|u(s)-v(s)|_{\mathscr{D}(A)}\,ds \\ \\ +2ML_f e^{w(t+\theta)}\displaystyle\int_0^{t+\theta} e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds, \quad \text{if} \quad t+\theta \geq 0. \end{cases}$$

If $w \geq 0$, then

$$\begin{aligned} e^{-wt}|u_t-v_t|_{\mathscr{D}(A)} &\leq M|\varphi_1-\varphi_2| + M\alpha(t)\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds + Mb(0)\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds \\ &\quad + M\beta(t)\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds + 2ML_f\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds \\ e^{-wt}|u_t-v_t|_{\mathscr{D}(A)} &\leq M|\varphi_1-\varphi_2| + M\left[b(0)+\alpha(t)+\beta(t)+2L_f\right]\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds. \end{aligned}$$

If $w < 0$, then

$$e^{-wt}|u_t-v_t|_{\mathscr{D}(A)} \leq Me^{-wr}|\varphi_1-\varphi_2| + Me^{-wr}\left[b(0)+\alpha(t)+\beta(t)+2L_f\right]\int_0^t e^{-ws}|u_s-v_s|_{\mathscr{D}(A)}\,ds.$$

By Gronwall's Lemma, the result follows. □

Proposition 3.8. Suppose the hypothesis of Theorem 3.4 and $\varphi \in \mathscr{C}$. Suppose there exist constants α_0 and β_0 such that $\int_0^t e^{-ws}(b(s)+c(s))ds \leq \alpha_0$, $b(t)e^{-wt} \leq \beta_0$ for $t \geq 0$, and $M(\alpha_0+\beta_0+b(0)+k)+w =^{def} \lambda < 0$ for some $w < 0$. Then the solutions of $Eq.(1.1)$ are exponentially asymptotically stable in the following sens: if u, v are the solutions of $Eq.(1.1)$ for $u_0 = \varphi_1, v_0 = \varphi_2$, respectively, then

$$|u_t-v_t|_{\mathscr{D}(A)} \leq Me^{-wr}|\varphi_1-\varphi_2|e^{\lambda t}, \quad \text{for} \quad t \geq 0.$$

Proof. The proof following Proposition 3.7 by obseving that $\alpha(t)$ and $\beta(t)$ satisfy $\alpha(t) \leq \alpha_0$ and $\beta(t) \leq \beta_0$. □

3.2 Existence of strict solutions

In this section we recall some fundamental results needed to establish our results. We consider the inhomogeneous initial value problem

$$(3.4)\qquad \begin{cases} u'(t) = Au(t) + F(t) \quad \text{for} \quad t \geq 0, \\ \\ u(0) = x \in \mathscr{X} \end{cases}$$

where $F : [0,a] \longrightarrow \mathscr{X}$, is continuous.

Definition 3.9. A continuous function $u : [0,+\infty[\to \mathscr{X}$ is said to be strict solution of $Eq.(3.4)$ if

(i) $u \in \mathscr{C}^1([0,+\infty[;\mathscr{X}) \cap \mathscr{C}([0,+\infty[;\mathscr{D}(A))$

(ii) u satisfies $Eq.(3.4)$ for all $t \geq 0$.

If u is a strict solution of $Eq.(3.4)$, then u is given by

$$(3.5)\qquad u(t) = T(t)x + \int_0^a T(t-s)F(s)ds \quad \text{for} \quad t \in [0,a].$$

The next Theorem provides sufficients conditions for the regularity of solution to $Eq.(3.4)$.

Theorem 3.10. [30]. Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. let $F \in L^1(0,a;\mathscr{X})$ be continuous on $[0,a]$ and let

$$v(t) = \int_0^t T(t-s)F(s)ds \quad t \in [0,a].$$

The $Eq.(3.4)$ has a strict solution u on $[0,a]$ for every $x \in \mathscr{D}(A)$ if one of the following conditions is satisfied;

(1) $v(t)$ is continuously differentiable on $[0,a]$.

(2) $v(t) \in \mathscr{D}(A)$ for $0 < t < a$ and $Av(t)$ is continuous on $[0,a]$. If $Eq.(3.4)$ has a strict solution u on $[0,a]$ for some $x \in \mathscr{D}(A)$ then v satisfies both (1) and (2).

From Theorem 3.10 we dram the following useful Lemma.

Lemma 3.11. [30]. Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. If $F \in L^1([0,a];\mathscr{D}(A))$ be continuous on $[0,a]$. If $F(s)\in\mathscr{D}(A)$ for $0<s<a$ and $AF\in L^1([0,a];\mathscr{D}(A))$ then for every $x\in\mathscr{D}(A)$ the $Eq.(3.4)$ has a strict solution u on $[0,a]$.

Theorem 3.12. Let $u\in\mathscr{C}([0,t_1];\mathscr{D}(A))$ the mild solution be defined by the formula (3.1). If $u_0\in\mathscr{D}(A)$ and $f\in L^1(\mathbb{R}^+\times\mathscr{C};\mathscr{D}(A))$ be continuous from $\mathbb{R}^+\times\mathscr{C}$ to $\mathscr{D}(A)$, then u is a strict solution of $Eq.(1.1)$.

Proof. It is just a consequence of Theorem 3.10. In fact, let us suppose

$$v(t)=\int_0^t T(t-s)\int_0^s g(s-r,u(r))drds+\int_0^t T(t-s)f(s,u_s)ds \quad \text{for} \quad t\geq 0.$$

We show that v satisfies the following two conditions

(i) v is continuously differentiable on $[0,t_1]$ and v' is continuous on $[0,t_1]$,

(ii) $v(t)\in\mathscr{D}(A)$ on $[0,t_1]$ and Av is continuous on $[0,t_1]$.

Based on the formula (3.1) we have: $v(t)=u(t)-T(t)\varphi(0)$ is differentiable for $t>0$ as the difference of two such differentiable functions and $v'(t)=u'(t)-T(t)A\varphi(0)$ is obviously continuous on $(0,t_1)$. Therefore (i) is satisfied. Also if $\varphi\in\mathscr{D}(A)$ $T(t)\varphi\in\mathscr{D}(A)$ for $t\geq 0$ and therefore $v(t)=u(t)-T(t)\varphi(0)\in\mathscr{D}(A)$ for $t>0$ and $Av(t)=Au(t)-AT(t)\varphi=u'(t)-\int_0^t g(t-s,u(s))ds-f(t,u_t)-T(t)A\varphi$ is continuous on $(0,t_1)$. Thus also (ii) is satisfaied.

On the other hand, it is easy to verify for $h>0$ the identify

$$\left(\frac{T(h)-I}{h}\right)v(t) = \frac{v(t+h)-v(t)}{h} -\frac{1}{h}\int_t^{t+h} T(t+h-s)\left[k(s)+f(s,u_s\right]ds. \tag{3.6}$$

From the continuity of $k(s)=\int_0^t g(t-s,u(s))ds$ and f it is clear that the second therm on the right-hand side of (3.6) has the limit $k(s)=\int_0^t g(t-s,u(s))ds+f(t,u_t)$ as $h\to 0$. If $v(t)$ is continuously differentiable on $(0,t_1)$ then it follows from (3.6) that $v(t)\in\mathscr{D}(A)$ for $0<t<t_1$ and $Av(t)=v'(t)-\left[\int_0^t g(t-s,u(s))ds+f(t,u_t)\right]$. Since $v(0)=0$ it follows that $u(t)=T(t)\varphi(0)+$

$v(t)$ is the solution of $Eq.(1.1)$ for $\varphi(0) \in \mathscr{D}(A)$. If $v(t) \in \mathscr{D}(A)$ it follows from (3.6) that $v(t)$ is differentiable from the right at t and the right derivative $\mathscr{D}^+v(t)$ of v satisfies $\mathscr{D}^+v(t) = Av(t) + \int_0^t g(t-s,u(s))ds + f(t,u_t)$. Since $\mathscr{D}^+v(t)$ is continuous, $v(t)$ is continuously differentiable and $v'(t) = Av(t) + k(t) + f(t,u_t)$. Since $v(0) = 0$, $u(t) = T(t)\varphi(0) + v(t)$ is the solution of $Eq.(1.1)$ for $\varphi \in \mathscr{D}(A)$ and the proof is complete.

4. **Application**

For illustration, we propose to study the existence of solutions for the following model

$$
(4.1) \qquad \begin{cases}
\dfrac{\partial}{\partial t}y(t,x) & = \dfrac{\partial^2}{\partial x^2}y(t,x) + \displaystyle\int_0^t \beta\Big(t-s, \dfrac{\partial^2}{\partial x^2}y(s,x)\Big)ds \\
& \quad + \displaystyle\int_{-r}^0 h(\theta, y(t+\theta,x))d\theta \quad \text{for} \quad t \geq 0, \quad 0 \leq x \leq 1, \\
y(t,0) & = y(t,1) = 0 \quad \text{for} \quad t \geq 0, \\
y(\theta,x) & = \varphi_0(\theta,x) \quad \text{for} \quad \theta \in [-r,0] \quad \text{and} \quad 0 \leq x \leq 1,
\end{cases}
$$

where $h : [-r,0] \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitzian with respect to the second argument, $\beta : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is bounded uniformly continuous, continuously differentiable in its first place and the derivative $\frac{\partial \beta}{\partial t}$ exists and is Lipschitzian continuous. The function $\varphi_0 : [-r,0] \times [0,1] \to \mathbb{R}$ will be specified later. To rewrite $Eq.(4.1)$ in the abstract from, we introduce the space $\mathscr{X} = L^2([0,1];\mathbb{R})$. Let $A : \mathscr{D}(A) \to \mathscr{X}$ be defined by

$$
\begin{cases}
\mathscr{D}(A) = H^2(0,1) \cap H_0^1(0,1), \\
Az = z''.
\end{cases}
$$

Let $g : \mathbb{R}^+ \times \mathscr{D}(A) \to \mathscr{X}$ by $g(t,z) = \beta(t,Az)$ for $t \geq 0$. Let $f : \mathbb{R}^+ \times \mathscr{C} \to \mathscr{X}$ be defined by

$$
f(t,\varphi)(x) = \int_{-r}^0 h(\theta, \varphi(\theta)(x))d\theta, \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{and} \quad t \geq 0.
$$

The initial data $\varphi(\theta)(x) = \varphi_0(\theta,x)$, for $\theta \in [-r,0]$ and $x \in [0,1]$.

Let us suppose $v(t) = y(t,.)$. Then $Eq.(4.1)$ takes the following abstract from

$$\begin{cases} \frac{d}{dt}v(t) = Av(t) + \int_0^t g(t-s, v(s))ds + f(t, v_t) \quad \text{for} \quad t \geq 0, \\ \\ v_0 = \varphi. \end{cases} \tag{4.2}$$

It is well known that A is the generator of C_0-semigroup $(T(t))_{t\geq 0}$ in $\mathscr{X}$. In addition, we suppose that

(i) $h \in \mathscr{C}([-r,0]\times\mathbb{R},\mathbb{R})$ is continuous and Lipshitzian with respect to the second argument.

(ii) $\beta \in \mathscr{C}(\mathbb{R}^+\times\mathbb{R},\mathbb{R})$ is bounded uniformly continuous, continuously differentiable in its first place and the derivative β_t exists and is Lipschitzian continuous.

(iii) The initial data $\varphi \in \mathscr{C}([-r,0]\times[0,1],\mathscr{D}(A))$, $\varphi_0(0,0) = \varphi_0(0,1) = 0$ is continuous from $[-r,0]\times[0,1]$ to $\mathscr{D}(A)$. From the assumption (i), f satisfies the hypothesis $(\mathbf{H_1})$. Moreover, from assumption (ii), it follows that g satisfies the hypothesis $(\mathbf{H_2})$. Finally, from assumption (iii) and Theorem 3.4, we deduce that $Eq.(4.1)$ has a unique mild solution which is defined for all $t \geq 0$. For the regularity, we impose the following conditions which imply the hypotheses of Theorem 3.12.

(iv) $h \in L^1([-r,0]\times\mathbb{R};\mathbb{R})$ be continuous on $[-r,0]\times\mathbb{R}$,

(v) $\varphi_0 \in \mathscr{C}([-r,0];\mathscr{D}(A))$ such that $\varphi_0(0,.) \in \mathscr{D}(A)$.

Consequently, by Theorem 3.12, we obtain the following existence result.

Proposition 4.1. Under the above assumptions, $Eq.(4.1)$ has a unique strict solution v and the solution u defined by $u(t,x) = v(t)(x)$ for $t \geq 0$ and $x \in [0,1]$ is a solution $Eq.(4.1)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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