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SOME GENERALISED FIXED POINT THEOREMS IN A PARTIALLY ORDERED SPACE ENDOWED WITH TWO METRICS

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Abstract. The purpose of this paper is to establish fixed point results for a single mapping in a partially ordered space, and to prove a common fixed point theorem for two self-maps satisfying some weak contractive inequalities. We introduce an application to illustrate the usability of our results.

Keywords: Banach contraction; fixed point; common fixed point; partially ordered space.

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1. Introduction and preliminaries

The fixed point theorem, on metric space, most cited in literature is Banach contraction mapping principle (see [5]), which asserts that if T is a self contractive mapping on complete metric space X then T has a unique fixed point. Mizoguchi and Takahashi (see [12]) generalise the contraction principle by the following theorem:

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Theorem 1.1. Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping satisfying:

$$d(Tx,Ty) \le \alpha(d(x,y))\, d(x,y) \qquad \forall (x,y) \in X^2$$

where $\alpha : [0,+\infty[\to [0,1[$ is a function such that $\lim\sup_{t \to r^+} \alpha(t) < 1$, for all $r \ge 0$. Then T has a unique fixed point.

The following theorem established by EL.Marhrani and K.Chaira in [4] is a generalisation of the above result to space with two metrics.

Theorem 1.2. Let X be a nonempty set, d and δ two metrics of X, and $T : X \to X$ a mapping such that:

(1) (X,d,δ) is an (M)-space.
(2) For all $x,y \in X$, one of the following conditions
 (i) : $d(x,Ty) \le \delta(x,y)$
 (ii) : $\delta(x,Ty) \le d(x,y)$
 implies

$$\begin{cases} d(Tx,Ty) \le \alpha(\delta(x,y))\, \delta(x,y) \\ \delta(Tx,Ty) \le \alpha(d(x,y))\, d(x,y) \end{cases}$$

Then T has a unique fixed point in X.

In recent times, There has been a rapid development of fixed point theory in partially ordered metric spaces (see A.Bege [2], A.C.M.Ran and M.C.Reurings [3], S.Carl and S.Heikkila [9], A.Abkar and B.S.Choudhury [1]). In parallel, some generalizations of the Banach contraction fixed point theorem in a space with two metrics were proved (see EL.Marhrani and K.Chaira [4]). In this work, we introduce a partial order in a space with two metrics and generalise the above theorem. But before stating our main results let us give some basic definitions:

Definition 1.3. [11]

- A partial order (or just an order) on a nonempty set X is a binary relation " $\preceq$ " on X that is reflexive, antisymmetric and transitive. The pair $(X,\preceq)$ is called a partially ordered set or poset.
- If $x \preceq y$ or $y \preceq x$ then x and y are said to be comparable

- A mapping $T : X \to X$ is said to be nondecreasing, monotone or order preserving if $Tx \preceq Ty$ whenever $x \preceq y$.

We say that a partially ordered metric space X verifies the property (P), if for every nondecreasing sequence $(x_n)_{n\in\mathbb{N}}$ in X: $(x_n)_{n\in\mathbb{N}}$ converges to x in X, implies that $x_n \preceq x$ for all $n \in \mathbb{N}$.

Definition 1.4. Let (X,d) be a metric space. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is a Cauchy sequence provided that for every $\varepsilon > 0$, there is a natural number N such that for all $n,m \geq N$, we have $d(x_n,x_m) \leq \varepsilon$.

Definition 1.5. [4] (X,d,δ) is called an (M)-space if for every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in the metric spaces (X,d) and (X,δ), there exist $x^*,y^* \in X$ such that,

$$\lim_{n\to+\infty} d(x_n,x^*) = \lim_{n\to+\infty} \delta(x_n,y^*) = 0.$$

2. Main results

Consider a function $\alpha : [0,+\infty[\to [0,1[$ such that for all $r \geq 0$, $\lim_{t\to r^+} \sup \alpha(t) < 1$.

Theorem 2.1. Let $(X,\preceq)$ be a nonempty poset endowed with two metrics d and δ and let $T : X \to X$ be an order preserving mapping such that:

(1) (X,d,δ) is an (M)-space.

(2) For all comparable elements x and y in X, one of the following assertions

(i) $d(x,Ty) \leq \delta(x,y)$

(ii) $\delta(x,Ty) \leq d(x,y)$

implies

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))\,\delta(x,y), \\ \delta(Tx,Ty) \leq \alpha(d(x,y))\,d(x,y). \end{cases}$$

If there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$ and X verifies the property (P) for d and δ, then T admits at least a fixed point in X.

Proof. We divide this proof into two steps.

Step.1. Let x_0 be the element whose existence is assumed in the above theorem and defining a sequence $(x_n)_n$ in X by $x_n = T^n x_0$ for each $n \in \mathbb{N}$.

Since T is order preserving, we have:

$$x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots$$

Since $d(x_{n+1},Tx_n) = d(x_{n+1},x_{n+1}) = 0 \leq \delta(x_{n+1},x_n)$ and $x_n \preceq x_{n+1}$, then

$$\begin{cases} d(Tx_n,Tx_{n+1}) \leq \alpha(\delta(x_n,x_{n+1}))\,\delta(x_n,x_{n+1}) \\ \delta(Tx_n,Tx_{n+1}) \leq \alpha(d(x_n,x_{n+1}))\,d(x_n,x_{n+1}) \end{cases}$$

So,

$$\begin{cases} d(x_{n+1},x_{n+2}) \leq \alpha(\delta(x_n,x_{n+1}))\,\delta(x_n,x_{n+1}) \\ \delta(x_{n+1},x_{n+2}) \leq \alpha(d(x_n,x_{n+1}))\,d(x_n,x_{n+1}) \end{cases}$$

Since $0 \leq \alpha(\delta(x_n,x_{n+1})) < 1$ and $0 \leq \alpha(d(x_n,x_{n+1})) < 1$, then

$$\begin{cases} d(x_{n+1},x_{n+2}) \leq \alpha(\delta(x_n,x_{n+1}))\alpha(d(x_{n-1},x_n))\,d(x_{n-1},x_n) \leq d(x_{n-1},x_n) \\ \delta(x_{n+1},x_{n+2}) \leq \alpha(d(x_n,x_{n+1}))\alpha(\delta(x_{n-1},x_n))\,\delta(x_{n-1},x_n) \leq \delta(x_{n-1},x_n) \end{cases}$$

Then, the sequences $(d(x_{2p},x_{2p+1}))_p$, $(d(x_{2p+1},x_{2p+2}))_p$,$(\delta(x_{2p},x_{2p+1}))_p$ and $(\delta(x_{2p+1},x_{2p+2}))_p$ are decreasing and bounded below. So, they converge, respectively, to l_1, l_2, l_3 and l_4.

Since $\lim_{t\to l_2^+} \sup \alpha(t) < 1$ and $\lim_{t\to l_3^+} \sup \alpha(t) < 1$, there exist $k_1 \in [0,1[$ and an integer $p_1 \in \mathbb{N}$ such that for all $p \geq p_1$,

$$d(x_{2p+1},x_{2p+2}) \leq k_1\, d(x_{2p-1},x_{2p})$$

Since $\lim_{t\to l_1^+} \sup \alpha(t) < 1$ and $\lim_{t\to l_4^+} \sup \alpha(t) < 1$, there exist $k_2 \in [0,1[$ and an integer $p_2 \in \mathbb{N}$ such that for all $p \geq p_2$,

$$d(x_{2p+2},x_{2p+3}) \leq k_2\, d(x_{2p},x_{2p+1})$$

It follows that the series $\sum_{p\geq 0} d(x_{2p},x_{2p+1})$ and $\sum_{p\geq 1} d(x_{2p-1},x_{2p})$ converge, then, the series $\sum_{n\geq 0} d(x_n,x_{n+1})$ converges, and by the same arguments, we show that the series $\sum_{n\geq 0} \delta(x_n,x_{n+1})$ converges also. Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for d and δ in the (M)-space X. So, there exist x^* and y^* in X such that $\lim_{n\to+\infty} d(x_n,x^*) = 0$ and $\lim_{n\to+\infty} \delta(x_n,y^*) = 0$

Step.2. Let us prove that x^* and y^* are fixed points of T.

Consider the sets A and B defined by

$$A = \{n \in \mathbb{N}\ \ /d(x^*,Tx_n) \leq \delta(x^*,x_n)\}$$

and

$$B = \{n \in \mathbb{N} \ \ / \delta(x^*, Tx_n) \leq d(x^*, x_n)\}$$

If we suppose that A and B are finite, there exists a finite integer $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\begin{cases} d(Tx_n, x^*) > \delta(x_n, x^*) \\ \delta(Tx_n, x^*) > d(x_n, x^*), \end{cases}$$

which implies that $d(x_{n+2}, x^*) > d(x_n, x^*)$, for all $n \geq N$.

Thereby, $(d(x_{2n}, x^*))_{n \geq N}$ is an increasing nonnegative sequence, which contradicts the fact that $\lim_n d(x_n, x^*) = 0$. Hence, A or B is infinite.

Then, there exists a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ such that

$$d(Tx_{\varphi(n)}, x^*) \leq \delta(x_{\varphi(n)}, x^*) \qquad \forall n \in \mathbb{N}$$

or

$$\delta(Tx_{\varphi(n)}, x^*) \leq d(x_{\varphi(n)}, x^*) \qquad \forall n \in \mathbb{N}$$

Since the sequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ is increasing and convergent by d to x^*, it follows, for every $n \in \mathbb{N}$, that $x_{\varphi(n)} \preceq x^*$ and

$$\begin{cases} d(x_{\varphi(n)+1}, Tx^*) \leq \alpha(\delta(x_{\varphi(n)}, x^*))\, \delta(x_{\varphi(n)}, x^*) \\ \delta(x_{\varphi(n)+1}, Tx^*) \leq \alpha(d(x_{\varphi(n)}, x^*))\, d(x_{\varphi(n)}, x^*) \end{cases}$$

And by passing to the limit we can assert the existence of $k \in [0, 1[$ such that,

$$\begin{cases} d(x^*, Tx^*) \leq k\, \delta(y^*, x^*) \\ \delta(y^*, Tx^*) = 0 \end{cases} \tag{1}$$

Then, $Tx^* = y^*$, and if we replace in (1), we obtain

$$d(x^*, Tx^*) \leq k\, \delta(Tx^*, x^*)$$

Consider the sets C and D defined by

$$C = \{n \in \mathbb{N} \ \ / d(y^*, Tx_n) \leq \delta(y^*, x_n)\}$$

and

$$D = \{n \in \mathbb{N} \ \ / \delta(y^*, Tx_n) \leq d(y^*, x_n)\}$$

By the same arguments as above, we can assume that C or D is infinite. So, there exists a subsequence $(x_{\psi(n)})_{n\in\mathbb{N}}$ such that

$$d(Tx_{\psi(n)},y^*) \leq \delta(x_{\psi(n)},y^*) \qquad \forall n \in \mathbb{N}$$

or

$$\delta(Tx_{\psi(n)},y^*) \leq d(x_{\psi(n)},y^*) \qquad \forall n \in \mathbb{N}$$

Since the sequence $(x_{\psi(n)})_{n\in\mathbb{N}}$ is increasing and convergent by δ to y^*, it follows, for every $n \in \mathbb{N}$, that $x_{\psi(n)} \preceq y^*$ and

$$\begin{cases} d(x_{\psi(n)+1},Ty^*) \leq \alpha(\delta(x_{\psi(n)},y^*))\,\delta(x_{\psi(n)},y^*) \\ \delta(x_{\psi(n)+1},Ty^*) \leq \alpha(d(x_{\psi(n)},y^*))\,d(x_{\psi(n)},y^*) \end{cases}$$

And by passing to the limit we can assert the existence of $k^{'} \in [0,1[$ such that:

$$\begin{cases} d(x^*,Ty^*) = 0 \\ \delta(y^*,Ty^*) \leq k^{'}\,d(x^*,y^*) \end{cases} \qquad (2)$$

Then, $Ty^* = x^*$, and if we replace in (2) we obtain

$$\delta(y^*,Ty^*) \leq k^{'}\,d(x^*,Tx^*)$$

In the end, we have

$$\begin{cases} d(x^*,Tx^*) \leq k\,\delta(Tx^*,x^*) \\ \delta(Tx^*,x^*) \leq k^{'}\,d(x^*,Tx^*) \end{cases}$$

Thus,

$$d(x^*,Tx^*) \leq kk^{'}\,d(x^*,Tx^*)$$

which implies that $Tx^* = x^*$. Thereby, $Ty^* = y^*$ and $x^* = y^*$.

Theorem 2.2. With the same conditions of the theorem 2.1 and if we assume that any pair $\{x,y\} \subseteq X$ admits an upper bound or a lower bound in X, then T admits a unique fixed point in X.

Proof. Let x and y be two fixed points of T in X, and let z be an upper bound for the pair $\{x,y\}$. Then, $x \preceq z$ and $y \preceq z$, and since T is order preserving, we have for all $n \in \mathbb{N}$,

$$\begin{cases} x \preceq T^n z = z_n \\ y \preceq T^n z = z_n \end{cases}$$

For every $n \in \mathbb{N}$, one of the two following cases is verified

(i) $d(z_n, Tx) \leq \delta(z_n, x)$

(ii) $\delta(z_n, Tx) \leq d(z_n, x)$

Then, for every $n \in \mathbb{N}$,

$$\begin{cases} d(x, z_{n+1}) \leq \alpha(\delta(x, z_n))\, \delta(x, z_n) \\ \delta(x, z_{n+1}) \leq \alpha(d(x, z_n))\, d(x, z_n) \end{cases}$$

thus, for every $n \in \mathbb{N}$,

$$\begin{cases} d(x, z_{n+1}) \leq \alpha(\delta(x, z_n))\, \alpha(d(x, z_{n-1}))\, d(x, z_{n-1}) \\ \delta(x, z_{n+1}) \leq \alpha(d(x, z_n))\, \alpha(\delta(x, z_{n-1}))\, \delta(x, z_{n-1}) \end{cases}$$

which implies that the sequences $(d(x, z_{2n+1}))_{n\in\mathbb{N}}$, $(d(x, z_{2n}))_{n\in\mathbb{N}}$, $(\delta(x, z_{2n+1}))_{n\in\mathbb{N}}$ and $(\delta(x, z_{2n}))_{n\in\mathbb{N}}$ converge respectively to d_1, d_2, δ_1 and δ_2

Then, there exist $r_1, r_2 \in [0,1[$ and a rank $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\alpha(d(x, z_{2n-1})) \leq r_1 \quad \text{and} \quad \alpha(\delta(x, z_{2n})) \leq r_2$$

and then, for all $n \geq N$,

$$d(x, z_{2n+1}) \leq r_1 r_2\, d(x, z_{2n-1})$$

Hence, $\lim_n d(x, z_{2n+1}) = 0$. Analogously, we prove that $\lim_n d(y, z_{2n+1}) = 0$.

By acting the limit on the triangular inequality:

$$d(x,y) \leq d(x, z_{2n+1}) + d(y, z_{2n+1})$$

we obtain $d(x,y) = 0$ and so, $x = y$.

If z is a lower bound for the pair $\{x,y\}$, we copy exactly the above proof.

Corollary 2.3. Let (X,d) be a complete metric space endowed with a partial order " $\preceq$ " such that every pair has an upper bound or a lower bound and let $T : X \to X$ be an order preserving mapping such that for all comparable elements x and y in X,

$$d(x,Ty) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq \alpha(d(x,y))\, d(x,y)$$

If there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$ and X verifies the property (P), then T admits a unique fixed point in X.

Now, using two self-mappings on an (M)-space, we obtain the following:

Theorem 2.4. Let X be a nonempty set endowed with a partial order " $\preceq$ " and two metrics d and δ and let $T,S : X \to X$ be two self-mappings such that:

(1) (X,d,δ) is an (M)-space.

(2) For all comparable elements x and y in X, one of the following assertions

 (i) $d(x,Sy) \leq \delta(x,y)$

 (ii) $\delta(y,Tx) \leq d(x,y)$

 implies

$$\begin{cases} d(Tx,Sy) \leq \alpha(\delta(x,y)) \max\{d(x,y), \delta(x,Tx), d(y,Sy)\} \\ \delta(Tx,Sy) \leq \alpha(d(x,y)) \max\{\delta(x,y), d(x,Tx), \delta(y,Sy)\}. \end{cases}$$

If there exists an element $x_0 \in X$ such that

$$x_0 \preceq Sx_0 \preceq TSx_0 \preceq STSx_0 \preceq (TS)^2x_0 \preceq S(TS)^2x_0 \preceq (TS)^3x_0 \preceq \ldots$$

and X verifies the property (P), then S and T have a common fixed point.

Proof. We divide our proof on three steps.

Step.1. Let x_0 be an element of X whose the existence is assured by the conditions of the theorem and let us define the sequence $(x_n)_{n\in\mathbb{N}}$ as follows

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}.$$

We have, for all $n \in \mathbb{N}$,

$$x_{2n} \preceq x_{2n+1} \preceq x_{2n+2}$$

Since $d(x_{2n+1},Sx_{2n}) = 0 \leq \delta(x_{2n+1},x_{2n})$ and $x_{2n} \preceq x_{2n+1}$, then

$$\begin{aligned} d(x_{2n+2},x_{2n+1}) &= d(Tx_{2n+1},Sx_{2n}) \\ &\leq \alpha(\delta(x_{2n+1},x_{2n})) \max\{d(x_{2n+1},x_{2n}), \delta(x_{2n+1},Tx_{2n+1}), d(x_{2n},Sx_{2n})\} \end{aligned}$$

and

$$\begin{aligned}\delta(x_{2n+2},x_{2n+1}) &= \delta(Tx_{2n+1},Sx_{2n})\\ &\leq \alpha(d(x_{2n+1},x_{2n}))\max\{\delta(x_{2n+1},x_{2n}),d(x_{2n+1},Tx_{2n+1}),\delta(x_{2n},Sx_{2n})\}\end{aligned}$$

Thus,

$$\begin{cases} d(x_{2n+2},x_{2n+1}) \leq \alpha(\delta(x_{2n+1},x_{2n}))\max\{d(x_{2n+1},x_{2n}),\delta(x_{2n+1},x_{2n})\}\\ \delta(x_{2n+2},x_{2n+1}) \leq \alpha(d(x_{2n+1},x_{2n}))\max\{\delta(x_{2n+1},x_{2n}),d(x_{2n+1},x_{2n})\}\end{cases}$$

Since $\delta(x_{2n},Tx_{2n-1}) = 0 \leq d(x_{2n},x_{2n-1})$ and $x_{2n-1} \preceq x_{2n}$, then

$$\begin{aligned}d(x_{2n},x_{2n+1}) &= d(Tx_{2n-1},Sx_{2n})\\ &\leq \alpha(\delta(x_{2n-1},x_{2n}))\max\{d(x_{2n-1},x_{2n}),\delta(x_{2n-1},Tx_{2n-1}),d(x_{2n},Sx_{2n})\}\end{aligned}$$

and

$$\begin{aligned}\delta(x_{2n},x_{2n+1}) &= \delta(Tx_{2n-1},Sx_{2n})\\ &\leq \alpha(d(x_{2n-1},x_{2n}))\max\{\delta(x_{2n-1},x_{2n}),d(x_{2n-1},Tx_{2n-1}),\delta(x_{2n},Sx_{2n})\}\end{aligned}$$

Thus,

$$\begin{cases} d(x_{2n},x_{2n+1}) \leq \alpha(\delta(x_{2n-1},x_{2n}))\max\{d(x_{2n-1},x_{2n}),\delta(x_{2n-1},x_{2n})\}\\ \delta(x_{2n},x_{2n+1}) \leq \alpha(d(x_{2n-1},x_{2n}))\max\{\delta(x_{2n-1},x_{2n}),d(x_{2n-1},x_{2n})\}\end{cases}$$

If we put, for every $n \in \mathbb{N}$,

$$u_n = \max\{d(x_{n+1},x_n),\delta(x_{n+1},x_n)\}$$

and

$$\alpha_n = \max\{\alpha(d(x_{n+1},x_n)),\alpha(\delta(x_{n+1},x_n))\}$$

we obtain for every $n \in \mathbb{N}$,

$$u_{n+1} \leq \alpha_n\, u_n$$

Thereby, the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing and bounded below and accordingly it converges to some $l \geq 0$

Therefore, the two sequences $(d(x_{n+1},x_n))_n$ and $(\delta(x_{n+1},x_n))_n$ are bounded and by Weierstrass, there exists an increasing mapping $\varphi:\mathbb{N}\to\mathbb{N}$ such that $(d(x_{\varphi(n)+1},x_{\varphi(n)}))_n$ converges to some $l_d \geq 0$ and $(\delta(x_{\varphi(n)+1},x_{\varphi(n)}))_n$ converges to some $l_\delta \geq 0$

Since $\lim_{t\to l_d^+}\sup \alpha(t) < 1$ and $\lim_{t\to l_\delta^+}\sup \alpha(t) < 1$, there exist $r \in [0,1[$ and a positive integer N such that

$$u_{\varphi(n)+1} \leq r\, u_{\varphi(n)}, \qquad \text{for all } n \geq N$$

By passing to the limit, we have $l = 0$. And so,

$$\lim_{n\to+\infty} d(x_{n+1},x_n) = \lim_{n\to+\infty} \delta(x_{n+1},x_n) = 0$$

Knowing that $\lim_{t\to 0^+}\sup \alpha(t) < 1$, we can assume the existence of $k \in [0,1[$ and a positive integer N' such that

$$u_{n+1} \leq k \times u_n, \qquad \text{for all } n \geq N'$$

Thus, the series $\sum u_n$ converges. Thereby, the series $\sum d(x_{n+1},x_n)$ and $\sum \delta(x_{n+1},x_n)$ converge, which implies that $(x_n)_n$ is a Cauchy sequence for d and δ and then, there exist $x^*, y^* \in X$ such that

$$\lim_{n\to+\infty} d(x_n,x^*) = \lim_{n\to+\infty} \delta(x_n,y^*) = 0$$

Step.2. Let us prove that $x^* = y^*$

Suppose that $x^* \neq y^*$ and consider the set

$$A = \{n \in \mathbb{N} \ \ / \delta(y^*, Tx_{2n+1}) \leq d(y^*, x_{2n+1})\}$$

There are two cases to distinguish.

Case.1. A is finite.

There exists a positive integer p such that

$$\delta(y^*, x_{2n+2}) > d(y^*, x_{2n+1}), \text{ for every } n \geq p$$

and by passing to the limit, we obtain $0 \geq d(x^*, y^*)$, which is a contradiction.

Case.2. A is infinite.

There exists an increasing mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\delta(y^*, Tx_{2\sigma(n)+1}) \leq d(y^*, x_{2\sigma(n)+1})$$

and since $x_{2\sigma(n)+1} \preceq y^*$, then

$$d(Tx_{2\sigma(n)+1}, Sy^*) \leq \alpha(\delta(x_{2\sigma(n)+1}, y^*)) \max\{d(x_{2\sigma(n)+1}, y^*), \delta(x_{2\sigma(n)+1}, Tx_{2\sigma(n)+1}), d(y^*, Sy^*)\}$$

and

$$\delta(Tx_{2\sigma(n)+1},Sy^*) \leq \alpha(d(x_{2\sigma(n)+1},y^*)) \max\{\delta(x_{2\sigma(n)+1},y^*),d(x_{2\sigma(n)+1},Tx_{2\sigma(n)+1}),\delta(y^*,Sy^*)\}$$

It follows that there exist $k_1,k_2 \in [0,1[$ such that

$$\begin{cases} d(x^*,Sy^*) \leq k_1 \max\{d(x^*,y^*),d(y^*,Sy^*)\} \\ \delta(y^*,Sy^*) \leq k_2\,\delta(y^*,Sy^*) \end{cases}$$

and so, $Sy^* = y^*$ and $d(x^*,y^*) \leq k_1\,d(x^*,y^*)$, which is a contradiction too. Hence $x^* = y^*$

Step.3. Let us prove that $Sx^* = Tx^* = x^*$

Consider the two sets :

$$\begin{cases} A = \{n \in \mathbb{N} \ \ /\delta(x^*,Tx_{2n+1}) \leq d(x^*,x_{2n+1})\} \\ B = \{n \in \mathbb{N} \ \ /d(x^*,Sx_{2n}) \leq \delta(x^*,x_{2n})\} \end{cases}$$

We can assert that A or B is infinite.

If A and B are finite, there exists a positive integer q such that, for all $n \geq q$,

$$d(x^*,x_{2n+1}) > \delta(x^*,x_{2n}) > d(x^*,x_{2n-1}).$$

thus, the sequence $(d(x^*,x_{2n+1}))_{n\geq N}$ is strictly increasing to 0, which is a false assertion.

If we assume that A is infinite, then, as the above, there exists $k_2 \in [0,1[$ such that

$$\delta(x^*,Sx^*) \leq k_1\,\delta(x^*,Sx^*)$$

Then $Sx^* = x^*$

If we assume that B is infinite, we obtain, by the same way, $Tx^* = x^*$. Then x^* is a common fixed point for T and S.

One can remark that $\mathfrak{F}_T = \mathfrak{F}_S$, when $\mathfrak{F}_T$ is the set of fixed points of T and $\mathfrak{F}_S$ is the set of fixed points of S. Indeed, If $x \in \mathfrak{F}_T$ then, $d(x,Tx) \leq \delta(x,x)$ which implies that

$$d(x,Sx) \leq \alpha(0)d(x,Sx)$$

And Since $0 \leq \alpha(0) < 1$, thus $d(x,Sx) = 0$. So $x \in \mathfrak{F}_S$.

If $x \in \mathfrak{F}_S$ then, $\delta(x,Sx) \leq d(x,x)$ which implies that $d(Tx,x) \leq \alpha(0)d(Tx,x)$. Then $d(Tx,x) = 0$ and so $x \in \mathfrak{F}_T$. Hence we have the equality.

Corollary 2.5. Let (X,d,δ) be an (M)-space endowed with a partial order "$\preceq$" such that every pair has an upper bound, and let $T,S : X \longrightarrow X$ be two self-mappings such that for all comparable elements x and y in X,

$$\begin{cases} d(Tx,Sy) \leq \alpha(\delta(x,y)) \max\{d(x,y),\delta(x,Tx),d(y,Sy)\} \\ \delta(Tx,Sy) \leq \alpha(d(x,y)) \max\{\delta(x,y),d(x,Tx),\delta(y,Sy)\}. \end{cases}$$

If, for every $x \in X$, $x \preceq Sx$ and $x \preceq Tx$ and X verifies the property (P), then S and T have a unique common fixed.

Proof. 1. The existence: One can see that for all $x \in X$,

$$x \preceq Sx \preceq TSx \preceq STSx \preceq (TS)^2x \preceq S(TS)^2x \preceq (TS)^3x \preceq S(TS)^3x \preceq \ldots$$

Then, accordingly to the theorem 2.4, T and S have a common fixed point in X.

2. The uniqueness: Let x and y be two common fixed points of T and S, let z be an upper bound for the pair $\{x,y\}$ and let us define the sequence $(z_n)_{n\in\mathbb{N}}$ as follows:

$$z_0 = z \text{ and for every } n \in \mathbb{N},\ z_{2n+1} = Sz_{2n} \text{ and } z_{2n+2} = Tz_{2n+1}$$

then,

$$z_{2n} \preceq z_{2n+1} \preceq z_{2n+2}, \text{ for every } n \in \mathbb{N}$$

As we have seen in the previous proof, $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then, there exist $z_d, z_\delta \in X$ such that,

$$\lim_{n\to+\infty} d(z_n,z_d) = \lim_{n\to+\infty} \delta(z_n,z_\delta) = 0$$

One can see that, for every $n \in \mathbb{N}$, $x \preceq z \preceq z_{2n}$, then, for every $n \geq N$,

$$d(x,z_{2n+1}) \leq \alpha(\delta(x,z_{2n})) \max\{d(x,z_{2n}),d(z_{2n},z_{2n+1})\}$$

Since $\lim\limits_{n\to+\infty} d(x,z_{2n}) = d(x,z_d) = d_1$ and $\limsup\limits_{t\to d_1^+} \alpha(t) < 1$, there exist $k \in [0,1[$ and $p \in \mathbb{N}$ such that, for every $n \geq p$,

$$d(x,z_{2n+1}) \leq k \max\{d(x,z_{2n}),d(z_{2n},z_{2n+1})\}$$

By passing to the limit we obtain $d(x,z_d) = 0$, which follows that $x = z_d$.

And by the same way, we prove that $y = z_d$. Then $x = y$.

Example 2.6. Consider the space $X=[0,1]$ ordered by " $\preceq$ " which is the reverse order of the usual order between the reals ($x\preceq y \Leftrightarrow x\geq y$) and endowed with two distances d and δ defined as follows:

$$d(x,y)=|x-y|$$

and

$$\delta(x,y)=\begin{cases} x+y & si \quad x\neq y \\ 0 & si \quad x=y \end{cases}$$

Consider the function $\alpha : t\longmapsto \frac{3}{4}+\frac{1}{8}e^{-t}$ and the two self-mappings:

$$T:x\longmapsto T(x)=\tfrac{x}{4} \text{ and } S:x\longmapsto S(x)=\tfrac{x}{2}$$

Denote:

$$(S)\begin{cases} d(Tx,Sy)\leq \alpha(\delta(x,y))\max\{d(x,y),\delta(x,Tx),d(y,Sy)\} \\ \delta(Tx,Sy)\leq \alpha(d(x,y))\max\{\delta(x,y),d(x,Tx),\delta(y,Sy)\} \end{cases}$$

Let x and y be two elements in $[0,1]$. There are four cases to distinguish:

Case.1. $x=2y$. Then (S) is obviously verified.

Case.2. $x\prec y$, i.e., $x>y$, and $x\neq 2y$. Then

$$(S)\Leftrightarrow\begin{cases} |\frac{x}{4}-\frac{y}{2}|\leq \alpha(x+y))\max\{x-y,\frac{5x}{4}\} & (1) \\ \frac{x}{4}+\frac{y}{2}\leq \alpha(x-y)\max\{x+y,\frac{3x}{4},\frac{3y}{2}\} & (2) \end{cases}$$

If we set $t=\frac{y}{x}$ we will have $0\leq t<1$ and,

$$(1)\Leftrightarrow |1-2t|\leq 4\alpha(x+y)(1-t) \text{ ou } |1-2t|\leq 5\alpha(x+y)$$

which is verified. And

$$(2)\Leftrightarrow 2t+1\leq 4\alpha(x-y)(1+t) \text{ ou } 2t+1\leq 3\alpha(x-y) \text{ ou } 2t+1\leq 6\alpha(x-y)t$$

which is also verified. So the system (S) is verified.

Case.3. $y\prec x$, i.e., $x<y$. The system (S) is equivalent to

$$\begin{cases} |\frac{x}{4}-\frac{y}{2}|)\leq \alpha(x+y))\max\{y-x,\frac{5x}{4},\frac{y}{2}\} & (3) \\ \frac{x}{4}+\frac{y}{2}\leq \alpha(x-y)\max\{x+y,\frac{3y}{2}\} & (4) \end{cases}$$

If we set $t=\frac{x}{y}$, $\beta=\alpha(x+y)$ and $\gamma=\alpha(x-y)$ then,

$$(3) \Leftrightarrow (4\beta-1)t-4\beta+2 \leq 0 \text{ ou } 2-t \leq 5\beta t \text{ ou } 2-2\beta \leq t$$

If $t < 2-2\beta$ then $(4\beta-1)t-4\beta+2 \leq 8\beta(\frac{3}{4}-\beta) \leq 0$.

Hence (3) is true.

$$(4) \Leftrightarrow 2+t \leq 4\gamma(t+1) \text{ ou } 2+t \leq 6\gamma$$

which is also true. Thus, the system (S) is verified.

Case.4. $x=y$. The system (S) is equivalent to

$$\begin{cases} x \leq 5x\alpha(0) \\ x \leq 2x\alpha(0) \end{cases}$$

which is true.

In all cases the system (S) is verified, and one can see that:

- (X,d,δ) is an (M)-space.
- X verifies the property (P).
- for every $x \in X$, we have $x \preceq Tx$ and $x \preceq Sx$.

Then, the assertions of the above corollary are verified and the mappings T and S have a unique common fixed point which is 0.

If we assume that $T=S$, we obtain the following result:

Corollary 2.7. Let (X,d,δ) be an (M)-space endowed with a partial order " $\preceq$ " such that every pair has an upper bound, and let $T: X \longrightarrow X$ be a self-mapping such that for all comparable elements x and y in X, one of the following assertions:

(i) $d(x,Ty) \leq \delta(x,y)$

(ii) $\delta(y,Tx) \leq d(x,y)$

implies the system:

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y)) \max\{d(x,y),\delta(x,Tx),d(y,Ty)\} \\ \delta(Tx,Ty) \leq \alpha(d(x,y)) \max\{\delta(x,y),d(x,Tx),\delta(y,Ty)\} \end{cases}$$

If for every element $x \in X$, $x \preceq Tx$ and X verifies the property (P), then T admits a unique fixed point in X.

Proof. 1. The existence: Since for every $x \in X$,

$$x \preceq Tx \preceq T^2x \preceq \ldots \preceq T^n x \preceq T^{n+1}x \preceq \ldots$$

then, accordingly to the theorem 2.4, T admits a fixed point in X.

2. The uniqueness: Let x and y be two common fixed points of T, let z be an upper bound for the pair $\{x, y\}$ and let us define the sequence $(z_n)_{n\in\mathbb{N}}$ as follows:

$$z_0 = z \text{ and for every } n \in \mathbb{N},\ z_{n+1} = Tz_{2n}$$

As we have seen in the previous proof, $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exist $z_d, z_\delta \in X$ such that,

$$\lim_{n\to+\infty} d(z_n, z_d) = \lim_{n\to+\infty} \delta(z_n, z_\delta) = 0$$

One can see that, for every $n \in \mathbb{N}$, $x \preceq z \preceq z_n$ and $y \preceq z \preceq z_n$.

Consider the sets

$$F = \{n \in \mathbb{N} \ \ / d(z_n, x) \leq \delta(z_n, x)\}$$

and

$$G = \{n \in \mathbb{N} \ \ / \delta(z_n, x) \leq d(z_n, x)\}$$

If we suppose that F and G are finite, there exists a positive integer N such that

$$d(z_n, x) > \delta(z_n, x) > d(z_n, x), \text{ for all } n \geq N$$

which is absurd. Then F or G is infinite. So, there exist an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$d(z_{\varphi(n)+1}, x) \leq \alpha(\delta(z_{\varphi(n)}, x)) \max\{d(z_{\varphi(n)}, x), \delta(z_{\varphi(n)}, z_{\varphi(n)+1})\},$$

or for every $n \in \mathbb{N}$,

$$d(x, z_{\varphi(n)+1}) \leq \alpha(\delta(x, z_{\varphi(n)})) \max\{d(x, z_{\varphi(n)}), d(z_{\varphi(n)}, z_{\varphi(n)+1})\}$$

By passing to the limit in the two cases, we can assert the existence of a real k in $[0,1[$ such that, $d(x, z_d) \leq k\, d(x, z_d)$, which implies that $x = z_d$.

And by the same way, we obtain $y = z_d$. Then $x = y$.

Remark 2.8. An alternative of the above result is obtained if we assume that:

- every pair $\{x, y\} \subseteq X$ admits a lower bound in X.
- for all $x \in X$, $Tx \preceq x$

- for every decreasing sequence $(x_n)_{n\in\mathbb{N}}$, if it converges to z either by d or by δ, then $z \preceq x_n$ for all $n \in \mathbb{N}$

Example 2.9. Consider the space $X = [0,1]$ ordered by " $\preceq$ " which is the reverse order of the usual order between the reals ($x \preceq y \Leftrightarrow x \geq y$) and endowed with two distances d and δ defined as follows:

$$d(x,y) = |x-y| \text{ and } \delta(x,y) = 2|x-y|$$

Let us consider the self-mapping

$$T : x \longmapsto T(x) = \begin{cases} \frac{x}{8} & si \quad x \in [0,1[\\ 0 & si \quad x = 1 \end{cases}$$

and $\alpha(t) = \frac{2}{15} + \frac{1}{106}e^{-t}$.

Denote $\begin{cases} (i) \;\; d(x,Ty) \leq \delta(x,y) \\ (ii) \;\; \delta(y,Tx) \leq d(x,y) \end{cases}$ and

$$(S)\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y)) \max\{d(x,y), \delta(x,Tx), d(y,Ty)\} \\ \delta(Tx,Ty) \leq \alpha(d(x,y)) \max\{\delta(x,y), d(x,Tx), \delta(y,Ty)\} \end{cases}$$

Let x and y two elements in $[0,1]$. There are three cases to distinguish:

Case.1. If $x,y \in [0,1[$ then, the system (S) is equivalent to

$$(S) \Leftrightarrow \begin{cases} \frac{1}{8}|x-y| \leq \alpha(2|x-y|) \max\{|x-y|, 2|x-\frac{x}{8}|, |y-\frac{y}{8}|\} \\ \frac{2}{8}|x-y| \leq \alpha(|x-y|) \max\{2|x-y|, |x-\frac{x}{8}|, 2|y-\frac{y}{8}|\} \end{cases}$$

which is always true since $\alpha(t) \geq \frac{1}{8}$, for all $t \in \mathbb{R}^+$.

Case.2. If $x \in [0,1[$ and $y = 1$ then

$$((i) \; or \; (ii)) \Leftrightarrow x \in [0, \frac{2}{3}]$$

And for all $x \in [0, \frac{2}{3}]$ the system (S) is equivalent to

$$\begin{cases} \left(\frac{x}{8} \leq \alpha(2(1-x))\frac{14}{8}x \quad or \quad \frac{x}{8} \leq \alpha(2(1-x))\right) \\ \frac{x}{8} \leq \alpha(1-x) \end{cases}$$

which is true.

Case.3. If $y \in [0,1[$ and $x=1$, the system (S) becomes

$$\begin{cases} \frac{y}{8} \leq \alpha(2(1-y)) \max\{1-y, 2, \frac{7y}{8}\} \\ \frac{y}{4} \leq \alpha(1-y) \max\{2(1-y), 1, \frac{7y}{4}\} \end{cases}$$

and, $((i) \; or \; (ii)) \Leftrightarrow y \in [0, \frac{8}{15}] \cup [0, \frac{1}{3}] = [0, \frac{8}{15}]$

For all $y \in [0, \frac{8}{15}]$, the system (S) is equivalent to

$$\begin{cases} \frac{y}{16} \leq \alpha(2(1-y)) \\ \left(\frac{y}{8} \leq \alpha(1-y)(1-y) \quad or \quad \frac{y}{4} \leq \alpha(1-y) \quad or \quad \frac{y}{4} \leq \frac{7y\alpha(1-y)}{4}\right) \end{cases}$$

which is true. And, for every $y \in]\frac{454}{795}, 1]$, both of (i) and (ii) are false, and if we assume that

$$\left(\frac{y}{8} \leq \alpha(1-y)(1-y) \quad or \quad \frac{y}{4} \leq \alpha(1-y) \quad or \quad \frac{y}{4} \leq \frac{7y\alpha(1-y)}{4}\right)$$

then, $\left(\frac{227}{1590} \geq \alpha(1-y) > \frac{454}{2728} \quad or \quad e^{y-1} > 1\right)$, which is a contradiction.

Thus, the system (S) is false.

In all cases one of the assertions (i) or (ii) implies the system (S, and then T admits a unique fixed point in X which is 0.

If we assume, in the above theorem, that $d = \delta$ and α is a constant function, we obtain a generalisation of contraction type Kannan [6]

Theorem 2.10. Let (X,d) be a complete metric space endowed with a partial order " $\preceq$ " such that any pair $\{x,y\} \subseteq X$ admits an upper bound or a lower bound, and $T : X \longrightarrow X$ be an order preserving mapping such that, for all comparable elements x and y in X,

$$d(x,Ty) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq r\,(d(x,Tx) + d(y,Ty))$$

where $0 \leq r < \frac{1}{2}$.

If there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$ and X verifies the property (P), then T admits a unique fixed point.

Proof. 1. The existence: Since, for all $x, y \in X$,

$$d(x,Tx) + d(y,Ty) \leq 2 \max\{d(x,y), d(x,Tx), d(y,Ty)\}$$

then, for all comparable elements x and y in X,

$$d(x,Ty) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq 2r\max\{d(x,y),d(x,Tx),d(y,Ty)\}$$

Since $x_0 \preceq Tx_0$ and T is an order preserving, then

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \ldots \preceq T^nx_0 \preceq T^{n+1}x_0 \preceq \ldots$$

And by applying the theorem 2.4, T admits a fixed point in X.

2. The uniqueness: Let x and y be two fixed points of T in X, and let z be an upper bound for the pair $\{x,y\}$, then $x \preceq z$ and $y \preceq z$. Since T is order preserving, we have, for all $n \in \mathbb{N}$,

$$\begin{cases} x \preceq T^nz = z_n \\ y \preceq T^nz = z_n \end{cases}$$

Since $d(z_n,Tx) \leq d(z_n,x)$ and $x \preceq z_n$, then, for every $n \in \mathbb{N}$,

$$d(z_{n+1},x) \leq r\,d(z_n,z_{n+1}) \leq r\,d(z_n,x) + r\,d(z_{n+1},x),$$

which implies that

$$d(z_{n+1},x) \leq \tfrac{r}{1-r}d(z_n,x), \text{ for all } n \in \mathbb{N}$$

Since $0 \leq \frac{r}{1-r} < 1$, then $\lim\limits_{n} d(z_n,x) = 0$

By the same way, we can prove that $\lim\limits_{n} d(z_n,y) = 0$, and by acting the limit on the triangular inequality $d(x,y) \leq d(z_n,x) + d(z_n,y)$, we conclude that $x = y$.

If z is a lower bound for the pair $\{x,y\}$, we copy exactly the above proof.

3. Application

Consider the space $X = \{x \in \mathscr{C}^1([0,1],\mathbb{R}) \,/\, x(0) = 0\}$ endowed with two metrics d_∞ and δ_∞ defined for all $(x,y) \in X$ as follows:

$$d_\infty(x,y) = \|x-y\|_\infty = \sup_{t\in[0,1]} |x(t)-y(t)|$$

and

$$\delta_\infty(x,y) = \|x'-y'\|_\infty = \sup_{t\in[0,1]} |x'(t)-y'(t)|$$

X is partially ordered by the order defined as follows:

$$x \preceq y \Leftrightarrow x(t) \leq y(t) \qquad \forall t \in [0,1]$$

Let us consider the following **integral equations system:**

$$(IES): \begin{cases} x(t) = \int_0^1 f(t,y(s))\,\mathrm{d}s + a(t) & \forall t \in [0,1] \\ y(t) = \int_0^1 g(t,x(s))\,\mathrm{d}s + a(t) & \forall t \in [0,1] \end{cases}$$

when $a \in X$ and $f,g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two mappings such that

(i). f and g are of the class C^1 on $[0,1] \times \mathbb{R}$ and are nondecreasing with respect to the second coordinate

(ii). for every $x \in \mathbb{R}$, $f(0,x) = g(0,x) = 0$

(iii). there exists an element $x_0 \in X$ such that for all $s \in [0,1]$,

$$x_0 \preceq f(.,x_0(s)) + a \text{ and } x_0 \preceq g(.,x_0(s)) + a$$

Let us consider the two mappings T and S defined in X as follows:

$$\begin{cases} Tx(t) = \int_0^1 f(t,x(s))\,\mathrm{d}s + a(t) \\ & t \in [0,1] \\ Sx(t) = \int_0^1 g(t,x(s))\,\mathrm{d}s + a(t) \end{cases}$$

From (i) and (ii), we have for all $x \in X$, Tx and Sx are in X.

Lemma 3.1. Consider the set E of the elements $x \in X$ verifying:

$$x \preceq Tx \text{ and } x \preceq Sx$$

The space $(E, d_\infty, \delta_\infty)$ is an (M)-space.

Proof. Since $x_0 \in E$, then E is nonempty. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in E for d and δ. Since (X,d,δ) is a (M)-space (see[4]), there exist $x^*, y^* \in X$ such that

$$\lim_{n\to+\infty} d(x_n,x^*) = \lim_{n\to+\infty} \delta(x_n,y^*) = 0$$

Since the sequence $(x_n)_{n\in\mathbb{N}}$ converges uniformly to x^*, we have:

$$\lim_{n\to+\infty} \int_0^1 f(t,x_n(s))ds = \int_0^1 f(t,x^*(s))ds$$

and

$$\lim_{n\to+\infty}\int_0^1 g(t,x_n(s))ds = \int_0^1 g(t,x^*(s))ds$$

By applying the limit on the two following inequalities,

$$x_n(t) \le \int_0^1 f(t,x_n(s))\,\mathrm{d}s + a(t) \text{ and } x_n(t) \le \int_0^1 g(t,x_n(s))\,\mathrm{d}s + a(t)$$

we obtain,

$$x^*(t) \le Tx^*(t) \text{ and } x^*(t) \le Sx^*(s) \text{ for all } t \in [0,1]$$

Then, $x^* \in E$.

Since $(x'_n)_{n\in\mathbb{N}}$ converges uniformly to $(y^*)'$ and $x_n(0) = 0$ for every $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ converges uniformly to y^*. Thus, $y^* = x^* \in E$ and we conclude that E is an (M)-space.

Theorem 3.2. Consider a function $G\colon[0,1]\to[0,1]$ and a nondecreasing function $\alpha : [0,+\infty[\to [0,1[$ such that for all $r \ge 0$, $\lim_{t\to r^+}\sup \alpha(t) < 1$

If, for every $s,t \in [0,1]$ and for all comparable elements $x,y \in X$, one of the following assertions

(i) $|x(s) - Sy(s)| \le \delta_\infty(x,y)$

(ii) $|y'(s) - (Tx)'(s)| \le d_\infty(x,y)$

implies the system

$$\begin{cases} |f(t,x(s)) - g(t,y(s))| \le \alpha(|x'(s) - y'(s)|)\,G(t)\,|x(s) - y(s)| \\ |\frac{\partial f}{\partial t}(t,x(s)) - \frac{\partial g}{\partial t}(t,y(s))| \le \alpha(|x(s) - y(s)|)\,G(t)\,(|x'(s) - y'(s)|) \end{cases}$$

Then the system (IES) admits at least a solution which belongs to the diagonal of X^2.

Proof. Since for each $t \in [0,1]$, $f(t,.)$ and $g(t,.)$ are nondecreasing in $\mathbb{R}$, the mappings T and S are order preserving in X. When $x \in E$, we have $x \preceq Tx$ and $x \preceq Sx$. Then $Tx \prec T^2x$ and $Sx \prec S^2x$, which implies that $Tx \in E$ and $Sx \in E$. So T and S are two self-mappings in E.

Let x and y be two comparable elements in E. If we assume that

$$d_\infty(x,Sy) \le \delta_\infty(x,y) \text{ or } \delta_\infty(y,Tx) \le d_\infty(x,y)$$

then, for every $s \in [0,1]$, one of the following assertions is verified

$$\begin{cases} |x(s) - Sy(s)| \le \delta_\infty(x,y) \\ |y'(s) - (Tx)'(s)| \le d_\infty(x,y) \end{cases}$$

Which implies that

$$\begin{cases} |f(t,x(s)) - g(t,y(s))| \leq \alpha(|x'(s) - y'(s)|)\, G(t)\, |x(s) - y(s)| \\ |\frac{\partial f}{\partial t}(t,x(s)) - \frac{\partial g}{\partial t}(t,y(s))| \leq \alpha(|x(s) - y(s)|)\, G(t)\, (|x'(s) - y'(s)|) \end{cases}$$

And since α is nondecreasing, we have

$$\begin{cases} |f(t,x(s)) - g(t,y(s))| \leq \alpha(\delta_\infty(x,y))\, G(t)\, d_\infty(x,y) \\ |\frac{\partial f}{\partial t}(t,x(s)) - \frac{\partial g}{\partial t}(t,y(s))| \leq \alpha(d_\infty(x,y))\, G(t)\, \delta_\infty(x,y) \end{cases}$$

Since

$$\|Tx - Sy\|_\infty = \sup_{t\in[0,1]} |Tx(t) - Sy(t)| \leq \sup_{t\in[0,1]} \int_0^1 |f(t,x(s)) - g(t,y(s))|\, \mathrm{d}s$$

then,

$$d_\infty(Tx,Sy) \leq \alpha(\delta_\infty(x,y)) d_\infty(x,y)$$

And since,

$$\|(Tx)' - (Sy)'\|_\infty = \sup_{t\in[0,1]} |(Tx)'(t) - (Sy)'(t)| \leq \sup_{t\in[0,1]} \int_0^1 |\frac{\partial f}{\partial t}(t,x(s)) - \frac{\partial g}{\partial t}(t,y(s))|\, \mathrm{d}s$$

then,

$$\delta_\infty(Tx,Sy) \leq \alpha(d_\infty(x,y)) \delta_\infty(x,y)$$

Therefore, for all comparable elements $x, y \in E$, one of the following assertions

(i) $d_\infty(x,Sy) \leq \delta_\infty(x,y)$

(ii) $\delta_\infty(y,Ty) \leq d_\infty(x,y)$

implies the system

$$\begin{cases} d_\infty(Tx,Sy) \leq \alpha(\delta_\infty(x,y)) \max\{d_\infty(x,y), \delta_\infty(x,Tx), d_\infty(y,Sy)\} \\ \delta_\infty(Tx,Sy) \leq \alpha(d_\infty(x,y)) \max\{\delta_\infty(x,y), d_\infty(x,Tx), \delta_\infty(y,Sy)\} \end{cases}$$

In addition, $(E, d_\infty, \delta_\infty)$ is an (M)-space, E verifies the property (P), and

$$x_0 \preceq Sx_0 \preceq TSx_0 \preceq STSx_0 \preceq (TS)^2 x_0 \preceq S(TS)^2 x_0 \preceq \ldots$$

Then, accordingly to the theorem 2.4, T and S have a common fixed point in E, i.e., there exists an element $x^* \in E$ such that (x^*, x^*) verifies the system (IES) and so, $\int_0^1 f(t,x^*(s)\, \mathrm{d}s$=$\int_0^1 g(t,x^*(s)\, \mathrm{d}s$ for all $t \in [0,1]$.

Then, the sysetm (IES) admits at least a solution in X^2 which belongs to the diagonal of X^2.

Conflict of Interests

The authors declare that there is no conflict of interests.

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