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CLASSES OF FUNCTIONS ON COMMON FIXED POINTS FOR TWO MAPPINGS OF INTEGRAL TYPE WITH SEMI-IMPLICIT CONTRACTIVE CONDITIONS IN METRIC SPACES

ARSLAN HOJAT ANSARI[1], YONGJIE PIAO[2,*], NAWAB HUSSAIN[3]

[1]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

[2]Department of Mathematics, College of Science, Yanbian University, Yanji, China

[3]Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia

Abstract. In this paper, by using the classes $\mathscr{C}$ and Φ_u of functions to introduce a generalization of the known class Ψ of 5-dimensional functions, we discuss the existence problems of common fixed points for two mappings of integral type with semi-implicit contractive conditions and give more general results and some particular forms.

Keywords: class $\mathscr{C}$; class Ψ^*; class Φ_u; semi-implicit; sub-additive; common fixed point.

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1. Introduction and Preliminaries

Throughout this paper, we assume that $\mathbb{R}^+ = [0,+\infty)$ and

$\Phi = \{\phi : \phi : \mathbb{R}^+ \to \mathbb{R}^+$satisfying that ϕ is Lebesgue integral, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^{\varepsilon} \phi(t)dt > 0$ for each $\varepsilon > 0\}$

The famous Banach's contraction principle is as follows:

*Corresponding author

E-mail address: sxpyj@ybu.edu.cn

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Theorem 1.1.[1] Let f be a self mapping on a complete metric space (X,d) satisfying

$$d(fx,fy) \leq cd(x,y),\ \forall\, x,y \in X, \tag{1.1}$$

where $c \in [0,1)$ is a constant. Then f has a unique fixed point $\hat{x} \in X$ such that $\lim_{n\to\infty} f^n x = \hat{x}$ for each $x \in X$.

It is known that the Banach contraction principle has a lot of generalizations and various applications in many directions, see, for examples, [2-12] and the references cited therein. Especially, in 1962, Rakotch[13] extended the Banach contraction principle with replacing the contraction constant c in (1.1) by a contraction function γ and obtained the next theorem:

Theorem 1.2.[13] Let f be a self-mapping on a complete metric space (X,d) satisfying

$$d(fx,fy) \leq \gamma(d(x,y))d(x,y),\ \forall\, x,y \in X, \tag{1.2}$$

where $\gamma : \mathbb{R}^+ \to [0,1)$ is a monotonically decreasing function. Then f has a unique fixed point $\hat{x} \in X$ such that $\lim_{n\to\infty} f^n x = \hat{x}$ for each $x \in X$.

In 2002, Branciari[14] gave an integral version of Theorem 1.1 as follows:

Theorem 1.3.[14] Let f be a self-mapping on a complete metric space (X,d) satisfying

$$\int_0^{d(fx,fy)} \phi(t)dt \leq c \int_0^{d(x,y)} \phi(t)dt,\ \forall\, x,y \in X, \tag{1.3}$$

where $c \in (0,1)$ is a constant and $\phi \in \Phi$. Then f has a unique fixed point $\hat{x} \in X$ such that $\lim_{n\to\infty} f^n x = \hat{x}$ for each $x \in X$.

In 2011, Liu and Li[15] modified the method of Rakotch to generalize the Branciari's fixed point theorem with replacing the contraction constant c in (1.3) by contraction functions α and β and established the following fixed point theorem:

Theorem 1.4.[15] Let f be a self-mapping on a complete metric space (X,d) satisfying

$$\int_0^{d(fx,fy)} \phi(t)dt \leq \alpha(d(x,y)) \int_0^{d(x,fx)} \phi(t)dt + \beta(d(x,y)) \int_0^{d(y,fy)} \phi(t)dt, \forall\, x,y \in X, \tag{1.4}$$

where $\phi \in \Phi$ and $\alpha, \beta : \mathbb{R}^+ \to [0,1)$ are two functions with

$$\alpha(t) + \beta(t) < 1, \forall\, t \in \mathbb{R}^+;\ \ \limsup_{s\to 0^+} \beta(s) < 1;\ \limsup_{s\to t^+} \frac{\alpha(s)}{1-\beta(s)} < 1,\ \forall\, t > 0.$$

Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

In [16], Jin and Piao discussed the existence problems of unique common fixed points for two mappings of integral type with variable coefficient in metric spaces and obtained the more general results, the main results generalize and improve Theorem 1.4. Also they introduce the following two definitions and obtain a unique common fixed point theorem for two mappings of integral type with semi-implicit contractive conditions:

The function $\phi \in \Phi$ is called to be sub-additive if and only if for all $a, b \in \mathbb{R}^+$,

$$\int_0^{a+b} \phi(t)dt \leq \int_0^a \phi(t)dt + \int_0^b \phi(t)dt.$$

Let $\psi \in \Psi^{[16]}$ if and only if $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ is a continuous and nondecreasing function about the 4th and 5th variables and the following conditions hold:

(i) there exists $h_1 \in (0,1)$ such that $u \leq \psi(v,u,v,0,u+v)$ implies $u \leq h_1 v$;

(ii) there exists $h_2 \in (0,1)$ such that $u \leq \psi(v,v,u,u+v,0)$ implies $u \leq h_2 v$;

(iii) $\psi(t,0,0,t,t) < t$, $\psi(0,t,0,0,t) < t$ and $\psi(0,0,t,t,0) < t$ for all $t > 0$.

Theorem 1.5.[16] Let (X,d) be a complete metric space, $f, g : X \to X$ two mappings. If for each $x, y \in X$,

$$\int_0^{d(fx,gy)} \phi(t)dt \leq \psi\Big(\int_0^{d(x,y)} \phi(t)dt, \int_0^{d(x,fx)} \phi(t)dt, \int_0^{d(y,gy)} \phi(t)dt, \int_0^{d(x,gy)} \phi(t)dt, \int_0^{d(y,fx)} \phi(t)dt\Big), \tag{1.5}$$

where $\phi \in \Phi$ is sub-additive and $\psi \in \Psi$. Then f and g have a unique common fixed point.

The aim of this paper is to use two classes $\mathscr{C}$ and Φ_u of real functions to introduce a real generalization of the above class Ψ and to discuss the unique existence problems of common fixed points for two self-mappings of integral type with a new semi-implicit limitation in a non-complete metric space. The obtained results further generalize and improve the corresponding conclusions in the literature.

To do this, we introduce the definitions of $\mathscr{C}$ and Φ_u and give a well-known lemma.

The following concept of class $\mathscr{C}$ of functions was introduced by A.H.Ansari **[17]**.

Definition 1.6.[17]A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies following axioms:

(1)$F(s,t) \leq s$;

(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0,\infty)$.

Note for some F we have that $F(0,0)=0$.

We denote C-class functions as $\mathscr{C}$.

Example 1.7.[17] The following functions $F:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathscr{C}$, for all $s,t\in[0,\infty)$:

(1) $F(s,t)=s-t$, $F(s,t)=s\Rightarrow t=0$;

(2) $F(s,t)=ms$, $0<m<1$, $F(s,t)=s\Rightarrow s=0$;

(3) $F(s,t)=\frac{s}{(1+t)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;

(4) $F(s,t)=\log(t+a^s)/(1+t)$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;

(5) $F(s,t)=\ln(1+a^s)/2$, $a>e$, $F(s,1)=s\Rightarrow s=0$;

(6) $F(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1, r\in(0,\infty)$, $F(s,t)=s\Rightarrow t=0$;

(7) $F(s,t)=s\log_{t+a}a$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;

(8) $F(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t)=s\Rightarrow t=0$;

(9) $F(s,t)=s\beta(s)$, $\beta:[0,\infty)\to[0,1)$ is continuous, $F(s,t)=s\Rightarrow s=0$;

(10) $F(s,t)=s-\varphi(s), F(s,t)=s\Rightarrow s=0$,here $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0\Leftrightarrow t=0$;

(11) $F(s,t)=sh(s,t), F(s,t)=s\Rightarrow s=0$,here $h:[0,\infty)\times[0,\infty)\to[0,\infty)$is a continuous function such that $h(t,s)<1$ for all $t,s>0$;

(12) $F(s,t)=s-(\frac{2+t}{1+t})t$, $F(s,t)=s\Rightarrow t=0$;

(13) $F(s,t)=\sqrt[n]{\ln(1+s^n)}$, $F(s,t)=s\Rightarrow s=0$;

(14) $F(s,t)=\phi(s), F(s,t)=s\Rightarrow s=0$,here $\phi:[0,\infty)\to[0,\infty)$ is a upper semicontinuous function such that $\phi(0)=0$, and $\phi(t)<t$ for $t>0$,

(15) $F(s,t)=\frac{s}{(1+s)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$;

(16) $F(s,t)=\vartheta(s)$; $\vartheta:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is a generalized Mizoguchi-Takahashi type function , $F(s,t)=s\Rightarrow s=0$;

(17) $F(s,t)=\frac{s}{\Gamma(1/2)}\int_0^\infty\frac{e^{-x}}{\sqrt{x}+t}\,dx$, where Γ is the Euler Gamma function.

Definition 1.8. Let Φ_u be a set of all functions $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the following conditions:

(1) φ is continuous;

(2) $\varphi(t)>0$ if $t>0$ and $\varphi(0)\geq 0$.

Lemma 1.9.[18] Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ for each $k \in \mathbb{N}$ such that $d(x_{m(k)},x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1},x_{n(k)}) < \varepsilon$ and the following result holds

$$\lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)}) = \varepsilon.$$

Remark 1.10. Under the conditions of Lemma 1.9, We easily obtian the following result:

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)},x_{n(k)-1}) = \varepsilon.$$

2. Common fixed points

Lemma 2.1.[15] Let $\phi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = \int_0^a \phi(t)dt.$$

Lemma 2.2.[15] Let $\phi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = 0 \Longleftrightarrow \lim_{n\to\infty} r_n = 0.$$

Now, we give a new and real generalization of the class Ψ in [16] as follows.

Let $\psi \in \Psi^*$ if and only if $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ is a continuous and nondecreasing function about the 4th and 5th variables and the following conditions hold:

(i) there exists $F_1 \in \mathscr{C}, \varphi_1 \in \Phi_u$, such that $u \leq \psi(v,u,v,0,u+v)$ implies $u \leq F_1(v,\varphi_1(v))$;

(ii) there exists $F_2 \in \mathscr{C}, \varphi_2 \in \Phi_u$, such that $u \leq \psi(v,v,u,u+v,0)$ implies $u \leq F_2(v,\varphi_2(v))$;

(iii) $\psi(t,0,0,t,t) < t$, $\psi(0,t,0,0,t) < t$ and $\psi(0,0,t,t,0) < t$ for all $t > 0$.

Example 2.1. Define $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ as follows

$$\psi(x_1,x_2,x_3,x_4,x_5) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5, \forall x_1,x_2,x_3,x_4,x_5 \in \mathbb{R}^+,$$

where $a_i \geq 0$ for all $i = 1,2,3,4,5$ and $\sum_{i=1}^{3} a_i + 2\sum_{i=4}^{5} a_i < 1$, and $\varphi_1(t) = h_1t$, $\varphi_2(t) = h_2t$ for all $t \in [0,\infty)$, where $h_1 \in (0, \frac{1-(a_1+a_2+a_3+2a_5)}{1-a_2-a_5})$ and $h_2 \in (0, \frac{1-(a_1+a_2+a_3+2a_4)}{1-a_3-a_4})$ are two constants, and $F_1(s,t) = F_2(s,t) = s-t$ for all $s,t \geq 0$. Then it is easy to check that $\psi, F_1, F_2, \varphi_1, \varphi_2$ satisfy all of the conditions of $\psi \in \Psi^*$.

We also know that the definition of $\psi \in \Psi^*$ is a proper generalization of $\psi \in \Psi$.

The function $\phi \in \Phi$ is called to be strictly increasing about integral type if for any $x, y \in [0, \infty)$ with $x < y$,

$$\int_0^x \phi(t)dt < \int_0^y \phi(t)dt.$$

Example 2.2. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi(t) = \frac{1}{1+t}$ for each $t \in \mathbb{R}^+$. Then obviously $\phi \in \Phi$ and for all $a, b \in \mathbb{R}^+$,

$$\int_0^{a+b} \phi(t)dt = \ln^{(1+a+b)} \leq \ln^{(1+a+b+ab)} = \ln^{(1+a)(1+b)} = \int_0^a \phi(t)dt + \int_0^b \phi(t)dt.$$

And for $0 \leq x < y$,

$$\int_0^x \frac{1}{1+t}dt = \ln^{(1+x)} < \ln^{(1+y)} = \int_0^y \frac{1}{1+t}dt.$$

Hence $\phi(t) = \frac{1}{1+t}$ is a sub-additive and strictly increasing function about integral type.

The following results is the main common fixed point theorem.

Theorem 2.3. Let (X, d) be a metric space, $f, g : X \to X$ two mappings. Suppose that for each $x, y \in X$,

$$\int_0^{d(fx,gy)} \phi(t)dt \leq \psi\Big(\int_0^{d(x,y)} \phi(t)dt, \int_0^{d(x,fx)} \phi(t)dt, \int_0^{d(y,gy)} \phi(t)dt, \int_0^{d(x,gy)} \phi(t)dt, \int_0^{d(y,fx)} \phi(t)dt\Big), \tag{2.1}$$

where $\phi \in \Phi$ is sub-additive and strictly increasing about the integral type and $\psi \in \Psi^*$. If fX or gX is complete, then f and g have a unique common fixed point.

Proof. We take any element $x_0 \in X$ and consider the sequence $\{x_k\}$ constructed by $x_{2k+1} = fx_{2k}$ and $x_{2k+2} = gx_{2k+1}$ for all $k = 0, 1, 2, \cdots$. Let $d_n = d(x_n, x_{n+1})$ for all $n = 0, 1, 2, \cdots$.

Since

$$\begin{aligned}
&\int_0^{d_{2n}}\phi(t)dt=\int_0^{d(fx_{2n},gx_{2n-1})}\phi(t)dt\\
\leq&\psi\Big(\int_0^{d(x_{2n},x_{2n-1})}\phi(t)dt,\int_0^{d(x_{2n},fx_{2n})}\phi(t)dt,\int_0^{d(x_{2n-1},gx_{2n-1})}\phi(t)dt,\int_0^{d(x_{2n},gx_{2n-1})}\phi(t)dt,\\
&\int_0^{d(x_{2n-1},fx_{2n})}\phi(t)dt\Big)\\
=&\psi\Big(\int_0^{d_{2n-1}}\phi(t)dt,\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n-1}}\phi(t)dt,0,\int_0^{d(x_{2n-1},x_{2n+1})}\phi(t)dt\Big)\\
\leq&\psi\Big(\int_0^{d_{2n-1}}\phi(t)dt,\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n-1}}\phi(t)dt,0,\int_0^{d_{2n-1}+d_{2n+1}}\phi(t)dt\Big)\\
\leq&\psi\Big(\int_0^{d_{2n-1}}\phi(t)dt,\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n-1}}\phi(t)dt,0,\int_0^{d_{2n-1}}\phi(t)dt+\int_0^{d_{2n}}\phi(t)dt\Big).
\end{aligned}$$

So by (i),

$$\int_0^{d_{2n}}\phi(t)dt\leq F_1\Big(\int_0^{d_{2n-1}}\phi(t)dt,\varphi_1(\int_0^{d_{2n-1}}\phi(t)dt)\Big). \tag{2.2}$$

Similarly,

$$\begin{aligned}
&\int_0^{d_{2n+1}}\phi(t)dt=\int_0^{d(x_{2n+1},x_{2n+2})}\phi(t)dt=\int_0^{d(fx_{2n},gx_{2n+1})}\phi(t)dt\\
\leq&\psi\Big(\int_0^{d(x_{2n},x_{2n+1})}\phi(t)dt,\int_0^{d(x_{2n},fx_{2n})}\phi(t)dt,\int_0^{d(x_{2n+1},gx_{2n+1})}\phi(t)dt,\int_0^{d(x_{2n},gx_{2n+1})}\phi(t)dt,\\
&\int_0^{d(x_{2n+1},fx_{2n})}\phi(t)dt\Big)\\
=&\psi\Big(\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n+1}}\phi(t)dt,\int_0^{d(x_{2n},x_{2n+2})}\phi(t)dt,0\Big)\\
\leq&\psi\Big(\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n+1}}\phi(t)dt,\int_0^{d_{2n}+d_{2n+1}}\phi(t)dt,0\Big)\\
\leq&\psi\Big(\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n}}\phi(t)dt,\int_0^{d_{2n+1}}\phi(t)dt,\int_0^{d_{2n}}\phi(t)dt+\int_0^{d_{2n+1}}\phi(t)dt,0\Big).
\end{aligned}$$

So by (ii),

$$\int_0^{d_{2n+1}}\phi(t)dt\leq F_2\Big(\int_0^{d_{2n}}\phi(t)dt,\varphi_2(\int_0^{d_{2n}}\phi(t)dt)\Big). \tag{2.3}$$

Combining (2.3) and (2.4) and using the property of F_1 and F_2, we have

$$\int_0^{d_{n+1}}\phi(t)dt\leq\int_0^{d_n}\phi(t)dt,\ \forall n\in\mathbb{N}. \tag{2.4}$$

If there exists $n_0 \in \mathbb{N}$ such that $d_{n_0+1} > d_{n_0}$, then by the strictly increasing property of ϕ about integral type, we obtain

$$\int_0^{d_{n_0+1}} \phi(t)dt > \int_0^{d_{n_0}} \phi(t)dt,$$

which is a contradiction with (2.4), hence we have

$$d_{n+1} \leq d_n, \forall\, n = 0, 1, 2, \cdots. \tag{2.5}$$

So there is $u \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n = u$. If $u > 0$, then by Lemma 2.1 and (2.2),

$$\begin{aligned}\int_0^u \phi(t)dt &= \lim_{n\to\infty}\int_0^{d_{2n}} \phi(t)dt \leq F_1(\lim_{n\to\infty}\int_0^{d_{2n-1}} \phi(t)dt, \lim_{n\to\infty}\varphi_1(\int_0^{d_{2n-1}} \phi(t)dt))\\ &= F_1(\int_0^u \phi(t)dt, \varphi_1(\int_0^u \phi(t)dt)).\end{aligned}$$

So $\int_0^u \phi(t)dt = 0$ or $\varphi_1(\int_0^u \phi(t)dt) = 0$ by the property of F_1, thus $\int_0^u \phi(t)dt = 0$, which is a contradiction with the condition of ϕ. Therefore, $u = 0$, i.e., $\lim_{n\to\infty} d_n = 0$.

We claim that $\{x_n\}$ is Cauchy. Otherwise, by Lemma 1.9 and Remark 1.10, there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k)$ such that the parity of $m(k)$ and $n(k)$ is different and the following result holds

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \varepsilon. \tag{2.6}$$

If $m(k)$ is even and $n(k)$ is odd, then by Lemma 2.1, (2.1), (2.6) and (iii),

$$\begin{aligned}0 &< \int_0^{\varepsilon} \phi(t)dt = \lim_{k\to\infty}\int_0^{d(x_{m(k)+1}, x_{n(k)+1})} \phi(t)dt = \lim_{k\to\infty}\int_0^{d(fx_{m(k)}, gx_{n(k)})} \phi(t)dt\\ &\leq \lim_{k\to\infty}\psi\Big(\int_0^{d(x_{m(k)}, x_{n(k)})} \phi(t)dt, \int_0^{d(x_{m(k)}, fx_{m(k)})} \phi(t)dt, \int_0^{d(x_{n(k)}, gx_{n(k)})} \phi(t)dt,\\ &\qquad\qquad \int_0^{d(x_{m(k)}, gx_{n(k)})} \phi(t)dt, \int_0^{d(fx_{m(k)}, x_{n(k)})} \phi(t)dt\Big)\\ &= \psi\Big(\int_0^{\varepsilon} \phi(t)dt, 0, 0, \int_0^{\varepsilon} \phi(t)dt, \int_0^{\varepsilon} \phi(t)dt\Big)\\ &< \int_0^{\varepsilon} \phi(t)dt.\end{aligned}$$

This is a contradiction. Similarly, we obtain the same contradiction for the case that $m(k)$ is odd and $n(k)$ is even. Hence, $\{x_n\}$ is a Cauchy sequence.

If fX is complete, then $\{x_n\}$ is a Cauchy sequence and $x_{2n+1} \in fX$(for all $n = 0,1,2,\cdots$) implies there exists $x^* \in fX$ such that $x_{2n+1} \to x^*$. Hence

$$d(x_{2n+2},x^*) \leq d(x_{2n+2},x_{2n+1}) + d(x_{2n+1},x^*)$$

implies $x_{2n+2} \to x^*$, therefore $x_n \to x^*$ as $n \to \infty$. Similarly we have $x_n \to y^*$ for some $y^* \in gX$ for the case that gX is complete. Hence we can assume that $x_n \to x^* \in fX \cup gX$ as $n \to \infty$ for any case.

If $fx^* \neq x^*$, then $d(fx^*,x^*) > 0$, hence by Lemma 2.1 and (2.1) and (iii),

$$\begin{aligned}
0 < \int_0^{d(fx^*,x^*)} \phi(t)dt &= \lim_{n\to\infty} \int_0^{d(fx^*,gx_{2n+1})} \phi(t)dt \\
&\leq \lim \psi\Big(\int_0^{d(x^*,x_{2n+1})} \phi(t)dt, \int_0^{d(x^*,fx^*)} \phi(t)dt, \int_0^{d(x_{2n+1},gx_{2n+1})} \phi(t)dt, \\
&\qquad \int_0^{d(x^*,gx_{2n+1})} \phi(t)dt, \int_0^{d(fx^*,x_{2n+1})} \phi(t)dt\Big) \\
&= \psi\Big(0, \int_0^{d(fx^*,x^*)} \phi(t)dt, 0, 0, \int_0^{d(fx^*,x^*)} \phi(t)dt\Big) \\
&< \int_0^{d(fx^*,x^*)} \phi(t)dt.
\end{aligned}$$

This is a contradiction, hence $fx^* = x^*$. Similarly, we obtain $gx^* = x^*$. Therefore, x^* is a common fixed point of f and g.

If y^* is another common fixed point of f and g, then $d(x^*,y^*) > 0$, hence by (2.1) and (iii),

$$\begin{aligned}
0 < \int_0^{d(x^*,y^*)} \phi(t)dt &= \int_0^{d(fx^*,gy^*)} \phi(t)dt \\
&\leq \psi\Big(\int_0^{d(x^*,y^*)} \phi(t)dt, \int_0^{d(x^*,fx^*)} \phi(t)dt, \int_0^{d(y^*,gy^*)} \phi(t)dt, \int_0^{d(x^*,gy^*)} \phi(t)dt, \int_0^{d(fx^*,y^*)} \phi(t)dt\Big) \\
&= \psi\Big(\int_0^{d(x^*,y^*)} \phi(t)dt, 0, 0, \int_0^{d(x^*,y^*)} \phi(t)dt, \int_0^{d(x^*,y^*)} \phi(t)dt\Big) \\
&< \int_0^{d(x^*,y^*)} \phi(t)dt.
\end{aligned}$$

This is also a contradiction, hence x^* is the unique common fixed point of f and g.

Using Theorem 2.3 and Example 2.2, we have the next result.

Theorem 2.4. Let (X,d) be a metric space, $f,g : X \to X$ two mappings. Suppose that for all $x,y \in X$,

$$\ln^{(1+d(fx,gy))} \leq \psi\Big(\ln^{(1+d(x,y))}, \ln^{(1+d(x,fx))}, \ln^{(1+d(y,gy))}, \ln^{(1+d(x,gy))}, \ln^{(1+d(y,fx))}\Big),$$

where $\psi \in \Psi^*$. If fX or gX is complete, then f and g have a unique common fixed point.

Combining Theorem 2.4 and Example 2.1, we obtain the following result.

Theorem 2.5. Let (X,d) be a metric space, $f,g : X \to X$ two mappings. Suppose that for each $x,y \in X$,

$$1+d(fx,gy) \leq (1+d(x,y))^{a_1}(1+d(x,fx))^{a_2}(1+d(y,gy))^{a_3}(1+d(x,gy))^{a_4}(1+d(y,fx))^{a_5},$$

where $a_i \geq 0$ for all $i = 1,2,3,4,5$ and $a_1+a_2+a_3+2a_4+2a_5 < 1$. If fX or gX is complete, then f and g have a unique common fixed point u.

From Theorem 2.3, we obtain the next more general common fixed point theorem.

Theorem 2.6. Let (X,d) be a metric space, $m,n \in \mathbb{N}$ and $f,g : X \to X$ two mappings. Suppose that for each $x,y \in X$,

$$\begin{aligned} &\int_0^{d(f^m x, g^n y)} \phi(t)dt \\ \leq &\psi\Big(\int_0^{d(x,y)} \phi(t)dt, \int_0^{d(x,f^m x)} \phi(t)dt, \int_0^{d(y,g^n y)} \phi(t)dt, \int_0^{d(x,g^n y)} \phi(t)dt, \int_0^{d(y,f^m x)} \phi(t)dt\Big), \end{aligned} \tag{2.7}$$

where $\phi \in \Phi$ is sub-additive and strictly increasing about integral type and $\psi \in \Psi^*$. If $f^m X$ or $g^n X$ is complete, then f and g have a unique common fixed point.

Proof. Let $F = f^m$ and $G = g^n$, then F and G satisfy all of the conditions of Theorem 2.3, hence there exists an unique element $u \in X$ such that $f^m u = Fu = u = Gu = g^n u$. If $fu \neq u$, then by (2.7) and (iii),

$$\begin{aligned} 0 < &\int_0^{d(fu,u)} \phi(t)dt = \int_0^{d(f^m fu, g^n u)} \phi(t)dt \\ \leq &\psi\Big(\int_0^{d(fu,u)} \phi(t)dt, \int_0^{d(fu,f^m fu)} \phi(t)dt, \int_0^{d(u,g^n u)} \phi(t)dt, \int_0^{d(fu,g^n u)} \phi(t)dt, \int_0^{d(u,f^m fu)} \phi(t)dt\Big) \\ = &\psi\Big(\int_0^{d(fu,u)} \phi(t)dt, 0, 0, \int_0^{d(fu,u)} \phi(t)dt, \int_0^{d(u,fu)} \phi(t)dt\Big) \\ < &\int_0^{d(fu,u)} \phi(t)dt, \end{aligned}$$

which is a contradiction. Hence $fu = u$. Similarly, we can obtain $gu = u$. So u is the common fixed point of f and g. The uniqueness is obvious.

Conflict of Interests

The authors declare that there is no conflict of interests.

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