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ERROR BOUNDS FOR RANDOM GENERALIZED MIXED VARIATIONAL INEQUALITY PROBLEMS WITH RANDOM FUZZY MAPPING

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Abstract. In this paper, we define some new notions of gap functions of random generalized mixed variational inequality problems in a fuzzy environment. Further, we compute error bounds for random generalized mixed variational inequality problems in terms of the residual gap function, the regularized gap function and the D-gap function. The results obtained are new and generalize a number of known results for generalized mixed variational inequality problems with fuzzy mappings.

Keywords: random fuzzy mapping; sub-differential; strongly g-monotone; regularized gap function; error bound.

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1. Introduction

The theory of gap function was introduced for the study of a convex optimization problem and subsequently applied to variational inequality problems. One of the classical approaches in the analysis of a variational inequality problem is to transform it into an equivalent optimization problem via the notion of a gap function. Recently, some efforts has been made to develop gap

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functions for various classes of variational inequality problems; see for example [1, 2, 3, 7, 8, 9, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23]. Besides these, gap functions also turned out to be very useful in designing new globally convergent algorithms and in analyzing the rate of convergence of some iterative methods and also in deriving the error bounds.

In 1965, Zadeh [24] introduced the fuzzy set theory. The applications of the fuzzy set theory can be found in control engineering and optimization problems of mathematical sciences. In the recent past, variational inequalities in the setting of fuzzy mappings have been introduced and studied which are closely related with fuzzy optimization problems. As a result, variational inequality problems have been generalized and extended in various directions using novel techniques of fuzzy theory.

In 1989, Chang and Zhu [5] introduced the concepts of variational inequality for fuzzy mappings. The concept of a random fuzzy mapping was first introduced by Huang [11] while studying a new class of random multi-valued nonlinear generalized variational inclusions. For some related work, we refer to [6, 10, 11]. Recently, Dai [6] introduced a new class of generalized mixed variational-like inequalities for random fuzzy mappings and established an existence theorem and an iterative algorithm for finding the solution of problems.

Throughout the paper, let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively. Let $\mathscr{F}$ be a collection of all fuzzy sets over H. A mapping $T : H \to \mathscr{F}(H)$ is called a fuzzy mapping on H. If T is a fuzzy mapping on H, then $T(x)$ (denoted by T_x) is a fuzzy set on H and $T_x(y)$ is the membership function of y in T_x. Let $A \in \mathscr{F}(H), q \in [0,1]$. Then the set $(A)_q = \{x \in H : A(x) \geq q\}$ is called a q-cut set of A.

In this paper, we denote by (Ω,Σ) a measurable space, where Ω is a set and Σ is a σ-algebra of subsets of Ω and also we denote by $\mathscr{B}(H), 2^H, CB(H)$ and $H(\cdot,\cdot)$ the class of Borel σ-fields in H, the family of all nonempty subsets of H, the family of all nonempty closed bounded subsets of H, and the Hausd$\ddot{o}$rff metric on $CB(H)$, respectively.

Let $\hat{T} : \Omega \times H \to \mathscr{F}(H)$ be a random fuzzy mapping satisfying the following:

Condition(I): There exists a mapping $c : H \to [0,1]$ such that

$$(\hat{T}_{t,x})_{c(x)} \in CB(H),\ \forall\ (t,x) \in \Omega \times H.$$

By using the random fuzzy mapping $\hat{T}$, we can define a random multi-valued mapping T as follows:

$$T : \Omega \times H \to CB(H),\ (t,x) \to (\hat{T}_{t,x})_{c(x)},\ \forall\ (t,x) \in \Omega \times H.$$

T is called the random multi-valued mapping induced by the random fuzzy mapping $\hat{T}$.

Given mapping $c : H \to [0,1]$, the random fuzzy mapping $\hat{T} : \Omega \times H \to \mathscr{F}(H)$ satisfies the condition (I) and random operator $g : \Omega \times H \to H$ with $Im\ g \cap dom\ \partial\phi \neq \varnothing$, we consider the following random generalized mixed variational inequality problem (for short, RGMVIP):

Find measurable mappings $x, w : \Omega \to H$, such that for all $t \in \Omega,\ y(t) \in H$,

$$\hat{T}_{t,x(t)}(w(t)) \geq c(x(t)),$$

$$\big\langle w(t), y(t) - g(t,x(t))\big\rangle + \phi\big(g(t,x(t)), y(t)\big) - \phi\big(g(t,x(t)), g(t,x(t))\big) \geq 0, \qquad (1.1)$$

where $\partial\phi$ denotes the sub-differential of a proper, convex, and lower semi-continuous function $\phi(\cdot,\cdot) : H \times H \to \mathbb{R} \cup \{+\infty\}$ with its effective domain being closed.

The set of measurable mappings (x,w) is called a random solution of the RGMVIP (1.1).

Special cases:

(i) If c is zero operator and $T : H \to H$ is a single valued mapping and $g \equiv I$, the identity operator then the RGMVIP (1.1) reduces to the generalized mixed variational inequality problem, denoted by GMVIP, which consists in finding $x \in H$ such that

$$\langle Tx, y - x\rangle + \phi(x,y) - \phi(x,x) \geq 0, \forall\ y \in H. \qquad (1.2)$$

(ii) If $\phi(x,y) = \phi(y), \forall\ x$ then problem (1.2) reduces to mixed variational inequality problem, denoted by MVIP, which consists in finding $x \in H$ such that

$$\langle Tx, y - x\rangle + \phi(y) - \phi(x) \geq 0, \forall\ y \in H, \qquad (1.3)$$

which was studied by Tang and Huang [20]. In this paper, he introduced two regularized gap functions for the MVIP (1.3) and studied their differentiable properties.

(iii) If the function $\phi(\cdot)$ is an indicator function of a closed set K in H, then MVIP (1.3) reduces to a classical variational inequality problem, denoted by VIP, which consists in finding $x \in K$ such that

$$\langle Tx, y-x \rangle \geq 0, \forall\, y \in K, \tag{1.4}$$

which was studied by [1, 3, 7, 9, 15, 20]. They derived local and global error bounds for the VIP (1.4) in terms of the regularized gap functions and the D-gap functions.

Rest of the paper is organized as follows: In Section 2, we give some basic definitions and results which will be used in the paper. Furthermore, by using the residual vector we obtain the error bound for the solution of RGMVIP (1.1). In Section 3, we introduce a regularized gap function for RGMVIP (1.1) and derive the error bounds with and without Lipschitz continuity assumption. In Section 4, we introduce the D-gap function and derive global error bounds in terms of the D-gap function for the solution of RGMVIP (1.1).

2. Preliminaries

Definition 2.1. [11] A mapping $x : \Omega \to H$ is said to be measurable if for any $B \in \mathscr{B}(H), \{t \in \Omega : x(t) \in B\} \in \Sigma$.

Definition 2.2. [11] A mapping $f : \Omega \times H \to H$ is called a random operator if for any $x \in H,\ f(t,x) = x(t)$ is measurable. A random operator f is said to be continuous if for any $t \in \Omega$, the mapping $f(t,.) : H \to H$ is continuous.

Definition 2.3. [11] A multi-valued mapping $T : \Omega \to 2^H$ is said to be measurable if for any $B \in \mathscr{B}(H),\ T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \varnothing\} \in \Sigma$.

Definition 2.4. [11] A mapping $w : \Omega \to H$ is called a measurable selection of a multi-valued measurable mapping $T : \Omega \to 2^H$ if w is measurable and for any $t \in \Omega,\ w(t) \in T(t)$.

Definition 2.5. [11] A mapping $T : \Omega \times H \to 2^H$ is called a random multi-valued mapping if for any $x \in H,\ T(.,x)$ is measurable. A random multi-valued mapping $T : \Omega \times H \to CB(H)$ is said to be H-continuous if for any $t \in \Omega,\ T(t,.)$ is continuous in the Hausd$\ddot{o}$rff metric.

Definition 2.6. [11] A fuzzy mapping $T : \Omega \to \mathscr{F}(H)$ is called measurable, if for any $\nu \in (0,1],\ (T(.))_\nu : \Omega \to 2^H$ is a measurable multi-valued mapping.

Definition 2.7. [11] A fuzzy mapping $T : \Omega \times H \to \mathscr{F}(H)$ is called a random fuzzy mapping, if for any $x \in H, T(.,x) : \Omega \to \mathscr{F}(H)$ is a measurable fuzzy mapping.

Definition 2.8. A bi-function $\phi : H \times H \to \mathbb{R}$ is said to be skew symmetric if,

$$\phi(x(t),x(t)) - \phi(x(t),y(t)) - \phi(y(t),x(t)) + \phi(y(t),y(t)) \geq 0,\ \forall\, x,y \in H,\ t \in \Omega.$$

Definition 2.9. A function $G : H \to \mathbb{R}$ is said to be a gap function for the RGMVIP (1.1), if it satisfies the following properties:

(i) $G(x) \geq 0$, $\forall\, x \in H$;

(ii) $G(x_o) = 0$, if and only if $x_o \in H$ solves the RGMVIP (1.1).

Now we first recall the following well-known results and concepts.

For the VIP (1.4), it is well known that $x \in K$ is a solution, if and only if

$$0 = x - P_K[x - \alpha T(x)],$$

where P_K is the orthogonal projector onto K and $\alpha > 0$ is arbitrary. Hence, the norm of the right hand side of the above equation can serve as a gap function for VIP (1.4), which is commonly known as the natural residual vector.

Then, motivated by the proximal map given in [17], we derive a similar characterization for the RGMVIP (1.1) in the random fuzzy environment by defining the mapping $P^{\phi}_{\alpha(t)} : \Omega \times H \to dom\ \phi$, as

$$P^{\phi,z}_{\alpha(t)}(t,z) = \arg\min_{y(t)\in H} \left\{ \phi(g(t,x(t)),y(t)) + \frac{1}{2\alpha(t)}\|y(t) - z(t)\|^2 \right\},\ z(t) \in H,\ t \in \Omega,\ \alpha > 0,$$

where $\alpha : \Omega \to (0,+\infty)$ a measurable function which is the so called proximal mapping in H for a random fuzzy mapping. Note that the objective function above is proper strongly convex. Since $dom\ \phi$ is closed, $P^{\phi,z}_{\alpha(t)}(t,z)$ is well defined and single-valued.

For any measurable function $\alpha : \Omega \to (0,+\infty)$ define the residual vector

$$R^{\phi,x}_{\alpha(t)}(t,x(t)) = g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)],\ x(t) \in H. \tag{2.1}$$

Next, we show that $R^{\phi,x}_{\alpha(t)}(t,x(t))$ plays the role of the natural residual vector in random fuzzy mapping for the RGMVIP (1.1).

Lemma 2.1. *For any measurable function $\alpha : \Omega \to (0,+\infty)$ and for each $t \in \Omega$, the measurable mapping $x : \Omega \to H$ is a solution of the RGMVIP (1.1) if and only if $R^{\phi,x}_{\alpha(t)}(t,x(t)) = 0$.*

Proof. Let $R^{\phi,x}_{\alpha(t)}(t,x(t)) = 0$, which implies that $g(t,x(t)) = P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]$.
It is equivalent to

$$g(t,x(t)) = \arg\min_{y(t)\in H}\left\{\phi(g(t,x(t)),y(t)) + \frac{1}{2\alpha(t)}\|y(t) - (g(t,x(t)) - \alpha(t)w(t))\|^2\right\}.$$

By the optimality conditions (which are necessary and sufficient, by convexity), the latter is equivalent to

$$\begin{aligned}0 &\in \partial\phi(g(t,x(t)),y(t)) + \frac{1}{\alpha(t)}\big[g(t,x(t)) - (g(t,x(t)) - \alpha(t)w(t))\big]\\ &= \partial\phi(g(t,x(t)),y(t)) + w(t),\end{aligned}$$

which implies

$$-w(t) \in \partial\phi(g(t,x(t)),y(t)).$$

This in turn is equivalent, by the definition of the sub-gradient, to

$$\phi\big(g(t,x(t)),y(t)\big) \geq \phi\big(g(t,x(t)),g(t,x(t))\big) - \Big\langle w(t),y(t) - g(t,x(t))\Big\rangle,\ \forall\, y(t)\in H,\ t\in\Omega,$$

which implies that $x(t)$ solves the RGMVIP (1.1).

Definition 2.10. A random multi-valued operator $T:\Omega\times H\to CB(H)$ is said to be strongly g-monotone, if there exists a measurable function $\theta:\Omega\to(0,+\infty)$ such that

$$\Big\langle w_1(t) - w_2(t), g(t,x_1(t)) - g(t,x_2(t))\Big\rangle \geq \theta(t)\|x_1(t) - x_2(t)\|^2,$$
$$\forall\, w_i(t)\in T(t,x_i),\ \forall\, x_i(t)\in H,\ i=1,2,\ \forall\, t\in\Omega.$$

Definition 2.11. A random operator $g:\Omega\times H\to H$ is said to be Lipschitz continuous, if there exists a measurable function $L:\Omega\to(0,+\infty)$ such that

$$\|g(t,x_1(t)) - g(t,x_2(t))\| \leq L(t)\|x_1(t) - x_2(t)\|,\ \forall\, x_i(t)\in H, i=1,2,\ \forall\, t\in\Omega.$$

Definition 2.12. A random multi-valued mapping $T:\Omega\times H\to CB(H)$ is said to be $\hat{H}$-Lipschitz continuous, if there exists a measurable function $\lambda:\Omega\to(0,+\infty)$ such that

$$\hat{H}(T(t,x(t)),T(t,x_o(t))) \leq \lambda(t)\|x(t) - x_o(t)\|,\ \forall\, x(t),x_o(t)\in H.$$

Definition 2.13. $P^{\phi,x}$ is said to be non-expansive, if

$$\|P^{\phi,x}(v) - P^{\phi,x}(w)\| \leq \|v - w\|, \forall x,v,w\in H.$$

Now, we give the following lemmas.

Lemma 2.2. [4] *Let $T : \Omega \times H \to CB(H)$ be a $\hat{H}$-Lipschitz continuous random multi-valued mapping, then for measurable mapping $x : \Omega \to H$, the multi-valued mapping $T(\cdot, x(\cdot)) : \Omega \to CB(H)$ is measurable.*

Lemma 2.3. [4] *Let $T_1, T_2 : \Omega \to CB(H)$ be two measurable multi-valued mappings, $\varepsilon > 0$ be a constant, and $w_1 : \Omega \to H$ be a measurable selection of T_1, then there exists a measurable selection $w_2 : \Omega \to H$ of T_2 such that for all $t \in \Omega$,*

$$\|w_1(t) - w_2(t)\| \leq (1+\varepsilon)\hat{H}(T_1(t), T_2(t)).$$

Lemma 2.4. *Let $x : \Omega \to H$ be a measurable mapping and $\alpha : \Omega \to (0, +\infty)$ be a measurable function, then for all $x(t) \in H$ and for each $t \in \Omega$, $R^{\phi,x}_{\alpha(t)}(t, x(t))$ is a gap function for the RGMVIP (1.1).*

Next we show that the RGMVIP (1.1) has a unique solution.

Theorem 2.1. *Suppose that for each $t \in \Omega$, $x_o(t) \in H$ is a solution of the RGMVIP (1.1). Let (Ω, Σ) be a measurable space, and H be a real Hilbert space. Let the random fuzzy mapping $\hat{T} : \Omega \times H \to \mathscr{F}(H)$ satisfy the condition (I) and $T : \Omega \times H \to CB(H)$ be the random multi-valued mapping induced by the random fuzzy mapping $\hat{T}$. Let $g : \Omega \times H \to H$ be a random mapping and $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a real valued function such that*

- (i) *for each $t \in \Omega$, the measurable mapping T is strongly g-monotone and $\hat{H}$-Lipschitz continuous with the measurable functions $\theta, \lambda : \Omega \to (0, +\infty)$, respectively;*
- (ii) *for each $t \in \Omega$, the mapping $g(t, .)$ is Lipschitz continuous with the measurable function $L : \Omega \to (0, +\infty)$;*
- (iii) *if there exists a measurable function $K : \Omega \to (0, +\infty)$ such that*

$$\left\|P^{\phi,x}_{\alpha(t)}(z) - P^{\phi,x_o}_{\alpha(t)}(z)\right\| \leq K(t)\left\|x(t) - x_o(t)\right\|, \; \forall\, x(t), x_o(t), z(t) \in H,$$

with

$$K(t) < 1 - \sqrt{L^2(t) - 2\alpha(t)\theta(t) + \alpha^2(t)(1+\varepsilon)\lambda(t)}$$

then, RGMVIP (1.1) has a unique solution.

Proof. (Uniqueness). Let the two measurable mappings $x_1, x_2 : \Omega \to H$ be the two solutions of the RGMVIP (1.1) such that $x_1(t) \neq x_2(t) \in H$. Then, we have

$$\langle w_1(t), y(t) - g(t, x_1(t)) \rangle + \phi\big(g(t, x_1(t)), y(t)\big) - \phi\big(g(t, x_1(t)), g(t, x_1(t))\big) \geq 0, \tag{2.2}$$

$$\langle w_2(t), y(t) - g(t, x_2(t)) \rangle + \phi\big(g(t, x_2(t)), y(t)\big) - \phi\big(g(t, x_2(t)), g(t, x_2(t))\big) \geq 0. \tag{2.3}$$

Taking $y(t) = g(t, x_2(t))$ in (2.2) and $y(t) = g(t, x_1(t))$ in (2.3), adding the resultants, we have

$$\Big\langle w_1(t) - w_2(t), g(t, x_2(t)) - g(t, x_1(t)) \Big\rangle \geq 0.$$

Since T is strongly g-monotone with measurable function $\theta : \Omega \to (0, +\infty)$, therefore

$$0 \leq \Big\langle w_1(t) - w_2(t), g(t, x_2(t)) - g(t, x_1(t)) \Big\rangle \leq -\theta(t) \big\| x_1(t) - x_2(t) \big\|^2,$$

which implies that $x_1(t) = x_2(t)$, $\forall\, t \in \Omega$, the uniqueness of the solution of the RGMVIP (1.1).

Now, by using normal residual vector $R^{\phi,x}_{\alpha(t)}(t, x(t))$, we derive the error bounds for the solution of RGMVIP (1.1).

Theorem 2.2. *Suppose that for each $t \in \Omega, x_o(t) \in H$ is a solution of the RGMVIP (1.1). Let (Ω, Σ) be a measurable space, and H be a real Hilbert space. Suppose that $g : \Omega \times H \to H$ be a random mapping and $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a real valued function. Let the random fuzzy mapping $\hat{T} : \Omega \times H \to \mathscr{F}(H)$ satisfy the condition (I), and a random multi-valued mapping $T : \Omega \times H \to CB(H)$ induced by the random fuzzy mapping such that*

- (i) *for each $t \in \Omega$, the measurable mapping T is strongly g-monotone and $\hat{H}$-Lipschitz continuous with the measurable functions $\theta, \lambda : \Omega \to (0, +\infty)$, respectively;*
- (ii) *for each $t \in \Omega$, the mapping $g(t, .)$ is Lipschitz continuous with the measurable function $L : \Omega \to (0, +\infty)$;*
- (iii) *if there exists a measurable function $K : \Omega \to (0, +\infty)$ such that*

$$\big\| P^{\phi,x}_{\alpha(t)}(z) - P^{\phi,x_o}_{\alpha(t)}(z) \big\| \leq K(t) \big\| x(t) - x_o(t) \big\|, \ \forall\, x(t), x_o(t), z(t) \in H,$$

then for any $x(t) \in H$, $t \in \Omega$ and $\alpha(t) > \left[\frac{L(t)K(t)}{\theta(t) - K(t)\lambda(t)(1+\varepsilon)}\right]$, we have

$$\big\| x(t) - x_o(t) \big\| \leq \left[\frac{\alpha(t)\lambda(t)(1+\varepsilon) + L(t)}{\alpha(t)\theta(t) - L(t)K(t) - K(t)\alpha(t)\lambda(t)(1+\varepsilon)} \right] \big\| R^{\phi,x}_{\alpha(t)}(t, x(t)) \big\|.$$

Proof. Let for each $t \in \Omega,\ x_o(t) \in H$ be a solution of the RGMVIP (1.1), then

$$\Big\langle w_o(t), y(t) - g(t, x_o(t))\Big\rangle + \phi\big(g(t, x_o(t)), y(t)\big) - \phi\big(g(t, x_o(t)), g(t, x_o(t))\big) \geq 0.$$

Substituting $y(t) = P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]$ in the above inequality, we have

$$\begin{aligned} &\Big\langle w_o(t), P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)] - g(t, x_o(t))\Big\rangle \\ &\qquad + \phi\big(g(t, x_o(t)), P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\big) \\ &\qquad - \phi\big(g(t, x_o(t)), g(t, x_o(t))\big) \geq 0. \end{aligned} \tag{2.4}$$

For any fixed $x(t) \in H$ and measurable function $\alpha : \Omega \to (0, +\infty)$, we observe that

$$\begin{aligned} g(t, x(t)) - \alpha(t) w(t) &\in \big(I + \alpha(t)\partial\phi\big)\big(I + \alpha(t)\partial\phi\big)^{-1}(g(t, x(t)) - \alpha(t) w(t)) \\ &= \big(I + \alpha(t)\partial\phi\big) P^{\phi, x}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)], \end{aligned}$$

which is equivalent to

$$\begin{aligned} &-w(t) + \frac{1}{\alpha(t)}\big[g(t, x(t)) - P^{\phi, x}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\big] \\ &\qquad \in \partial\phi\big(P^{\phi, x}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\big). \end{aligned}$$

By the definition of a sub-differential, we have

$$\begin{aligned} &\Big\langle w(t) - \frac{1}{\alpha(t)}\big(g(t, x(t)) - P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\big), \\ &\qquad y(t) - P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\Big\rangle \\ &\qquad + \phi\big(P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)], y(t)\big) \\ &\qquad - \phi\big(P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)], P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\big) \geq 0. \end{aligned}$$

Taking $y(t) = g(t, x_o(t))$ in the above, we get

$$\begin{aligned} &\Big\langle w(t) - \frac{1}{\alpha(t)}\big(g(t, x(t)) - P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\big), \\ &\qquad g(t, x_o(t)) - P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\Big\rangle \\ &\qquad + \phi\big(P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)], g(t, x_o(t))\big) \\ &\qquad - \phi\big(P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)], P^{\phi, x_o}_{\alpha(t)}[g(t, x(t)) - \alpha(t) w(t)]\big) \geq 0. \end{aligned}$$

This implies that

$$\begin{aligned}
&\Big\langle -w(t)+\frac{1}{\alpha(t)}\big(g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big),\\
&\qquad P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x_o(t))\Big\rangle\\
&\qquad +\phi\big(P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)],g(t,x_o(t))\big)\\
&\qquad -\phi\big(P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)],P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big)\geq 0.
\end{aligned}\tag{2.5}$$

Adding (2.4) and (2.5), we get

$$\begin{aligned}
&\Big\langle w_o(t)-w(t)+\frac{1}{\alpha(t)}\big[g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big],\\
&\qquad P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x_o(t))\Big\rangle\\
&\qquad +\phi\big(g(t,x_o(t)),P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big)-\phi\big(g(t,x_o(t)),g(t,x_o(t))\big)\\
&\qquad +\phi\big(P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)],g(t,x_o(t))\big)\\
&\qquad -\phi\big(P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)],P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big)\geq 0.
\end{aligned}$$

Since ϕ is skew symmetric, we get

$$\begin{aligned}
&\Big\langle w_o(t)-w(t)+\frac{1}{\alpha(t)}\big[g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big],\\
&\qquad P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x_o(t))\Big\rangle\geq 0.
\end{aligned}$$

This can also be written as

$$\begin{aligned}
&\alpha(t)\Big\langle w_o(t)-w(t),P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x(t))\Big\rangle\\
&\qquad +\alpha(t)\Big\langle w_o(t)-w(t),g(t,x(t))-g(t,x_o(t))\Big\rangle\\
&\qquad +\Big\langle g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)],\\
&\qquad\qquad P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x(t))\Big\rangle\\
&\qquad +\Big\langle g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)],g(t,x(t))-g(t,x_o(t))\Big\rangle\geq 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\alpha(t)\Big\langle w_o(t)-w(t), P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x(t))\Big\rangle \\
&\qquad + \Big\langle g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)], g(t,x(t))-g(t,x_o(t))\Big\rangle \\
&\geq \alpha(t)\Big\langle w_o(t)-w(t), g(t,x_o(t))-g(t,x(t))\Big\rangle \\
&\quad + \Big\langle g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)], \\
&\qquad g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\Big\rangle.
\end{aligned}$$

By using the strong g-monotonicity of T, we get

$$\begin{aligned}
&\alpha(t)\Big\langle w_o(t)-w(t), P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x(t))\Big\rangle \\
&\quad + \Big\langle g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)], g(t,x(t))-g(t,x_o(t))\Big\rangle \\
&\quad \geq \alpha(t)\theta(t)\big\|x_o(t)-x(t)\big\|^2+\big\|R^{\phi,x_o}_{\alpha(t)}(t,x(t))\big\|^2.
\end{aligned}$$

Also the above inequality can be written as

$$\begin{aligned}
&\alpha(t)\Big\langle w_o(t)-w(t), P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x(t)) \\
&\qquad -P^{\phi,x}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]+P^{\phi,x}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\Big\rangle \\
&+\Big\langle g(t,x(t))-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-P^{\phi,x}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)] \\
&\qquad +P^{\phi,x}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)], g(t,x(t))-g(t,x_o(t))\Big\rangle \\
&\qquad \geq \alpha(t)\theta(t)\big\|x_o(t)-x(t)\big\|^2+\big\|R^{\phi,x_o}_{\alpha(t)}(t,x(t))\big\|^2.
\end{aligned}$$

By using the Cauchy-Schwarz inequality along with the triangular inequality, we have

$$\begin{aligned}
&\alpha(t)\big\|w_o(t)-w(t)\big\|\big\|P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-P^{\phi,x}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big\| \\
&\quad +\alpha(t)\big\|w_o(t)-w(t)\big\|\big\|P^{\phi,x}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-g(t,x(t))\big\| \\
&\quad +\big\|g(t,x(t))-P^{\phi,x}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big\|\big\|g(t,x_o(t))-g(t,x(t))\big\| \\
&\quad +\big\|P^{\phi,x}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]-P^{\phi,x_o}_{\alpha(t)}[g(t,x(t))-\alpha(t)w(t)]\big\| \\
&\quad \big\|g(t,x_o(t))-g(t,x(t))\big\| \\
&\quad \geq \alpha(t)\theta(t)\big\|x_o(t)-x(t)\big\|^2+\big\|R^{\phi,x_o}_{\alpha(t)}(t,x(t))\big\|^2.
\end{aligned}$$

Now using the $\hat{H}$-Lipschitz continuity of T, the Lipschitz continuity of g, and assumption (iii) on $P^{\phi,x}_{\alpha(t)}(.)$, we have

$$\begin{aligned}\alpha(t)\lambda(t)(1+\varepsilon)\|x_o(t)-x(t)\|K(t)\|x_o(t)-x(t)\|\\ +\alpha(t)\lambda(t)(1+\varepsilon)\|x_o(t)-x(t)\|\|R^{\phi,x}_{\alpha(t)}(t,x(t))\|\\ +\|R^{\phi,x}_{\alpha(t)}(t,x(t))\|L(t)\|x_o(t)-x(t)\|+K(t)L(t)\|x(t)-x_o(t)\|^2\\ \geq \alpha(t)\theta(t)\|x_o(t)-x(t)\|^2+\|R^{\phi,x_o}_{\alpha(t)}(t,x(t))\|^2.\end{aligned}$$

The above can be again written as

$$\begin{aligned}K(t)\alpha(t)\lambda(t)(1+\varepsilon)\|x_o(t)-x(t)\|^2+\alpha(t)\lambda(t)(1+\varepsilon)\|x_o(t)-x(t)\|\|R^{\phi,x}_{\alpha(t)}(t,x(t))\|\\ +L(t)\|R^{\phi,x}_{\alpha(t)}(t,x(t))\|\|x_o(t)-x(t)\|+K(t)L(t)\|x(t)-x_o(t)\|^2\\ \geq \alpha(t)\theta(t)\|x_o(t)-x(t)\|^2+\|R^{\phi,x}_{\alpha(t)}(t,x(t))\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\big[-K(t)\alpha(t)\lambda(t)(1+\varepsilon)-L(t)K(t)+\alpha(t)\theta(t)\big]\|x_o(t)-x(t)\|^2\\ \leq [\alpha(t)\lambda(t)(1+\varepsilon)+L(t)]\|x_o(t)-x(t)\|\|R^{\phi,x}_{\alpha(t)}(t,x(t))\|\end{aligned}$$

Hence,

$$\|x_o(t)-x(t)\|\leq\left[\frac{\alpha(t)\lambda(t)(1+\varepsilon)+L(t)}{\alpha(t)\theta(t)-L(t)K(t)-L(t)\alpha(t)\lambda(t)(1+\varepsilon)}\right]\|R^{\phi,x}_{\alpha(t)}(t,x(t))\|,$$
$$\forall\, x(t)\in H, t\in\Omega,$$

where

$$\alpha(t)>\left[\frac{L(t)K(t)}{\theta(t)-K(t)\lambda(t)(1+\varepsilon)}\right].$$

3. Regularized gap functions

In this section our main motivation is to overcome the non differentiability of residual vector $R^{\phi,x}_{\alpha(t)}$ i.e., the gap function defined by (2.1). Now by using an approach due to Fukushima [7], we construct another gap function associated with problem RGMVIP (1.1), which can be viewed as a regularized gap function. For $\alpha > 0$, the functions G_α is defined by

$$G_\alpha(x(t)) = \max_{y(t)\in H} \Big\{ \Big\langle w(t), g(t,x(t)) - y(t) \Big\rangle - \phi\big(g(t,x(t)), y(t)\big) + \phi\big(g(t,x(t)), g(t,x(t))\big) - \frac{1}{2\alpha}\big\|g(t,x(t)) - y(t)\big\|^2 \Big\} \tag{3.1}$$

which is finite valued everywhere and is differentiable whenever all operators involved in $G_\alpha(x(t))$, are differentiable.

Lemma 3.1. *For any $\alpha > 0, G_\alpha(x(t))$ can be written as*

$$\begin{aligned} G_\alpha(x(t)) = \Big\langle w(t), g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)] \Big\rangle - \phi\big(g(t,x(t)), \\ P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big) + \phi\big(g(t,x(t)), g(t,x(t))\big) \\ - \frac{1}{2\alpha}\big\|g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big\|^2, \ \forall\, x(t) \in H \end{aligned} \tag{3.2}$$

Proof. If $x(t) \notin dom\ \phi$, then equation (3.2) holds, because $\phi \cong +\infty$, while the other terms are all finite (recall that $P^{\phi,x}_{\alpha(t)}(t,z) \in dom\ \phi$, for any $z(t) \in H$). Consider now any $x(t) \in dom\ \phi$. Denote by $F(y)$ the function being maximized in (3.1). Let $z(t)$ be the (unique, by concavity of $F(y)$) element at which the maximum is realized in (3.1). Then $z(t)$ is uniquely characterized by the optimality condition

$$\begin{aligned} 0 \in \partial(-F(z)) &= w(t) + \partial\phi(.,z) + \frac{1}{\alpha}(z(t) - g(t,x(t)) \\ &= \partial\phi(.,z) + \frac{1}{\alpha(t)}[z(t) - (g(t,x(t)) - \alpha(t)w(t))]. \end{aligned}$$

But, this inclusion also uniquely characterizes the solution of the problem

$$\begin{aligned} z &= \arg\min_{y(t)\in H} \Big\{ \phi(g(t,x(t)), y(t)) + \frac{1}{2\alpha}\big\|y(t) - (g(t,x(t)) - \alpha(t)w(t))\big\|^2 \Big\} \\ &= P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], \end{aligned}$$

where the above equation follows from the definition of the proximal mapping $P^{\phi,x}_{\alpha(t)}(.)$.

Next we show the function $G_\alpha(x(t))$ for $\alpha > 0$ given by (3.1) is a gap function for RGMVIP (1.1).

Theorem 3.1. *If $\alpha > 0$, then we have*

$$G_\alpha(x(t)) \geq \frac{1}{2\alpha}\big\|R^{\phi,x}_{\alpha(t)}(t,x(t))\big\|^2, \ \forall\, x(t) \in H.$$

In particular, $G_\alpha(x(t)) = 0$, if and only if $x(t)$ is a solution of RGMVIP (1.1).

Proof. For any fixed $x(t) \in H, t \in \Omega$ and $\alpha > 0$.

$$\begin{aligned} g(t,x(t)) - \alpha(t)w(t) \in& (I+\alpha(t)\partial\phi)(I+\alpha(t)\partial\phi)^{-1}[g(t,x(t)) - \alpha(t)w(t)] \\ =& (I+\alpha(t)\partial\phi)P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], \end{aligned}$$

which is equivalent to

$$\begin{aligned} -w(t) + \frac{1}{\alpha(t)}\big[g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big] \\ \in \partial\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big). \end{aligned}$$

By the definition of a sub-differential, we have

$$\begin{aligned} \Big\langle w(t) - \frac{1}{\alpha(t)}\big[g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big], \\ y(t) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\Big\rangle \\ +\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], y(t)\big) \\ -\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big) \geq 0. \end{aligned}$$

Taking $y(t) = g(t,x(t))$ in the above inequality, we get

$$\begin{aligned} \Big\langle w(t) - \frac{1}{\alpha(t)}\big[g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big], \\ g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\Big\rangle \\ +\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], g(t,x(t))\big) \\ -\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big) \geq 0, \end{aligned}$$

or

$$\begin{aligned} \Big\langle w(t), g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\Big\rangle \\ +\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], g(t,x(t))\big) \\ -\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big) \\ \geq \frac{1}{\alpha(t)}\Big\langle R^{\phi,x}_{\alpha(t)}(t,x(t)), R^{\phi,x}_{\alpha(t)}(t,x(t))\Big\rangle. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) and by using the skew symmetry of ϕ, we get

$$
\begin{aligned}
G_\alpha(x(t)) \geq & \frac{1}{\alpha(t)}\left\langle R^{\phi,x}_{\alpha(t)}(t,x(t)), R^{\phi,x}_{\alpha(t)}(t,x(t))\right\rangle - \frac{1}{2\alpha(t)}\left\|R^{\phi,x}_{\alpha(t)}(t,x(t))\right\|^2 \\
= & \frac{1}{2\alpha(t)}\left\|R^{\phi,x}_{\alpha(t)}(t,x(t))\right\|^2.
\end{aligned}
$$

Clearly, we have $G_\alpha(x(t)) \geq 0,\ \forall\, x(t) \in H$.

Now, from the above conclusion, if $G_\alpha(x(t)) = 0$, then $R^{\phi,x}_{\alpha(t)}(t,x(t)) = 0$. Hence by Lemma 2.1, we see that $x(t) \in H$ is a solution of RGMVIP (1.1). Conversely, if $x(t) \in H$ is a solution of RGMVIP (1.1), then $g(t,x(t)) = P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]$, consequently, from (3.2) we have $G_\alpha(x(t)) = 0$.

As a consequence of Theorem 2.2 and Theorem 3.1, we have the following result on error bound in terms of $G_{\alpha(t)}(x(t))$ for RGMVIP (1.1).

Corollary 3.1. *Suppose that for each $t \in \Omega, x_o(t) \in H$ is a solution of the RGMVIP (1.1). Let (Ω,Σ) be a measurable space, and H be a real Hilbert space. Let the random fuzzy mapping $\hat{T} : \Omega \times H \to \mathscr{F}(H)$ satisfy the condition (I) and $T : \Omega \times H \to CB(H)$ be the random multi-valued mapping induced by the random fuzzy mapping $\hat{T}$. Let $g : \Omega \times H \to H$ be a random mapping and $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a real valued function such that*

- (i) *for each $t \in \Omega$, the measurable mapping T is strongly g-monotone and $\hat{H}$-Lipschitz continuous with the measurable functions $\theta, \lambda : \Omega \to (0,+\infty)$, respectively;*
- (ii) *for each $t \in \Omega$, the mapping $g(t,.)$ is Lipschitz continuous with the measurable function $L : \Omega \to (0,+\infty)$;*
- (iii) *if there exists a measurable function $K : \Omega \to (0,+\infty)$ such that*

$$
\|P^{\phi,x}_{\alpha(t)}(z) - P^{\phi,x_o}_{\alpha(t)}(z)\| \leq K(t)\|x(t) - x_o(t)\|, \forall x(t), x_o(t), z(t) \in H,
$$

then for any $x(t) \in H$, $t \in \Omega$ and $\alpha(t) > \left[\frac{L(t)K(t)}{\theta(t) - K(t)\lambda(t)(1+\varepsilon)}\right]$, we have

$$
\left\|x(t) - x_o(t)\right\| \leq \left[\frac{\alpha(t)\lambda(t)(1+\varepsilon) + L(t)}{\alpha(t)\theta(t) - L(t)K(t) - K(t)\alpha(t)\lambda(t)(1+\varepsilon)}\right]\sqrt{2\alpha}\sqrt{G_\alpha(x(t))}.
$$

Now, we derive the error bound for RGMVIP (1.1) without using the Lipschitz continuity of T.

Theorem 3.2. *Suppose that for each $t \in \Omega, x_o(t) \in H$ is a solution of the RGMVIP (1.1). Let (Ω, Σ) be a measurable space, and H be a real Hilbert space. Let the random fuzzy mapping $\hat{T} : \Omega \times H \to \mathscr{F}(H)$ satisfy the condition (I) and $T : \Omega \times H \to CB(H)$ be the random multi-valued mapping induced by the random fuzzy mapping $\hat{T}$. Let $g : \Omega \times H \to H$ be a random mapping and $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a real valued function such that for each $t \in \Omega$, the measurable mapping T is strongly g-monotone and $\hat{H}$-Lipschitz continuous with the measurable functions $\theta, \lambda : \Omega \to (0, +\infty)$, respectively, then for any $x(t) \in H, t \in \Omega$, we have*

$$\|x(t) - x_o(t)\| \leq \left[\frac{1}{\sqrt{\left(\theta(t) - \frac{L^2(t)}{2\alpha(t)}\right)}} \right] \sqrt{G_\alpha(x(t))}.$$

Proof. From (3.1), it can be written as

$$G_\alpha(x(t)) \geq \Big\langle w(t), g(t, x(t)) - g(t, x_o(t)) \Big\rangle - \phi\big(g(t, x(t)), g(t, x_o(t))\big) + \phi\big(g(t, x(t)), g(t, x(t))\big) - \frac{1}{2\alpha} \|g(t, x(t)) - g(t, x_o(t))\|^2.$$

By using the strong g-monotonicity of T, we have

$$\begin{aligned} G_\alpha(x(t)) \geq & \Big\langle w(t) - w_o(t), g(t, x(t)) - g(t, x_o(t)) \Big\rangle + \Big\langle w_o(t), g(t, x(t)) - g(t, x_o(t)) \Big\rangle \\ & - \phi\big(g(t, x(t)), g(t, x_o(t))\big) + \phi\big(g(t, x(t)), g(t, x(t))\big) \\ & - \frac{1}{2\alpha} \|g(t, x(t)) - g(t, x_o(t))\|^2 \end{aligned}$$

or

$$\begin{aligned} \geq & \, \theta(t) \|x(t) - x_o(t)\|^2 + \langle w_o(t), g(t, x(t)) - g(t, x_o(t)) \rangle \\ & - \phi\big(g(t, x(t)), g(t, x_o(t))\big) + \phi\big(g(t, x(t)), g(t, x(t))\big) \\ & - \frac{1}{2\alpha(t)} L^2(t) \|x(t)) - x_o(t))\|^2. \end{aligned} \tag{3.4}$$

Since $g(t, x_o(t))$ is a solution of RGMVIP (1.1).

$$\Big\langle w_o(t), y(t) - g(t, x_o(t)) \Big\rangle + \phi\big(g(t, x_o(t)), y(t)\big) - \phi\big(g(t, x_o(t)), g(t, x_o(t))\big) \geq 0.$$

Taking $y(t) = g(t, x(t))$ in the above inequality

$$\Big\langle w_o(t), g(t, x(t)) - g(t, x_o(t)) \Big\rangle + \phi\big(g(t, x_o(t)), g(t, x(t))\big) - \phi\big(g(t, x_o(t)), g(t, x_o(t))\big) \geq 0. \tag{3.5}$$

Combining (3.4) and (3.5) and using the skew symmetry of ϕ, we get

$$G_{\alpha}(x(t)) \geq \theta(t)\left\|x(t)-x_o(t)\right\|^2 - \frac{1}{2\alpha(t)}L^2(t)\left\|x(t)-x_o(t)\right\|^2$$
$$G_{\alpha}(x(t)) \geq \left(\theta(t)-\frac{L^2(t)}{2\alpha(t)}\right)\left\|x(t)-x_o(t)\right\|^2$$

which implies

$$\left\|x(t)-x_o(t)\right\|^2 \leq \frac{1}{\left(\theta(t)-\frac{L^2(t)}{2\alpha(t)}\right)}G_{\alpha}(x(t))$$
$$\left\|x(t)-x_o(t)\right\| \leq \frac{1}{\sqrt{\left(\theta(t)-\frac{L^2(t)}{2\alpha(t)}\right)}}\sqrt{G_{\alpha}(x(t))}$$

4. D-Gap Function

In this section, we consider another gap function associated with RGMVIP (1.1), which can be viewed as a difference of two regularized gap functions with distinct parameters, known as the D-gap function, which was introduced and studied by [18, 19, 23] for solving variational inequalities and complementarity problems.

For each $x(t) \in H$, the difference of two regularized gap functions $G_{\alpha}(x(t)) - G_{\beta}(x(t))$, where $\alpha > \beta > 0$ for RGMVIP (1.1) will not be well defined for $x(t) \notin dom\ \phi$, as both quantities are not finite. Nevertheless, we shall define the D-gap function by taking a formal difference of equations (3.1) for the two parameters $\alpha > \beta > 0$.

The D-gap function associated with RGMVIP (1.1) is given by

$$\begin{aligned} D_{\alpha,\beta}(x(t)) = \max_{y(t)\in H}\Big\{ & \Big\langle w(t), g(t,x(t))-y(t)\Big\rangle - \phi\big(g(t,x(t)),y(t)\big) \\ & +\phi\big(g(t,x(t)),g(t,x(t))\big) + \frac{1}{2\beta}\left\|g(t,x(t))-y(t)\right\|^2 \\ & - \frac{1}{2\alpha}\left\|g(t,x(t))-y(t)\right\|^2\Big\}, x(t)\in H, \alpha>\beta>0. \end{aligned} \tag{4.1}$$

The D-gap function defined by (4.1) can be written as

$$\begin{aligned} D_{\alpha,\beta}(x(t)) = &\Big\langle w(t), P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)] - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\Big\rangle \\ &-\phi\left(P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\right) \\ &+\phi\left(P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)], P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)]\right) \\ &+\frac{1}{2\beta}\left\|g(t,x(t)) - P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)]\right\|^2 \\ &-\frac{1}{2\alpha}\left\|g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\right\|^2. \end{aligned}$$

Further, it can be written as

$$\begin{aligned} D_{\alpha,\beta}(x(t)) = &\Big\langle w(t), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\Big\rangle \\ &-\phi\left(P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\right) \\ &+\phi\left(P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)], P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)]\right) \\ &+\frac{1}{2\beta}\left\|R^{\phi,x}_{\beta(t)}(t,x(t))\right\|^2 - \frac{1}{2\alpha}\left\|R^{\phi,x}_{\alpha(t)}(t,x(t))\right\|^2. \end{aligned} \tag{4.2}$$

Next, we derive global error bounds for RGMVIP (1.1).

Theorem 4.1. *For all $x(t) \in H, t \in \Omega,\ \alpha > \beta > 0$, we have*

$$\frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\left\|R^{\phi,x}_{\beta(t)}(t,x(t))\right\|^2 \leq \left\|D_{\alpha,\beta}(x(t))\right\| \leq \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\left\|R^{\phi,x}_{\alpha(t)}(t,x(t)))\right\|^2.$$

In particular $D_{\alpha,\beta}(x(t)) = 0$, if and only if, $x(t) \in H$ solves RGMVIP (1.1).

Proof. By the definition of a sub-differential, we have

$$\begin{aligned} &\Big\langle w(t) - \frac{1}{\alpha(t)}\left[g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\right], \\ &\qquad\qquad y(t) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\Big\rangle \\ &\quad+\phi\left(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], y(t)\right) \\ &-\phi\left(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\right) \geq 0. \end{aligned}$$

Taking $y(t) = P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)]$ in the above inequality, we get

$$\begin{aligned}
&\Big\langle w(t) - \frac{1}{\alpha(t)}\big[g(t,x(t)) - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big],\\
&\qquad P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)] - P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\Big\rangle\\
&+\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)]\big)\\
&\quad -\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big) \geq 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\Big\langle w(t), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\Big\rangle\\
&\quad \geq \frac{1}{\alpha(t)}\Big\langle R^{\phi,x}_{\alpha(t)}(t,x(t)), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\Big\rangle\\
&\qquad -\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)]\big)\\
&\qquad\quad +\phi\big(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\big).
\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) and using the skew symmetry of ϕ, we get

$$\begin{aligned}
D_{\alpha,\beta}(x(t)) &\geq \frac{1}{\alpha(t)}\Big\langle R^{\phi,x}_{\alpha(t)}(t,x(t)), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\Big\rangle\\
&\qquad + \frac{1}{2\beta}\big\|R^{\phi,x}_{\beta(t)}(t,x(t))\big\|^2 - \frac{1}{2\alpha(t)}\big\|R^{\phi,x}_{\alpha(t)}(t,x(t))\big\|^2\\
&= \frac{1}{2}\Big(\frac{1}{\beta} - \frac{1}{\alpha}\Big)\big\|R^{\phi,x}_{\beta(t)}(t,x(t))\big\|^2\\
&\quad + \frac{1}{\alpha}\Big\langle R^{\phi,x}_{\alpha(t)}(t,x(t)), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\Big\rangle\\
&\qquad - \frac{1}{2\alpha}\big\|R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\big\|^2\\
&\quad - \frac{1}{\alpha}\Big\langle R^{\phi,x}_{\beta(t)}(t,x(t)), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\Big\rangle\\
&= \frac{1}{2}\Big(\frac{1}{\beta} - \frac{1}{\alpha}\Big)\big\|R^{\phi,x}_{\beta(t)}(t,x(t))\big\|^2 + \frac{1}{2\alpha}\big\|R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\big\|^2\\
&\geq \frac{1}{2}\Big(\frac{1}{\beta} - \frac{1}{\alpha}\Big)\big\|R^{\phi,x}_{\beta(t)}(t,x(t))\big\|^2,
\end{aligned} \tag{4.4}$$

which implies the left most inequality in the assertion.

On the other hand,

$$-w(t) + \frac{1}{\beta}R^{\phi,x}_{\beta(t)}(t,x(t)) \in \partial\phi\big(P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)]\big),$$

which implies that

$$
\begin{aligned}
&\left\langle w(t), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\right\rangle \\
&\qquad\qquad \le \frac{1}{\beta}\left\langle R^{\phi,x}_{\beta(t)}(t,x(t)), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\right\rangle \\
&\qquad\qquad\qquad - \phi\left(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\beta(t)}[g(t,x(t)) - \beta(t)w(t)]\right) \\
&\qquad\qquad\quad + \phi\left(P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)], P^{\phi,x}_{\alpha(t)}[g(t,x(t)) - \alpha(t)w(t)]\right).
\end{aligned}
\tag{4.5}
$$

Similarly, to the analysis above, we then obtain

$$
\begin{aligned}
D_{\alpha,\beta}(x(t)) &\le \frac{1}{\beta}\left\langle R^{\phi,x}_{\beta(t)}(t,x(t)), R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\right\rangle \\
&\qquad + \frac{1}{2\beta}\left\|R^{\phi,x}_{\beta(t)}(t,x(t))\right\|^2 - \frac{1}{2\alpha}\left\|R^{\phi,x}_{\alpha(t)}(t,x(t))\right\|^2 \\
&= \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\left\|R^{\phi,x}_{\alpha(t)}(t,x(t))\right\|^2 \\
&\qquad - \frac{1}{2\beta}\left\|R^{\phi,x}_{\alpha(t)}(t,x(t)) - R^{\phi,x}_{\beta(t)}(t,x(t))\right\|^2 \\
&\le \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\left\|R^{\phi,x}_{\alpha(t)}(t,x(t))\right\|^2,
\end{aligned}
\tag{4.6}
$$

which implies the right most inequality in the assertion. Combining (4.4) and (4.6), we obtain the required result. The last assertion now follows from Lemma 2.1.

As a consequence of Theorem 2.2 and Theorem 4.1, we obtain the following result on the global error bound for RGMVIP (1.1).

Corollary 4.1. *Suppose that for each $t \in \Omega, x_o(t) \in H$ is a solution of the RGMVIP (1.1). Let (Ω, Σ) be a measurable space, and H be a real Hilbert space. Let the random fuzzy mapping $\hat{T} : \Omega \times H \to \mathscr{F}(H)$ satisfy the condition (I) and $T : \Omega \times H \to CB(H)$ be the random multi-valued mapping induced by the random fuzzy mapping $\hat{T}$. Let $g : \Omega \times H \to H$ be a random mapping and $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a real valued function such that*

- (i) *for each $t \in \Omega$, the measurable mapping T is strongly g-monotone and $\hat{H}$-Lipschitz continuous with the measurable functions $\theta, \lambda : \Omega \to (0, +\infty)$, respectively;*
- (ii) *for each $t \in \Omega$, the mapping $g(t,.)$ is Lipschitz continuous with the measurable function $L : \Omega \to (0, +\infty)$;*

(iii) *if there exists a measurable function $K:\Omega\to(0,+\infty)$ such that*

$$\|P^{\phi,x}_{\alpha(t)}(z)-P^{\phi,x_o}_{\alpha(t)}(z)\|\le K(t)\|x(t)-x_o(t)\|,\forall x(t),x_o(t),z(t)\in H,$$

then for any $x(t)\in H$, $t\in\Omega$ and $\alpha(t)>\left[\frac{L(t)K(t)}{\theta(t)-K(t)\lambda(t)(1+\varepsilon)}\right]$, we have

$$\left\|x(t)-x_o(t)\right\|\le\left[\frac{\alpha(t)\lambda(t)(1+\varepsilon)+L(t)}{\alpha(t)\theta(t)-L(t)K(t)-K(t)\alpha(t)\lambda(t)(1+\varepsilon)}\right]\sqrt{\frac{2\alpha\beta}{\alpha-\beta}}\sqrt{D_{\alpha,\beta}(x(t))}.$$

Now, we derive the global error bound for RGMVIP (1.1) without using the Lipschitz continuity of T.

Theorem 4.2. *Suppose that for each $t\in\Omega, x_o(t)\in H$ is a solution of the RGMVIP (1.1). Let (Ω,Σ) be a measurable space, and H be a real Hilbert space. Let the random fuzzy mapping $\hat{T}:\Omega\times H\to\mathscr{F}(H)$ satisfy the condition (I) and $T:\Omega\times H\to CB(H)$ be the random multi-valued mapping induced by the random fuzzy mapping $\hat{T}$. Let $g:\Omega\times H\to H$ be a random mapping and $\phi:H\times H\to\mathbb{R}\cup\{+\infty\}$ be a real valued function such that for each $t\in\Omega$, the measurable mapping T is strongly g-monotone and $\hat{H}$-Lipschitz continuous with the measurable functions $\theta,\lambda:\Omega\to(0,+\infty)$, respectively, then for any $x(t)\in H$, $t\in\Omega$, we have*

$$\left\|x(t)-x_o(t)\right\|\le\frac{1}{\sqrt{\theta(t)+\frac{L^2(t)}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)}}\sqrt{D_{\alpha,\beta}(x(t))}.$$

Proof. From (4.1), it can be written as

$$\begin{aligned}D_{\alpha,\beta}(x(t))\ge&\left\langle w(t),g(t,x(t))-g(t,x_o(t))\right\rangle\\&-\phi\left(g(t,x(t)),g(t,x_o(t))\right)+\phi\left(g(t,x(t)),g(t,x(t))\right)\\&+\frac{1}{2\beta}\left\|g(t,x(t))-g(t,x_o(t))\right\|^2-\frac{1}{2\alpha}\left\|g(t,x(t))-g(t,x_o(t))\right\|^2.\end{aligned}$$

By using the strong g-monotonicity of T, we get

$$\begin{aligned} D_{\alpha,\beta}(x(t)) \geq & \Big\langle w(t)-w_o(t), g(t,x(t))-g(t,x_o(t))\Big\rangle + \Big\langle w_o(t), g(t,x(t))-g(t,x_o(t))\Big\rangle \\ & -\phi\big(g(t,x(t)),g(t,x_o(t))\big)+\phi\big(g(t,x(t)),g(t,x(t))\big) \\ & +\frac{1}{2\beta}\big\|g(t,x(t))-g(t,x_o(t))\big\|^2-\frac{1}{2\alpha}\big\|g(t,x(t))-g(t,x_o(t)\big\|^2. \\ \geq & \Big\langle w_o(t), g(t,x(t))-g(t,x_o(t))\Big\rangle + \theta(t)\big\|x(t)-x_o(t)\big\|^2 \\ & -\phi\big(g(t,x(t)),g(t,x_o(t))\big)+\phi\big(g(t,x(t)),g(t,x(t))\big) \\ & +\frac{1}{2\beta}\big\|g(t,x(t))-g(t,x_o(t))\big\|^2-\frac{1}{2\alpha}\big\|g(t,x(t))-g(t,x_o(t)\big\|^2. \end{aligned} \tag{4.7}$$

Since $x_o(t)\in H$ is a solution of RGMVIP (1.1)

$$\Big\langle w_o(t), y(t)-g(t,x_o(t))\Big\rangle + \phi\big(g(t,x_o(t)),y(t)\big)-\phi\big(g(t,x_o(t)),g(t,x_o(t))\big)\geq 0.$$

Taking $y(t)=g(t,x(t))$ in the above inequality

$$\begin{aligned} &\Big\langle w_o(t), g(t,x(t))-g(t,x_o(t))\Big\rangle + \phi\big(g(t,x_o(t)),g(t,x(t))\big) \\ &\qquad -\phi\big(g(t,x_o(t)),g(t,x_o(t))\big)\geq 0. \end{aligned} \tag{4.8}$$

Combining (4.7) with (4.8) and using the skew symmetry of ϕ, we get

$$\begin{aligned} D_{\alpha,\beta}(x(t)) \geq & \,\theta(t)\big\|x(t)-x_o(t)\big\|^2+\frac{1}{2\beta}\big\|g(t,x(t))-g(t,x_o(t))\big\|^2 \\ & -\frac{1}{2\alpha}\big\|g(t,x(t))-g(t,x_o(t))\big\|^2. \\ \geq & \,\theta(t)\big\|x(t)-x_o(t)\big\|^2+\frac{1}{2\beta}L^2(t)\big\|x(t)-x_o(t)\big\|^2 \\ & -\frac{1}{2\alpha}L^2(t)\big\|x(t)-x_o(t)\big\|^2 \\ D_{\alpha,\beta}(x(t)) \geq & \left(\theta(t)+\frac{L^2(t)}{2\beta}-\frac{L^2(t)}{2\alpha}\right)\big\|x(t)-x_o(t)\big\|^2 \end{aligned}$$

which implies

$$\begin{aligned} \big\|x(t)-x_o(t)\big\| \leq & \frac{1}{\sqrt{\theta(t)+\frac{L^2(t)}{2\beta}-\frac{L^2(t)}{2\alpha}}}\sqrt{D_{\alpha,\beta}(t,x(t))} \\ \leq & \frac{1}{\sqrt{\theta(t)+\frac{L^2(t)}{2}\big(\frac{1}{\beta}-\frac{1}{\alpha}\big)}}\sqrt{D_{\alpha,\beta}(t,x(t))}. \end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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