Available online at http://scik.org

Adv. Fixed Point Theory, 7 (2017), No. 2, 243-265

ISSN: 1927-6303

COMMON FIXED POINTS OF TWO PAIRS OF SELFMAPS SATISFYING A CONTRACTIVE CONDITION WITH RATIONAL EXPRESSION

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Abstract. In this paper, we prove the existence of common fixed points of two pairs of selfmaps under the assumptions that these two pairs of maps are weakly compatible and satisfying a contractive condition involving rational expression in a complete metric space. The same is extended to a sequence of selfmaps. Also, we prove the same with different hypotheses on two pairs of selfmaps in which one pair is compatible, reciprocally continuous and the other one is weakly compatible. We also discuss the importance of rational expression in our contractive condition. Our theorems extend the results of Chandok [1] to two pairs of selfmaps.

Keywords: common fixed points; complete metric space; compatible maps; weakly compatible maps and reciprocally continuous maps.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambiant spaces of the operator under consideration on

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Received December 28, 2016

the other. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful results in fixed point theory. Banach contraction principle has been generalized in various ways either by using contractive conditions or by imposing some additional conditions on the ambiant spaces. In the direction of generalization of contraction conditions, in 1975, Dass and Gupta [2] established fixed point results using contraction conditions involving rational expressions.

The following theorem is due to Dass and Gupta [2].

Theorem 1.1. [2] Let (X,d) be a complete metric space and $T : X \to X$ a mapping such that there exist $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}$$

for all $x, y \in X$. Then T has a unique fixed point.

Definition 1.1. [3] Let A and B be selfmaps of a metric space (X,d). The pair (A,B) is said to be a compatible pair on X, if $\lim\limits_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = t$, for some $t \in X$.

Definition 1.2. [4] Let A and B be selfmaps of a metric space (X,d). The pair (A,B) is said to be weakly compatible, if they commute at their coincidence points. i.e., $ABx = BAx$ whenever $Ax = Bx, x \in X$.

Every compatible pair of maps is weakly compatible, but its converse need not true [4].

Definition 1.3. [5] Let A and B be selfmaps of a metric space (X,d). Then A and B are said to be reciprocally continuous, if $\lim\limits_{n\to\infty} ABx_n = At$ and $\lim\limits_{n\to\infty} BAx_n = Bt$, whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = t$, for some $t \in X$.

Clearly, if A and B are continuous then they are reciprocally continuous but its converse need not be true [5].

Recently, Chandok [1] established the following common fixed point result of selfmaps satisfying certain contraction condition involving rational expression.

Theorem 1.2. (Theorem 2.1, [1]) Let M be a subset of a metric space (X,d). Suppose that $T, f, g : M \to M$ satisfy

$$d(Tx, fy) \leq \alpha\left(\frac{d(gx,Tx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,Tx)}\right) + \beta(d(gx,gy))$$

for all $x, y \in M$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$.
Suppose also that $T(M) \cup f(M) \subseteq g(M)$ and $(g(M), d)$ is complete.
Then T, f and g have a coincidence point in M. Also, if the pairs (g, T) and (g, f) are weakly compatible, then T, f and g have a unique common fixed point in X.

Throughout this paper, we denote $\mathbb{R}_+ = [0, \infty)$ and

$$\Psi = \{\psi / \psi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is continuous}, \psi \text{ is nondecreasing}, \psi(t) < t \text{ for } t > 0 \text{ and } \psi(t) = 0 \Leftrightarrow t = 0\}.$$

In Section 2, we prove the existence of common fixed points for two pairs of selfmaps under the assumptions that these two pairs of maps are weakly compatible and satisfying a contractive condition involving rational expression in a complete metric space. Also, we prove the same with different hypotheses on two pairs of selfmaps in which one pair is compatible, reciprocally continuous and the other one is weakly compatible. In section 3, we draw some corollaries from our main results and provide examples in support of our results and discuss the importance of rational expression in our contractive condition (Example 2).

2. Main results

Let A, B, S and T be mappings from a metric space (X, d) into itself and satisfying

(1) $$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X).$$

Now by (A), for any $x_0 \in X$, there exists $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. In the same way for this x_1, we can choose a point $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$ and so on. In general, we can define a sequence $\{y_n\}$ in X such that

(2) $$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and} y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0, 1, 2, \ldots.$$

Lemma 2.1. Let (X,d) be a metric space. Assume that A,B,S and T are selfmaps of X which satisfy the following condition: there exists $\psi \in \Psi$ such that

$$d(Ax,By) \leq \begin{cases} \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}), \\ \qquad\qquad \text{if} \quad d(Sx,Ty)+d(Sx,By)+d(Ty,Ax) \neq 0 \\ 0, \qquad\qquad \text{if} \quad d(Sx,Ty)+d(Sx,By)+d(Ty,Ax) = 0 \end{cases} \tag{3}$$

for all $x,y \in X$. Then we have the following:

(i) If $A(X) \subseteq T(X)$ and the pair (B,T) is weakly compatible, and if z is a common fixed point of A and S then z is a common fixed point of A,B,S and T and it is unique.

(ii) If $B(X) \subseteq S(X)$ and the pair (A,S) is weakly compatible, and if z is a common fixed point of B and T then z is a common fixed point of A,B,S and T and it is unique.

Proof. First, we assume that (i) holds. Let z be a common fixed point of A and S.

Then $Az = Sz = z$. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = z$.

Therefore $Az = Sz = Tu = z$. We now prove that $Az = Bu$. Suppose that $Az \neq Bu$.

We consider,

$d(Az,Bu) \leq \psi(\max\{\frac{d(Sz,Az)d(Tu,Bu)}{d(Sz,Tu)+d(Sz,Bu)+d(Tu,Az)}, d(Sz,Tu)\}) = \psi(\max\{0,0\}) = \psi(0) = 0.$

Therefore, $Az = Bu$.

Hence $Az = Bu = Sz = Tu = z$.

Since the pair (B,T) is weakly compatible and $Tu = Bu$, we have

$BTu = TBu$. i.e., $Bz = Tz$.

Now we prove that $Bz = z$. If $Bz \neq z$, then

$$\begin{aligned} d(Bz,z) = d(z,Bz) &= d(Az,Bz) \\ &\leq \psi(\max\{\frac{d(Sz,Az)d(Tz,Bz)}{d(Sz,Tz)+d(Sz,Bz)+d(Tz,Az)}, d(Sz,Tz)\}) \\ &= \psi(\max\{0,d(z,Bz)\}) = \psi(d(z,Bz)) < d(z,Bz), \end{aligned}$$

a contradiction. Hence, $Bz = z$.

Therefore $Bz = Tz = z$.

Hence $Az = Bz = Sz = Tz = z$.

Therefore, z is a common fixed point of A,B,S and T.

If z' is also a common fixed point of A,B,S and T with $z \neq z'$, then

$$d(z,z') = d(Az,Bz') \leq \psi(\max\{\frac{d(Sz,Az)d(Tz',Bz')}{d(Sz,Tz')+d(Sz,Bz')+d(Tz',Az)}, d(Sz,Tz')\})$$
$$= \psi(\max\{0,d(z,z')\}) = \psi(d(z,z')) < d(z,z'),$$

a contradiction. Therefore, $z = z'$.

Hence, z is the unique common fixed point of A,B,S and T.

The proof of (ii) is similar to (i) and hence is omitted.

Lemma 2.2. Let A,B,S and T be selfmaps of a metric space (X,d) and satisfy (1) and the inequality (3). Then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by (2) is Cauchy in X.

Proof. Let $x_0 \in X$ and let $\{y_n\}$ be a sequence defined by (2).

Assume that $y_n = y_{n+1}$ for some n.

Case (i): n even.

We write $n = 2m, m \in \mathbb{N}$.

Now we consider

$$\begin{aligned}
d(y_{n+1},y_{n+2}) &= d(y_{2m+1},y_{2m+2}) \\
&= d(y_{2m+2},y_{2m+1}) \\
&= d(Ax_{2m+2},Bx_{2m+1}) \\
&\leq \psi(\max\{\frac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+1},Bx_{2m+1})}{d(Sx_{2m+2},Tx_{2m+1})+d(Sx_{2m+2},Bx_{2m+1})+d(Tx_{2m+1},Ax_{2m+2})}, \\
&\qquad d(Sx_{2m+2},Tx_{2m+1}) \\
&= \psi(\max\{\frac{d(y_{2m+1},y_{2m+2})d(y_{2m},y_{2m+1})}{d(y_{2m+1},y_{2m})+d(y_{2m+1},y_{2m+1})+d(y_{2m},y_{2m+2})}, \\
&\qquad d(y_{2m+1},y_{2m})\}) \\
&\leq \psi(\max\{\frac{d(y_{2m+1},y_{2m+2})d(y_{2m},y_{2m+1})}{d(y_{2m+1},y_{2m+2})}, d(y_{2m+1},y_{2m})\}) \\
&= \psi(d(y_{2m+1},y_{2m})) = \psi(0) = 0.
\end{aligned}$$

Therefore, $d(y_{2m+1},y_{2m+2}) = 0$ which implies that $y_{2m+2} = y_{2m+1} = y_{2m}$.

In general, we have $y_{2m+k} = y_{2m}$ for $k = 0,1,2,\ldots$.

Case (ii): n odd.

We write $n=2m+1$ for some $m\in\mathbb{N}$.

We consider

$$\begin{aligned}
d(y_{n+1},y_{n+2}) &= d(y_{2m+2},y_{2m+3})\\
&= d(Ax_{2m+2},Bx_{2m+3})\\
&\leq \psi(\max\{\frac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+3},Bx_{2m+3})}{d(Sx_{2m+2},Tx_{2m+3})+d(Sx_{2m+2},Bx_{2m+3})+d(Tx_{2m+3},Ax_{2m+2})},\\
&\qquad d(Sx_{2m+2},Tx_{2m+3})\})
\end{aligned}$$

$$\begin{aligned}
&= \psi(\max\{\frac{d(y_{2m+1},y_{2m+2})d(y_{2m+2},y_{2m+3})}{d(y_{2m+1},y_{2m+2})+d(y_{2m+1},y_{2m+3})+d(y_{2m+2},y_{2m+2})},\\
&\qquad d(y_{2m+1},y_{2m+2})\})\\
&\leq \psi(\max\{\frac{d(y_{2m+1},y_{2m+2})d(y_{2m+2},y_{2m+3})}{d(y_{2m+2},y_{2m+3})},d(y_{2m+1},y_{2m+2})\})\\
&= \psi(d(y_{2m+1},y_{2m+2}))=\psi(0)=0.
\end{aligned}$$

Therefore, $d(y_{n+2},y_{n+3})=0$ implies that $y_{2m+3}=y_{2m+2}=y_{2m+1}$.

In general, we have $y_{2m+k}=y_{2m+1}$ for $k=1,2,3,\dots$.

From Case (i) and Case (ii), we have $y_{n+k}=y_n$ for $k=0,1,2,\dots$.

Hence, $\{y_{n+k}\}$ is a constant sequence and hence $\{y_n\}$ is Cauchy.

Now we assume that $y_n\neq y_{n+1},$ for all $n\in\mathbb{N}$. If n is odd, then $n=2m+1$ for some $m\in\mathbb{N}$.

We now consider

$$\begin{aligned}
d(y_n,y_{n+1}) &= d(y_{2m+1},y_{2m+2})\\
&= d(y_{2m+2},y_{2m+1})\\
&= d(Ax_{2m+2},Bx_{2m+1})\\
&\leq \psi(\max\{\frac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+1},Bx_{2m+1})}{d(Sx_{2m+2},Tx_{2m+1})+d(Sx_{2m+2},Bx_{2m+1})+d(Tx_{2m+1},Ax_{2m+2})},\\
&\qquad d(Sx_{2m+2},Tx_{2m+1})\})\\
&= \psi(\max\{\frac{d(y_{2m+1},y_{2m+2})d(y_{2m},y_{2m+1})}{d(y_{2m+1},y_{2m})+d(y_{2m+1},y_{2m+1})+d(y_{2m},y_{2m+2})},\\
&\qquad d(y_{2m+1},y_{2m})\})
\end{aligned}$$

$$\leq \psi(\max\{\frac{d(y_{2m+1},y_{2m+2})d(y_{2m},y_{2m+1})}{d(y_{2m+1},y_{2m+2})},d(y_{2m+1},y_{2m})\})$$

$$= \psi(d(y_{2m+1},y_{2m})) < d(y_{2m+1},y_{2m})$$

Therefore, $d(y_n,y_{n+1}) < d(y_{n-1},y_n)$.

On the similar lines, if n is even, it follows that

$$d(y_n,y_{n+1}) \leq \psi(d(y_{n-1},y_n)) < d(y_{n-1},y_n). \tag{4}$$

Therefore, $\{d(y_n,y_{n+1})\}$ is a monotone decreasing sequence which bounded below by 0.

So, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(y_n,y_{n+1}) = r$.

If $r > 0$, then from (4), we have

$d(y_n,y_{n+1}) \leq \psi(d(y_{n-1},y_n))$.

Letting $n \to \infty$, we get

$r \leq \psi(r) < r$,

a contradiction.

Therefore

$$\lim_{n\to\infty} d(y_n,y_{n+1}) = 0. \tag{5}$$

We now prove that $\{y_n\}$ is Cauchy.

It is sufficient to show that $\{y_{2n}\}$ is Cauchy in X.

Otherwise, there is an $\varepsilon > 0$ and there exist sequences $\{2m_k\},\{2n_k\}$ with $2m_k > 2n_k > k$ such that

$$d(y_{2m_k},y_{2n_k}) \geq \varepsilon \text{ and } d(y_{2m_k-2},y_{2n_k}) < \varepsilon. \tag{6}$$

Now, we prove that (i) $\lim_{k\to\infty} d(y_{2m_k},y_{2n_k}) = \varepsilon$.

Since $\varepsilon \leq d(y_{2m_k},y_{2n_k})$ for all k, we have

$\varepsilon \leq \liminf_{k\to\infty} d(y_{2m_k},y_{2n_k})$.

Now for each positive integer k, by the triangular inequality, we get

$d(y_{2m_k},y_{2n_k}) \leq d(y_{2m_k},y_{2m_k-1}) + d(y_{2m_k-1},y_{2m_k-2}) + d(y_{2m_k-2},y_{2n_k})$.

On taking limit superior as $k \to \infty$, from (5) and (6), we have

$\limsup_{k\to\infty} d(y_{2m_k},y_{2n_k}) \leq \varepsilon$.

Hence, $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k})$ exists and $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \varepsilon$.

In similar way, it is easy to see that

(ii) $\lim\limits_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) = \varepsilon$; (iii) $\lim\limits_{k\to\infty} d(y_{2m_k}, y_{2n_k-1}) = \varepsilon$ and (iv) $\lim\limits_{k\to\infty} d(y_{2n_k-1}, y_{2m_k+1}) = \varepsilon$.

We now consider

$$
\begin{aligned}
d(y_{2n_k}, y_{2m_k+1}) &= d(Ax_{2n_k}, Bx_{2m_k+1}) \\
&\leq \psi(\max\{\frac{d(Sx_{2n_k}, Ax_{2n_k})d(Tx_{2m_k+1}, Bx_{2m_k+1})}{d(Sx_{2n_k}, Tx_{2m_k+1}) + d(Sx_{2n_k}, Bx_{2m_k+1}) + d(Tx_{2m_k+1}, Ax_{2n_k})}, \\
&\qquad d(Sx_{2n_k}, Tx_{2m_k+1})\}) \\
&= \psi(\max\{\frac{d(y_{2n_k-1}, y_{2n_k})d(y_{2m_k}, y_{2m_k+1})}{d(y_{2n_k-1}, y_{2m_k}) + d(y_{2n_k-1}, y_{2m_k+1}) + d(y_{2m_k}, y_{2n_k})}, \\
&\qquad d(y_{2n_k-1}, y_{2m_k})\}).
\end{aligned} \tag{7}
$$

On letting $k \to \infty$ in (7), we get $\varepsilon \leq \psi(\max\{0, \varepsilon\}) = \psi(\varepsilon) < \varepsilon$,

a contradiction.

Therefore, $\{y_n\}$ is a Cauchy sequence in X.

The following is the main result of this paper.

Theorem 2.3. Let A, B, S and T be selfmaps on a complete metric space (X, d) and satisfy (1) and the inequality (3). If the pairs (A, S) and (B, T) are weakly compatible and one of the range sets $S(X), T(X), A(X)$ and $B(X)$ is closed, then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by (2) is Cauchy in X and $\lim\limits_{n\to\infty} y_n = z$(say), $z \in X$ and z is the unique common fixed point of A, B, S and T.

Proof. By Lemma 2.2, the sequence $\{y_n\}$ is Cauchy in X.

Since X is complete, there exists $z \in X$ such that $\lim\limits_{n\to\infty} y_n = z$. Thus,

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \tag{8}$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{9}$$

We now consider the following four cases.

Case (i). $S(X)$ is closed.

In this case $z \in S(X)$ and there exists $u \in X$ such that $z = Su$.

Now we claim that $Au = z$. Suppose that $Au \neq z$.

We now consider

$d(Au, Bx_{2n+1}) \leq \psi(\max\{\frac{d(Su,Au)d(Tx_{2n+1},Bx_{2n+1})}{d(Su,Tx_{2n+1})+d(Su,Bx_{2n+1})+d(Tx_{2n+1},Au)}, d(Su, Tx_{2n+1})\})$.

On letting $n \to \infty$, using (8) and (9), we get

$d(Au, z) \leq \psi(\max\{0, 0\}) = \psi(0) = 0$.

Hence $Au = z$.

Therefore,

$$Au = z = Su. \tag{10}$$

Since the pair (A, S) is weakly compatible and $Au = Su$, we have

$ASu = SAu$. i.e., $Az = Sz$.

Now, we prove that $Az = z$.

If $Az \neq z$, then

$d(Az, Bx_{2n+1}) \leq \psi(\max\{\frac{d(Sz,Az)d(Tx_{2n+1},Bx_{2n+1})}{d(Sz,Tx_{2n+1})+d(Sz,Bx_{2n+1})+d(Tx_{2n+1},Az)}, d(Sz, Tx_{2n+1})\})$.

On letting $n \to \infty$, using (8) and (9), we get

$d(Az, z) \leq \psi(\max\{0, d(Az, z)\}) = \psi(d(Az, z)) < d(Az, z)$,

a contradiction. Hence, $Az = z$.

Therefore, $Az = Sz = z$.

Hence, z is a common fixed point of A and S.

By Lemma 2.1, we get that z is a unique common fixed point of A, B, S and T.

Case (ii). $T(X)$ is closed.

In this case $z \in T(X)$ and there exists $u \in X$ such that $z = Tu$.

Now we claim that $Bu = z$. Suppose that $Bu \neq z$.

We now consider

$d(Ax_{2n+2}, Bu) \leq \psi(\max\{\frac{d(Sx_{2n+2},Ax_{2n+2})d(Tu,Bu)}{d(Sx_{2n+2},Tu)+d(Sx_{2n+2},Bu)+d(Tu,Ax_{2n+2})}, d(Sx_{2n+2}, Tu)\})$.

On letting $n \to \infty$, using (8) and (9), we get

$d(z, Bu) \leq \psi(\max\{0, 0\}) = \psi(0) = 0$ and hence $Bu = z$.

Therefore,

$$Bu = z = Tu. \tag{11}$$

Since the pair (B,T) is weakly compatible and $Bu = Tu$, we have

$BTu = TBu$. i.e., $Bz = Tz$.

Now, we prove that $Bz = z$. If $Bz \neq z$, then

$d(Ax_{2n+2}, Bz) \leq \psi(\max\{\frac{d(Sx_{2n+2},Ax_{2n+2})d(Tz,Bz)}{d(Sx_{2n+2},Tz)+d(Sx_{2n+2},Bz)+d(Tz,Ax_{2n+2})}, d(Sx_{2n+2}, Tz)\})$.

On letting $n \to \infty$, using (8) and (9), we get

$d(z,Bz) \leq \psi(\max\{0, d(z,Bz)\}) = \psi(d(z,Bz)) < d(z,Bz)$,

a contradiction. Hence, $Bz = z$ and that $Bz = Tz = z$.

Therefore, z is a common fixed point of B and T.

Hence, by Lemma 2.1, we get that z is the unique common fixed point of A, B, S and T.

Case (iii). $A(X)$ is closed.

Since $z \in A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Tu$.

Now we show that $Bu = z$.

If $Bu \neq z$, then we consider

$d(Ax_{2n+2}, Bu) \leq \psi(\max\{\frac{d(Sx_{2n+2},Ax_{2n+2})d(Tu,Bu)}{d(Sx_{2n+2},Tu)+d(Sx_{2n+2},Bu)+d(Tu,Ax_{2n+2})}, d(Sx_{2n+2}, Tu)\})$.

On letting $n \to \infty$, using (8) and (9), we get

$d(z,Bu) \leq \psi(\max\{0,0\}) = \psi(0) = 0$ and hence $Bu = z$.

Therefore $Bu = z = Tu$. Thus (11) holds. Now by Case *(ii)*, the conclusion of the theorem follows.

Case (iv). $B(X)$ is closed.

Since $z \in B(X) \subseteq S(X)$, there exists $u \in X$ such that $z = Su$.

Now we show that $Au = z$.

If $Au \neq z$, then we consider

$d(Au, Bx_{2n+1}) \leq \psi(\max\{\frac{d(Su,Au)d(Tx_{2n+1},Bx_{2n+1})}{d(Su,Tx_{2n+1})+d(Su,Bx_{2n+1})+d(Tx_{2n+1},Au)}, d(Su, Tx_{2n+1})\})$.

On letting $n \to \infty$, using (8) and (9), we get

$d(Au,z) \leq \psi(\max\{0,0\}) = \psi(0) = 0$ and hence $Au = z$.

Therefore $Au = z = Su$. Thus (10) holds.

Now by Case *(i)*, the conclusion of the theorem follows.

Theorem 2.4. Let A, B, S and T be selfmaps on a metric space (X,d) and satisfy (1) and the inequality (3). If the pairs (A,S) and (B,T) are weakly compatible and either one of the set

$(S(X),d),(T(X),d),(A(X),d)$ (or) $(B(X),d)$ is complete, then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by (2) is Cauchy in X and $\lim_{n\to\infty} y_n = z$(say), $z \in X$ and z is the unique common fixed point of A,B,S and T

Proof. By Lemma 2.2, the sequence $\{y_n\}$ is Cauchy in X.

Since $S(X)$ is complete, there exists $z \in S(X)$ such that $\lim_{n\to\infty} y_n = z$. Thus,

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \tag{12}$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{13}$$

Since $z \in S(X)$, there exists $u \in X$ such that $z = Su$.

We now prove that $Au = z$. If $Au \neq z$, then

$d(Au,Bx_{2n+1}) \leq \psi(\max\{\frac{d(Su,Au)d(Tx_{2n+1},Bx_{2n+1})}{d(Su,Tx_{2n+1})+d(Su,Bx_{2n+1})+d(Tx_{2n+1},Au)}, d(Su,Tx_{2n+1})\})$.

On letting $n \to \infty$, using (12) and (13), we get

$d(Au,z) \leq \psi(\max\{0,0\}) = \psi(0) = 0$ and hence $Au = z$.

Therefore, $Au = z = Su$.

Since the pair (A,S) is weakly compatible and $Au = Su$, we have

$ASu = SAu$. i.e., $Az = Sz$.

Now, we prove that $Az = z$. If suppose that $Az \neq z$, then

$d(Az,Bx_{2n+1}) \leq \psi(\max\{\frac{d(Sz,Az)d(Tx_{2n+1},Bx_{2n+1})}{d(Sz,Tx_{2n+1})+d(Sz,Bx_{2n+1})+d(Tx_{2n+1},Az)}, d(Sz,Tx_{2n+1})\})$.

On letting $n \to \infty$, using (12) and (13), we get

$d(Az,z) \leq \psi(\max\{0,d(Az,z)\}) = \psi(d(Az,z)) < d(Az,z)$,

a contradiction. Hence, $Az = z$. Therefore $Az = Sz = z$.

Thus, z is a common fixed point of A and S.

By Lemma 2.1, we get z is the unique common fixed point of A,B,S and T.

In a similar way, it is easy to see that z is the unique common fixed point of A,B,S and T when either $T(X)$ or $A(X)$ or $B(X)$ is complete.

Theorem 2.5. Let A,B,S and T be selfmaps on a complete metric space (X,d) and satisfy (1) and the inequality (3). Further assume that either

(i) (A,S) is reciprocally continuous and compatible pair of maps, and (B,T) is a pair of weakly compatible maps (or)

(ii) (B,T) is reciprocally continuous and compatible pair of maps, and (A,S) is a pair of weakly compatible maps.

Then A,B,S and T have a unique common fixed point.

Proof. By Lemma 2.2, for each $x_0 \in X$, the sequence $\{y_n\}$ defined by (2) is Cauchy in X.
Since X is complete, then there exists $z \in X$ such that $\lim\limits_{n\to\infty} y_n = z$.
Consequently, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also converges to $z \in X$, we have

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z, \text{and} \tag{14}$$

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{15}$$

First, we assume that (i) holds.

Since (A,S) is reciprocal continuous, it follows that
$\lim\limits_{n\to\infty} ASx_{2n+2} = Az$ and $\lim\limits_{n\to\infty} SAx_{2n+2} = Sz$.
Since (A,S) is compatible, we have
$\lim\limits_{n\to\infty} d(ASx_{2n+2}, SAx_{2n+2}) = 0$
which implies that $\lim\limits_{n\to\infty} d(Az,Sz) = 0$ implies that $Az = Sz$.
Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Az = Tu$.

Therefore, $Az = Sz = Tu$.

Now, we prove that $Az = Bu$. Suppose that $Az \neq Bu$.

We now consider

$$\begin{aligned} d(Az,Bu) &\leq \psi(\max\{\frac{d(Sz,Az)d(Tu,Bu)}{d(Sz,Tu)+d(Sz,Bu)+d(Tu,Az)}, d(Sz,Tu)\}) \\ &= \psi(\max\{0,0\}) = \psi(0) = 0. \end{aligned}$$

This implies that $Az = Bu = Sz = Tu$.

Since every compatible pair is weakly compatible, we have (A,S) is weakly compatible and $Az = Sz$, we have $ASz = SAz$. i.e $,AAz = SAz$.

Now, we prove that $AAz = Az$. If possible, suppose that $AAz \neq Az$.

Now, we consider

$$d(AAz,Az) = d(AAz,Bu) \leq \psi(\max\{\frac{d(SAz,AAz)d(Tu,Bu)}{d(SAz,Tu)+d(SAz,Bu)+d(Tu,AAz)},d(SAz,Tu)\})$$

$$= \psi(\max\{0,d(AAz,Az)\}) \leq \psi(d(AAz,Az)) < d(AAz,Az),$$

a contradiction.

Therefore $AAz = Az$. Hence, $AAz = SAz = Az$, so that Az is a common fixed point of A and S.

Since (B,T) is weakly compatible and $Bu = Tu$, we have $BTu = TBu$.

Therefore $BAz = TAz$.

We now prove that $BAz = Az$. Suppose that $BAz \neq Az$.

We now consider

$$d(BAz,Az) = d(Az,BAz) \leq \psi(\max\{\frac{d(Sz,Az)d(TAz,BAz)}{d(Sz,TAz)+d(Sz,BAz)+d(TAz,Az)},d(Sz,TAz)\})$$

$$= \psi(\max\{0,d(Az,BAz)\}) = \psi(d(Az,BAz)) < d(Az,BAz),$$

a contradiction.

Hence, $BAz = Az$. Therefore $BAz = TAz = Az$.

Hence, $AAz = BAz = SAz = TAz = Az$.

Therefore Az is a common fixed point of A,B,S and T.

Now, we show that $Az = z$. If $Az \neq z$, then

$$d(Az,Bx_{2n+1}) \leq \psi(\max\{\frac{d(Sz,Az)d(Tx_{2n+1},Bx_{2n+1})}{d(Sz,Tx_{2n+1})+d(Sz,Bx_{2n+1})+d(Tx_{2n+1},Az)},d(Sz,Tx_{2n+1})\})$$

On letting $n \to \infty$, using (14) and (15), we get

$d(Az,z) \leq \psi(\max\{0,d(Az,z)\}) = \psi(d(Az,z)) < d(Az,z)$,

a contradiction.

Hence, $Az = z$. Therefore $Az = Bz = Sz = Tz = z$.

Hence, z is a common fixed point of A,B,S and T.

In a similar way, under the assumption (ii), we obtain the existence of common fixed point of A,B,S and T. Uniqueness of common fixed point follows from the inequality (3).

Theorem 2.6. Let A,B,S and T be selfmaps on a complete metric space (X,d) and satisfy (1) and the inequality (3). If either

(i) S is continuous, (A,S) compatible and (B,T) is weakly compatible (or)

(ii) T is continuous, (B,T) compatible and (A,S) is a pair of weakly compatible maps, then A,B,S and T have a unique common fixed point.

Proof. By Lemma 2.2, for each $x_0 \in X$, the sequence $\{y_n\}$ defined by (2) is Cauchy in X.
Since X is complete, then there exists $z \in X$ such that $\lim\limits_{n\to\infty} y_n = z$.
Consequently, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also converges to $z \in X$, we have

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z, \tag{16}$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{17}$$

First, we assume that (i) holds.

Since (A,S) is compatible pair, we have
$\lim\limits_{n\to\infty} d(SAx_{2n}, ASx_{2n}) = 0$, it follows that
$\lim\limits_{n\to\infty} SAx_{2n} = \lim\limits_{n\to\infty} ASx_{2n}$.
Since S is continuous, we have $Sz = \lim\limits_{n\to\infty} SAx_{2n} = \lim\limits_{n\to\infty} ASx_{2n}$
Now, we prove that $Sz = z$. If $Sz \neq z$, then consider
$d(ASx_{2n+2}, Bx_{2n+1}) \leq \psi(\max\{\frac{d(SSx_{2n+2},ASx_{2n+2})d(Tx_{2n+1},Bx_{2n+1})}{d(SSx_{2n+2},Tx_{2n+1})+d(SSx_{2n+2},Bx_{2n+1})+d(Tx_{2n+1},ASx_{2n+2})}, d(SSx_{2n+2}, Tx_{2n+1})\})$
On letting $n \to \infty$, using (16) and (17), we get
$d(Sz,z) \leq \psi(\max\{\frac{d(Sz,Sz)d(z,z)}{d(Sz,z)+d(Sz,z)+d(z,Sz)}, d(Sz,z)\}) = \psi(d(Sz,z)) < d(Sz,z)$,
a contradiction. Hence, $Sz = z$.

We now prove that $Az = z$. If possible, suppose that $Az \neq z$.

Now we consider
$d(Az, Bx_{2n+1}) \leq \psi(\max\{\frac{d(Sz,Az)d(Tx_{2n+1},Bx_{2n+1})}{d(Sz,Tx_{2n+1})+d(Sz,Bx_{2n+1})+d(Tx_{2n+1},Az)}, d(Sz, Tx_{2n+1})\})$
On taking limits as $n \to \infty$, using (16) and (17), we get
$d(Az,z) \leq \psi(\max\{\frac{d(z,Az)d(z,z)}{d(z,z)+d(z,z)+d(z,Az)}, d(z,z)\}) = \psi(\max\{0,0\}) = \psi(0) = 0$.
Therefore $d(Az,z) \leq 0$ which implies that $Az = z$.

Hence $Az = Sz = z$.

Therefore z is a common fixed point of A and S.

Hence, by Lemma 2.1, we get that z is a common fixed point of A,B,S and T.

In a similar way, under the assumption (ii), we can obtain the existence of common fixed point of A,B,S and T. Uniqueness of common fixed point follows from the inequality (3).

3. Corollaries and examples

In this section, we draw some corollaries from the main results of Section 2 and provide examples in support of our results.

The following is an example in support of Theorem 2.3.

Example 1. Let $X=[0,1]$ with usual metric. We define selfmaps $A,B,S,$ T on X by

$$A(x)=\begin{cases}\frac{x^2}{2} & \text{if } 0\leq x<\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x\leq 1\end{cases},\ B(x)=\begin{cases}\frac{x^2}{4} & \text{if } 0\leq x<\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x\leq 1\end{cases}$$

$$S(x)=\begin{cases}x^2 & \text{if } 0\leq x<\frac{1}{2}\\ 1 & \text{if } \frac{1}{2}\leq x\leq 1\end{cases}\ \text{and } T(x)=\begin{cases}\frac{x^2}{2} & \text{if } 0\leq x<\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x\leq 1.\end{cases}$$

We define $\psi:\mathbb{R}_+\to\mathbb{R}_+$ by $\psi(t)=\frac{t}{2},\ t\geq 0$. Then clearly $\psi\in\Psi$.

Here $A(X)=[0,\frac{1}{8}),B(X)=[0,\frac{1}{16})$,

$S(X)=[0,\frac{1}{4})\cup\{1\}$ and $T(X)=[0,\frac{1}{8})$.

So that $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$.

We now verify the inequality (3).

Without loss of generality assume that $x\geq y$.

<u>*Case* (i)</u>: $x,y\in[0,\frac{1}{2})$.

$d(Ax,By)=|\frac{x^2}{2}-\frac{y^2}{4}|;d(Sx,Ty)=|x^2-\frac{y^2}{2}|$.

We consider

$$\begin{aligned}d(Ax,By)=\frac{1}{2}|x^2-\frac{y^2}{2}|&=\frac{1}{2}d(Sx,Ty)\\ &=\psi(d(Sx,Ty))\\ &\leq\psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)},d(Sx,Ty)\}).\end{aligned}$$

<u>*Case* (ii)</u>: $x,y\in[\frac{1}{2},1]$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

<u>*Case* (iii)</u>: $x\in[0,\frac{1}{2}),y\in[\frac{1}{2},1]$.

$d(Ax,By)=\frac{x^2}{2};d(Sx,Ty)=x^2$.

We have

$$d(Ax,By) = \frac{x^2}{2} = \frac{1}{2}d(Sx,Ty) = \psi(d(Sx,Ty))$$
$$\leq \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

Case (iv): $x \in [\frac{1}{2},1], y \in [0,\frac{1}{2})$.
$d(Ax,By) = \frac{y^2}{4}; d(Sx,Ty) = (1-\frac{y^2}{2})$.
We now consider

$$d(Ax,By) = \frac{y^2}{4} \leq \frac{1}{2}(1-\frac{y^2}{2}) = \frac{1}{2}d(Sx,Ty)$$
$$= \psi(d(Sx,Ty))$$
$$\leq \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

From the above four cases, A,B,S and T satisfy the inequality (3).

Therefore A,B,S and T satisfy all the hypotheses of Theorem 2.3 and 0 is the unique common fixed point of A,B,S and T.

Corollary 3.1. Let $\{A_n\}_{n=1}^{\infty}, S$ and T be selfmaps on a complete metric space (X,d) satisfying $A_1 \subseteq S(X)$ and $A_1 \subseteq T(X)$. Assume that there exists $\psi \in \Psi$ such that

$$(18) \qquad d(A_1x,A_jy) \leq \begin{cases} \psi(\max\{\frac{d(Sx,A_1x)d(Ty,A_jy)}{d(Sx,Ty)+d(Sx,A_jy)+d(Ty,A_1x)}, d(Sx,Ty)\}), \\ \qquad \text{if} \quad d(Sx,Ty)+d(Sx,A_jy)+d(Ty,A_1x) \neq 0 \\ 0, \qquad \text{if} \quad d(Sx,Ty)+d(Sx,A_jy)+d(Ty,A_1x) = 0, \end{cases}$$

for all $x,y \in X$ and $j = 1,2,3,\ldots$. If the pairs (A_1,S) and (A_1,T) are weakly compatible and one of the range sets $A_1(X), S(X)$ and $T(X)$ is closed, then $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.

Proof. Under the assumptions on A_1,S and T, the existence of common fixed point z of A_1,S and T follows by choosing $A = B = A_1$ in Theorem 2.3.

Therefore $A_1z = Sz = Tz = z$.

Now, let $j \in \mathbb{N}$ with $j \neq 1$.

We now consider

$$\begin{aligned} d(z,A_jz) &= d(A_1z,A_jz) \\ &\leq \psi(\max\{\frac{d(Sz,A_1z)d(Tz,A_jz)}{d(Sz,Tz)+d(Sz,A_jz)+d(Tz,A_1z)},d(Sz,Tz)\}) \\ &= \psi(\max\{0,0\}) = \psi(0) = 0. \end{aligned}$$

Therefore $d(z,A_jz) \leq 0$ which implies that $A_jz = z$ for $j = 1,2,3,\dots$ and uniqueness of common fixed point follows from the inequality (18).

Hence, $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.

Corollary 3.2. Let $\{A_n\}_{n=1}^{\infty}, S$ and T be selfmaps on a metric space (X,d) satisfy the conditions $A_1 \subseteq S(X), A_1 \subseteq T(X)$ and (18). If the pairs (A_1,S) and (A_1,T) are weakly compatible and either $(A_1(X),d),(S(X),d)$ or $(T(X),d)$ is complete, then $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.

Proof. Under the assumptions on A_1, S and T, the existence of common fixed point z of A_1, S and T follows by choosing $A = B = A_1$ in Theorem 2.4.

Therefore $A_1z = Sz = Tz = z$.

Now, let $j \in \mathbb{N}$ with $j \neq 1$.

We now consider

$$\begin{aligned} d(z,A_jz) &= d(A_1z,A_jz) \\ &\leq \psi(\max\{\frac{d(Sz,A_1z)d(Tz,A_jz)}{d(Sz,Tz)+d(Sz,A_jz)+d(Tz,A_1z)},d(Sz,Tz)\}) \\ &= \psi(\max\{0,0\}) = \psi(0) = 0. \end{aligned}$$

Therefore $d(z,A_jz) \leq 0$ which implies that $A_jz = z$ for $j = 1,2,3,\dots$ and uniqueness of common fixed point follows from the inequality (18).

Hence, $\{A_n\}_{n=1}^{\infty}, S$ and T have a unique common fixed point in X.

The following is an example in support of Theorem 2.5. In this example we show the importance of rational expression in the inequality (3).

Example 2. Let $X = [0, \frac{42}{27}] \cup [\frac{46}{27}, 2]$ with usual metric. We define selfmaps $A, B, S,$ T on X by

$$A(x) = \begin{cases} \frac{2}{3} & \text{if } x \in [0, \frac{42}{27}] \cup [\frac{46}{27}, 2) \\ \frac{1}{2} & \text{if } x = 2 \end{cases}, \quad B(x) = \begin{cases} \frac{2}{3} & \text{if } x \in [0, \frac{42}{27}] \cup [\frac{46}{27}, 2) \\ \frac{1}{3} & \text{if } x = 2 \end{cases}$$

$$S(x)=\begin{cases}\frac{2}{3} & \text{if } 0\le x<\frac{2}{3}\\ \frac{4}{3}-x & \text{if } \frac{2}{3}\le x\le 1\\ 0 & \text{if } x\in(1,\frac{42}{27}]\cup[\frac{46}{27},2]\end{cases} \quad \text{and } T(x)=\begin{cases}\frac{1}{3} & \text{if } x=0\\ 1-\frac{x}{2} & \text{if } x\in(0,\frac{42}{27}]\cup[\frac{46}{27},2)\\ \frac{4}{3} & \text{if } x=2\end{cases}$$

We define $\psi:\mathbb{R}_+\to\mathbb{R}_+$ by $\psi(t)=\frac{3}{4}t,\ t\ge 0$. Then clearly $\psi\in\Psi$. Here $A(X)=\{\frac{1}{2},\frac{2}{3}\}$, $B(X)=\{\frac{1}{3},\frac{2}{3}\}, S(X)=[\frac{1}{3},\frac{2}{3}]\cup\{0\}$ and $T(X)=(0,\frac{4}{27}]\cup[\frac{5}{27},1)\cup\{\frac{4}{3}\}$ so that $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$.

We now verify the inequality (3).

Case (1): $x=y=0$.

$d(Ax,By)=0$ and trivially holds the inequality (3).

Case (2): $x=y=2$.

$d(Ax,By)=\frac{1}{6}; d(Sx,Ty)=\frac{4}{3}$.

We consider

$$\begin{aligned} d(Ax,By)=\frac{1}{6}\le\frac{3}{4}(\frac{4}{3}) &= \psi(d(Sx,Ty))\\ &\le \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)},d(Sx,Ty)\}).\end{aligned}$$

Case (3): $x,y\in(0,\frac{2}{3})$.

$d(Ax,By)=0$ and trivially holds the inequality (3).

Case (4): $x,y\in[\frac{2}{3},1]$.

$d(Ax,By)=0$ and trivially holds the inequality (3).

Case (5): $x,y\in(1,\frac{42}{27}]\cup[\frac{46}{27},2)$.

$d(Ax,By)=0$ and trivially holds the inequality (3).

Case (6): $x=0,y=2$.

$d(Ax,By)=\frac{1}{3}; d(Sx,Ty)=\frac{2}{3}$. We now consider

$$\begin{aligned} d(Ax,By)=\frac{1}{3}\le\frac{3}{4}(\frac{2}{3}) &= \psi(d(Sx,Ty))\\ &\le \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)},d(Sx,Ty)\}).\end{aligned}$$

Case (7): $x=0,y\in(0,\frac{2}{3})$.

$d(Ax,By)=0$ and trivially holds the inequality (3).

Case (8): $x=0,y\in[\frac{2}{3},1]$.

$d(Ax,By) = 0$ and trivially holds the inequality (3).

Case (9): $x = 0, y \in (1, \frac{42}{27}] \cup [\frac{46}{27}, 2)$.

$d(Ax,By) = 0$ and trivially holds the inequality (3).

Case (10): $x = 2, y = 0$.

$d(Ax,By) = \frac{1}{6}; d(Sx,Ty) = \frac{1}{3}$. We have

$$d(Ax,By) = \frac{1}{6} \le \frac{3}{4}(\frac{1}{3}) = \psi(d(Sx,Ty))$$
$$\le \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

Case (11): $x = 2, y \in (0, \frac{2}{3})$.

$d(Ax,By) = \frac{1}{6}; d(Sx,Ty) = 1 - \frac{y}{2}$.

We consider

$$d(Ax,By) = \frac{1}{6} \le \frac{3}{4}(1 - \frac{y}{2}) = \psi(d(Sx,Ty))$$
$$\le \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

Case (12): $x = 2, y \in [\frac{2}{3}, 1]$.

$d(Ax,By) = \frac{1}{6}; d(Sx,Ty) = 1 - \frac{y}{2}$. We have

$$d(Ax,By) = \frac{1}{6} \le \frac{3}{4}(1 - \frac{y}{2}) = \psi(d(Sx,Ty))$$
$$\le \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

Case (13): $x = 2, y \in (1, \frac{42}{27}] \cup [\frac{46}{27}, 2)$.

$d(Ax,By) = \frac{1}{6}; d(Sx,Ty) = |1 - \frac{y}{2}|$.

$d(Sx,Ax) = \frac{1}{2}; d(Ty,By) = |\frac{1}{3} - \frac{y}{2}|$.

$d(Sx,By) = \frac{2}{3}; d(Ty,Ax) = |\frac{1}{2} - \frac{y}{2}|$.

Subcase (i). $x = 2$ and $y \in (1, \frac{42}{27}]$.

$$d(Ax,By) = \frac{1}{6} \le \frac{3}{4}(1 - \frac{y}{2}) = \psi(d(Sx,Ty))$$
$$\le \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

Subcase (ii). $x = 2$ and $y \in [\frac{46}{27}, 2)$.

Here $\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)} = \frac{3y-2}{14}$.

We now consider

$$d(Ax,By)=\frac{1}{6}\leq\frac{3}{4}(\frac{3y-2}{14})=\psi(\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)})$$
$$\leq\psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)},d(Sx,Ty)\}).$$

Case (14): $x\in(0,\frac{2}{3}),y=0$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

Case (15): $x\in(0,\frac{2}{3}),y=2$.

$d(Ax,By)=\frac{1}{3};d(Sx,Ty)=\frac{2}{3}$.

We consider

$$d(Ax,By)=\frac{1}{3}\leq\frac{3}{4}(\frac{2}{3})=\psi(d(Sx,Ty))$$
$$\leq\psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)},d(Sx,Ty)\}).$$

Case (16): $x\in(0,\frac{2}{3}),y\in[\frac{2}{3},1]$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

Case (17): $x\in(0,\frac{2}{3}),y\in(1,\frac{42}{27}]\cup[\frac{46}{27},2)$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

Case (18): $x\in[\frac{2}{3},1],y=0$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

Case (19): $x\in[\frac{2}{3},1],y=2$.

$d(Ax,By)=\frac{1}{3};d(Sx,Ty)=x$.

We have

$$d(Ax,By)=\frac{1}{3}\leq\frac{3}{4}(x)=\psi(d(Sx,Ty))$$
$$\leq\psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)},d(Sx,Ty)\}).$$

Case (20): $x\in[\frac{2}{3},1],y\in(0,\frac{2}{3})$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

Case (21): $x\in[\frac{2}{3},1],y\in(1,\frac{42}{27}]\cup[\frac{46}{27},2)$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

Case (22): $x\in(1,\frac{42}{27}]\cup[\frac{46}{27},2),y=0$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

Case (23): $x\in(1,\frac{42}{27}]\cup[\frac{46}{27},2), y=2$.

$d(Ax,By)=\frac{1}{3}; d(Sx,Ty)=\frac{4}{3}$.

We now consider

$$d(Ax,By)=\frac{1}{3}\leq\frac{3}{4}(\frac{4}{3})=\psi(d(Sx,Ty))$$
$$\leq\psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)},d(Sx,Ty)\}).$$

Case (24): $x\in(1,\frac{42}{27}]\cup[\frac{46}{27},2), y\in(0,\frac{2}{3})$.

$d(Ax,By)=0$ and trivially holds the inequality in this case.

Case (25): $x\in(1,\frac{42}{27}]\cup[\frac{46}{27},2), y\in[\frac{2}{3},1]$.

$d(Ax,By)=0$ and trivially holds the inequality (3) in this case.

From the above all cases, A,B,S and T satisfy the inequality (3).

Therefore A,B,S and T satisfy all the hypotheses of Theorem 2.5 and $\frac{2}{3}$ is the unique common fixed point of A,B,S and T.

Observation: Subcase (*ii*) of Case (*13*) indicates the importance of the rational expression in the inequality (3), since in the absence of the rational expression, the inequality (3) fails to hold, for $d(Ax,By)=\frac{1}{6}\not\leq\psi(|1-\frac{y}{2}|)$ for any $\psi\in\Psi$.

Example 3. Let $X=[0,1]$ with usual metric. We define selfmaps $A,B,S,$ T on X by

$$A(x)=\begin{cases}\frac{x^2}{2} & \text{if } 0\leq x<\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x\leq 1,\end{cases}, B(x)=\begin{cases}\frac{x^2}{4} & \text{if } 0\leq x<\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x\leq 1,\end{cases}$$

$$S(x)=x^2 \text{ if } 0\leq x\leq 1 \text{ and } T(x)=\begin{cases}\frac{x^2}{2} & \text{if } 0\leq x<\frac{1}{2}\\ 0 & \text{if } \frac{1}{2}\leq x\leq 1.\end{cases}$$

We define $\psi:\mathbb{R}_+\to\mathbb{R}_+$ by $\psi(t)=\frac{t}{2}$, $t\geq 0$.

Then clearly $\psi\in\Psi$.

Here $A(X)=[0,\frac{1}{8}), B(X)=[0,\frac{1}{16}), S(X)=[0,1]$ and $T(X)=[0,\frac{1}{8})$.

Clearly $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$.

Without loss of generality assume that $x\geq y$.

Case (i): $x,y\in[0,\frac{1}{2})$.

$d(Ax,By)=|\frac{x^2}{2}-\frac{y^2}{4}|=\frac{x^2}{2}-\frac{y^2}{4}$;

$d(Sx,Ty)=|x^2-\frac{y^2}{2}|=x^2-\frac{y^2}{2}$.

We consider

$$d(Ax,By) = \frac{1}{2}|x^2 - \frac{y^2}{2}| = \psi(d(Sx,Ty))$$
$$\leq \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

Case (ii): $x,y \in [\frac{1}{2},1]$.

$d(Ax,By) = 0$ and trivially holds the inequality (3) in this case.

Case (iii): $x \in [0,\frac{1}{2}), y \in [\frac{1}{2},1]$.

$d(Ax,By) = \frac{x^2}{2}; d(Sx,Ty) = x^2$. We have

$$d(Ax,By) = \frac{x^2}{2} = \frac{1}{2}d(Sx,Ty) = \psi(d(Sx,Ty))$$
$$\leq \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

Case (iv): $x \in [\frac{1}{2},1], y \in [0,\frac{1}{2})$.

$d(Ax,By) = \frac{y^2}{4}; d(Sx,Ty) = (x^2 - \frac{y^2}{2})$. We now consider

$$d(Ax,By) = \frac{y^2}{4} \leq \frac{1}{2}(x^2 - \frac{y^2}{2}) = \psi(d(Sx,Ty))$$
$$\leq \psi(\max\{\frac{d(Sx,Ax)d(Ty,By)}{d(Sx,Ty)+d(Sx,By)+d(Ty,Ax)}, d(Sx,Ty)\}).$$

From the above four cases, A,B,S and T satisfy the inequality (3).

Therefore A,B,S and T satisfy all the hypotheses of Theorem 2.6 and 0 is the unique common fixed point of A,B,S and T.

Corollary 3.3. Let $T,f,g : X \to X$ satisfying

$$d(Tx,fy) \leq \psi(\max\{\frac{d(gx,Tx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,Tx)}, d(gx,gy)\})$$

for all $x,y \in X, T(X) \cup f(X) \subseteq g(X)$ and $(g(X),d)$ is complete. Then T,f and g have a coincidence point in X. If the pairs (T,g) and (f,g) are weakly compatible, then T,f and g have a unique common fixed point in X.

Proof. By choosing $A = T, B = f, S = T = g$ in Theorem 2.4, the conclusion of corollary follows.

Remark 3.4. Theorem 1.2, follows as a corollary to Corollary 3.3, by choosing $\psi(t) = kt$, with $k = \alpha + \beta < 1, t \geq 0$.

An open problem: Is the conclusion of Theorem 2.6 valid if we replace 'continuity of S' by 'continuity of A'? Similarly, is the conclusion of Theorem 2.6 valid if we replace 'continuity of T' by 'continuity of B'?

Conflict of Interests

The authors declare that there is no conflict of interests.

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