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COMMON FIXED POINT THEOREM FOR FOUR SELFMAPS OF A COMPLETE G-METRIC SPACE

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Abstract: In this paper, we prove a common fixed point theorem for four weakly compatible selfmaps of a complete G-metric space.

Keywords: G-metric space; fixed point; weakly compatible selfmaps; contractive modulus; upper semicontinuous; associated sequence of a point relative to four selfmaps.

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1. Introduction

Fixed point theory is interesting due its simplicity in approach and richness in mathematical content. The classical Banach Contraction Principle can be considered as the first ever fixed point theorem. Many authors have extended improved and generalized Banach's fixed point theorem in different ways. Fixed point theory has numerous applications.

In an attempt to generalize fixed point theorems on a metric space, Gahler [1,2] introduced the notion of 2-metric spaces while Dhage [3] initiated the notion of D- metric spaces. Subsequently several researchers have proved that most of their claims made are not valid. As a probable

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modification to D - metric spaces Shaban Sedghi, Nabi Shobe and Haiyun Zhou [4] introduced D*-metric spaces. In 2006, Zead Mustafa and Brailey Sims [5,6] initiated G -metric spaces. Of these two generalizations, the G -metric space evinced interest in many researchers.

As a generalization of commuting maps Sessa [7] introduced the concept of weakly commuting mappings which was further generalized by Jungck [8] by initiating the notion of compatibility. Later Jungck and Rhoades [9] introduced the notion of weakly compatible mappings

The purpose of this paper is to prove a common fixed point theorem for four weakly compatible selfmaps of a complete G -metric space.

2. Preliminaries

Definition 2.1: [6] Let X be a non-empty set and $G: X^3 \to [0, \infty)$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$

(G3) $G(x, x, y) < G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$

(G4) $G(x, y, z) = G(\sigma(x, y, z))$ for all $x, y, z \in X$, where $\sigma(x, y, z)$ is a permutation of the set $\{x, y, z\}$

and

(G5) $G(x, y, z) < G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

Then G is called a G - metric on X and the pair (X, G) is called a G - metric Space.

Example 2.2: Let *(X, d)* be a metric space. Define $G_m^d : X^3 \to [0, \infty)$ by

$G_m^d(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for $x, y, z \in X$ .Then (X, G_m^d) is a G-metric Space.

Lemma 2.3: [6] If (X, G) is a G-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$

Definition 2.4: Let (X, G) be a G-metric Space. A sequence $\{x_n\}$ in X is said to be

G-convergent if there is a $x_0 \in X$ such that to each $\varepsilon > 0$ there is a natural number N for which $G(x_n, x_n, x_0) < \varepsilon$ for all $n \geq N$.

Lemma 2.5: [6] Let (X,G) be a G-metric Space, then for a sequence $\{x_n\} \subseteq X$ and point $x \in X$ the following are equivalent.

(1) $\{x_n\}$ is G- convergent to x.

(2) $d_G(x_n, x) \to 0$ as $n \to \infty$ (that is $\{x_n\}$ converges to x relative to the metric d_G)

(3) $G(x_n, x_n, x) \to 0$ as $n \to \infty$

(4) $G(x_n, x, x) \to 0$ as $n \to \infty$

(5) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$

Definition 2.6: [6] Let (X,G) be a G-metric space, then a sequence $\{x_n\} \subseteq X$ is said to be G-Cauchy if for each $\varepsilon > 0$, there exists a natural number N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$.

Note that every G-convergent sequence in a G-metric space (X,G) is G-Cauchy.

Definition 2.7: [6] A G-metric space (X,G) is said to be G-complete if every G -Cauchy sequence in (X,G) is G-convergent in (X,G)

Definition 2.8: [10] Suppose f and g are self maps of a G-metric space (X,G). The pair f and g is said to be weakly compatible if $G(fgx, gfx, gfx) = 0$ whenever $G(fx, gx, gx) = 0$

Definition 2.9: A mapping $\phi : [0,\infty) \to [0,\infty)$ is said to be a contractive modulus if $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$

Example 2.10: The mapping $\phi : [0,\infty) \to [0,\infty)$ defined by $\phi(t) = \frac{t}{t+1}$ for $t \geq 0$ is a contractive modulus

Example 2.11: The mapping $\phi:[0,\infty)\rightarrow[0,\infty)$ defined by $\phi(t)=ct$ where $0\le c<1$ is a contractive modulus

Definition 2.12: A real valued function ϕ defined on $X\subseteq\mathbb{R}$ is said to be upper semi continuous, if $\limsup_{n\to\infty}\phi(t_n)\le\phi(t)$ for every sequence $\{t_n\}$ in with $t_n\rightarrow t$ as $n\rightarrow\infty$

Clearly every continuous function is upper semicontinuous, but not conversely.

Definition 2.13: Suppose f,g,h and p are selfmaps of a *G*-metric space such that $f(X)\subseteq h(X)$ and $g(X)\subseteq p(X)$. For x_0 in X , if $\{x_n\}$ is a sequence in X such that $fx_{2n}=hx_{2n+1}$ and $gx_{2n+1}=px_{2n+2}$ for $n\ge 0$. Then $\{x_n\}$ is called an associated sequence of x_0 relative to selfmaps f,g,h and p

3. Main results

Theorem 3.1. *suppose* f,g,h *and* P *are four selfmaps of a complete* G*-metric space* (X,G) *satisfying the following conditions*

(3.1.1) $f(X)\subseteq h(X)$ *and* $g(X)\subseteq p(X)$

(3.1.2) $G(fx,gy,gy)\le\phi(\lambda(x,y)$ *where* ϕ *is an upper semicontinuous contractive modulus and* $\lambda(x,y)=\max\{G(px,hy,hy),G(fx,hy,hy),G(hy,gy,gy)\}$

(3.1.3) one of $f(X),g(X),h(X),P(X)$ *is closed sub subset of* X

(3.1.4) (f,p) *and* (g,h) *are weakly compatible pairs*

Then f,g,h *and* p *have a unique common fixed point in* X .

Proof. Let $x_0\in X$ be an arbitrary point. Then we can construct a sequence $\{x_n\}$ in X such that $y_{2n}=fx_{2n}=hx_{2n+1},\ \ y_{2n+1}=gx_{2n+1}=px_{2n+2}$ *for* $n\ge 0$ (3.1.5)

From condition (3.1.2) we have

$G(y_{2n}, y_{2n+1}, y_{2n+1}) = G(fx_{2n}, gx_{2n+1}, gx_{2n+1}) \le \phi(\lambda(x_{2n}, x_{2n+1})$

Where

$$\lambda(x_{2n}, x_{2n+1}) = \max\left\{G(px_{2n}, hx_{2n+1}, hx_{2n+1}), G(fx_{2n}, hx_{2n+1}, hx_{2n+1}), G(hx_{2n+1}, gx_{2n+1}, gx_{2n+1})\right\}$$
$$= \max\{G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n}, y_{2n}, y_{2n}), G(y_{2n}, y_{2n+1}, y_{2n+1})\}$$

If $G(y_{2n-1}, y_{2n}, y_{2n}) < G(y_{2n}, y_{2n+1}, y_{2n+1})$ then $\lambda(x_{2n}, x_{2n+1}) = G(y_{2n}, y_{2n+1}, y_{2n+1})$

Therefore $G(y_{2n}, y_{2n+1}, y_{2n+1}) \le \phi(G(y_{2n}, y_{2n+1}, y_{2n+1})) < G(y_{2n}, y_{2n+1}, y_{2n+1})$

Which is contradiction since ϕ is contractive modulus

Hence $G(y_{2n}, y_{2n+1}, y_{2n+1}) \le G(y_{2n-1}, y_{2n}, y_{2n})$ (3.1.6)

Similarly, we can show that

$G(y_{2n+1}, y_{2n+2}, y_{2n+2}) \le G(y_{2n}, y_{2n+1}, y_{2n+1})$ (3.1.7)

From (3.1.6) and (3.1.7) we have $G(y_n, y_{n+1}, y_{n+1}) \le G(y_{n-1}, y_n, y_n)$

Hence $G(y_n, y_{n+1}, y_{n+1}) \le \phi(G(y_{n-1}, y_n, y_n))$ (3.1.8)

The sequence $\{G(y_n, y_{n+1}, y_{n+1})\}$ is monotonic decreasing, hence there exists a real number $r \ge 0$ such that $\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = r$

Therefore as $n \to \infty$ equation (3.1.8) gives $r \le \phi(r)$ which is possible only if $r = 0$

Thus $\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0$

We now show that $\{y_n\}$ is a Cauchy sequence

It sufficient to show that $\{y_{2n}\}$ is a Cauchy.

Suppose $\{y_{2n}\}$ is not a Cauchy, then there exists, an $\varepsilon > 0$, for which we can find sub sequences $\{y_{2n_k}\}, \{y_{2m_k}\}$ of $\{y_{2n}\}$ such that $m_k, n_k > k$ and

$G(y_{2m_k}, y_{2n_k}, y_{2n_k}) \ge \varepsilon$, and $G(y_{2m_k}, y_{2n_k-2}, y_{2n_k-2}) < \varepsilon$

Now $\varepsilon \le G(y_{2m_k}, y_{2n_k}, y_{2n_k}) \le G(y_{2m_k}, y_{2n_k-2}, y_{n_k-2}) + G(y_{2n_k-2}, y_{2n_k-1}, y_{2n_k-1}) + G(y_{2n_k-1}, y_{2n_k}, y_{2n_k})$

on letting $k \to \infty$ we have $\lim_{k\to\infty} G(y_{2m_k}, y_{2n_k}, y_{2n_k}) = \varepsilon$

Moreover, we have $\left|G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) - G(y_{2m_k}, y_{2n_k}, y_{2n_k})\right| \le 2G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1})$

on letting $k \to \infty$ we get $\lim_{k\to\infty} G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) = \varepsilon$

Also $\left|G(y_{2m_k-1}, y_{2n_k}, y_{2n_k}) - G(y_{2m_k}, y_{2n_k}, y_{2n_k})\right| \le 2G(y_{2m_k-1}, y_{2m_k}, y_{2m_k})$

on letting $k \to \infty$ we get $\lim_{k\to\infty} G(y_{2m_k-1}, y_{2n_k}, y_{2n_k}) = \varepsilon$

And $\left|G(y_{2m_k-1}, y_{2n_k+1}, y_{2n_k+1}) - G(y_{2m_k-1}, y_{2n_k}, y_{2n_k})\right| \le 2G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1})$

on letting $k \to \infty$ we get $\lim_{k\to\infty} G(y_{2m_k-1}, y_{2n_k+1}, y_{2n_k+1}) = \varepsilon$

Now by (3.1.2)

$$G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) = G(fx_{2m_k}, gx_{2n_k+1}, gx_{2n_k+1}) \le \phi(\lambda(x_{2m_k}, x_{2n_k+1})) \qquad (3.1.9)$$

Where

$$\begin{aligned}\lambda(x_{2m_k}, x_{2n_k+1}) &= \max\{G(px_{2m_k}, hx_{2n_k+1}, hx_{2n_k+1}), G(fx_{2m_k}, hx_{2n_k+1}, hx_{2n_k+1}), G(hx_{2n_k+1}, gx_{2n_k+1}, gx_{2n_k+1})\}\\ &= \max\{G(y_{2m_k-1}, y_{2n_k}, y_{2n_k}), G(y_{2m_k}, y_{2n_k}, y_{2n_k}), G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1})\}\end{aligned}$$

on letting $k \to \infty$ we have

$$\lim_{k\to\infty} \lambda(x_{2m_k}, x_{2n_k+1}) = \max\{\varepsilon, \varepsilon, 0\} = \varepsilon$$

therefore from (3.1.9) we have $\varepsilon \le \phi(\varepsilon)$ this is a contradiction since $\varepsilon > 0$

Therefore $\{y_{2n}\}$ is a Cauchy sequence in X .

Since X is complete G-metric space, then there exists a point $z \in X$ such that

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} hx_{2n+1} = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} px_{2n+2} = z \qquad (3.1.10)$$

Suppose that $p(X)$ is a closed subset of X there exists a point $u \in X$ such that $z = pu$

We now show that $fu = z$. If $fu \ne z$ then $G(fu, z, z) > 0$

Now from (3.1.2) we have

$$G(fu, z, z) = G(fu, gx_{2n+1}, gx_{2n+1}) \le \phi(\lambda(u, x_{2n+1}))$$

Where

$\lambda(u, x_{2n+1}) = \max\{G(pu, hx_{2n+1}, hx_{2n+1}), G(fu, hx_{2n+1}, hx_{2n+1}), G(hx_{2n+1}, gx_{2n+1}, gx_{2n+1})\}$

letting $n \to \infty$ we have

$$\lim_{n\to\infty} \lambda(u, x_{2n+1}) = \max\{G(pu, z, z), G(fu, z, z), G(z, z, z)\}$$
$$= \max\{G(z, z, z), G(fu, z, z), G(z, z, z)\}$$
$$= G(fu, z, z)$$

Hence $G(fu, z, z) \le \phi(G(fu, z, z)) < G(fu, z, z)$

which is a contradiction, thus $G(fu, z, z) = 0$ implies $fu = z$

Therefore $fu = pu = z$ (3.1.11)

Since the pair (f, p) is weakly compatible then $fpu = pfu$ which gives $fz = pz$

If $fz \ne z$ then $G(fz, z, z) > 0$

Now from (3.1.2) $G(fz, z, z) = G(fz, gx_{2n+1}, gx_{2n+1}) \le \phi(\lambda(u, x_{2n+1}))$

Where $\lambda(z, x_{2n+1}) = \max\{G(pz, hx_{2n+1}, hx_{2n+1}), G(fz, hx_{2n+1}, hx_{2n+1}), G(hx_{2n+1}, gx_{2n+1}, gx_{2n+1})\}$

On letting $n \to \infty$ we have

$$\lim_{n\to\infty} \lambda(z, x_{2n+1}) = \max\{G(pz, z, z), G(fz, z, z), G(z, z, z)\}$$
$$= \max\{G(fz, z, z), G(fz, z, z), G(z, z, z)\}$$
$$= G(fz, z, z)$$

Hence $G(fz, z, z) \le \phi(G(fz, z, z)) < G(fz, z, z)$

which is a contradiction thus $G(fz, z, z) = 0$ implies $fz = z$

Therefore $fz = z = pz$ showing that z is common fixed point of f, p

Since $f(X) \subseteq h(X)$ there exists a point $v \in X$ such that $hv = z$

We now prove that $gv = z$. If $gv \ne z$ then $G(z, gv, gv) > 0$

By (3.1.2) we have $G(z, gv, gv) = G(fz, gv, gv) \le \phi(\lambda(z, v))$

Where

$$\lambda(z,v) = \max\{G(pz,hv,hv), G(fz,hv,hv), G(hv,gv,gv)\}$$
$$= \max\{G(z,z,z), G(z,z,z), G(z,gv,gv)\}$$
$$= G(z,gv,gv)$$

Hence $G(z,gv,gv) \le \phi(G(z,gv,gv)) < G(z,gv,gv)$

which is a contradiction thus $G(z,gv,gv) = 0$ gives $gv = z$

Therefore $hv = gv = z$ (3.1.12)

Since the pair (g,h) is weakly compatible then $gz = hz$

If $gz \ne z$ then $G(z,gz,gz) > 0$

Now from (3.1.2)

$$G(z,gz,gz) = G(fz,gz,gz) \le \phi(\lambda(z,z))$$

Where

$$\lambda(z,z) = \max\{G(pz,hz,hz), G(fz,hz,hz), G(hz,gz,gz)\}$$
$$= \max\{G(z,gz,gz), G(z,gz,gz), G(gz,gz,gz)\}$$
$$= G(z,gz,gz)$$

Hence $G(z,gz,gz) \le \phi(G(z,gz,gz)) < G(z,gz,gz)$

which is a contradiction thus $G(z,gz,gz) = 0$ implies $gz = z$

Therefore $hz = z = gz$, showing that z is common fixed point of g,h

Thus z is common fixed point of f,g,h and p

The proof is similar if one of $f(X), g(X), h(X)$ is a closed subset of X with appropriate changes.

We now prove the uniqueness,

if possible assume w is other fixed point of f,g,h and p

From (3.1.2)

$$G(z,w,w) = G(fz,gw,gw) \le \phi(\lambda(z,w))$$

Where

$$\lambda(z,w) = \max\left\{G(pz,hw,hw), G(fz,hw,hw), G(hw,gw,gw)\right\}$$
$$= \max\{G(z,w,w), G(z,w,w), G(w,w,w)\}$$
$$= G(z,w,w)$$

Hence $G(z,w,w) \le \phi(G(z,w,w)) < G(z,w,w)$ which is a contradiction, hence $z = w$

Proving that z is the unique common fixed point of f, g, h and p

As an illustration, we have the following

EXAMPLE 3.2: Let $X = [0,1]$ with $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for $x, y, z \in X$.

Then G is a G-metric on X.

Define $f : X \to X, g : X \to X, h : X \to X, p : X \to X$ by

$$fx = \begin{cases} \frac{1}{15}, \ if \ x \in [0, \frac{1}{2}) \\ \frac{1}{2}, if \ x \in [\frac{1}{2}, 1] \end{cases} \qquad gx = \begin{cases} \frac{1}{10}, \ if \ x \in [0, \frac{1}{2}) \\ \frac{1}{2}, if \ x \in [\frac{1}{2}, 1] \end{cases} \qquad hx = \begin{cases} \frac{1}{20} \ if \ x \in [0, \frac{1}{2}) \\ \frac{1}{2} \quad if \ x = \frac{1}{2} \\ \frac{1}{15} \ if \ x \in (\frac{1}{2}, 1] \end{cases} \qquad px = \begin{cases} \frac{19}{20} \ if \ x \in [0, \frac{1}{2}) \\ \frac{1}{2} \quad if \ x = \frac{1}{2} \\ \frac{1}{10} \ if \ x \in (\frac{1}{2}, 1] \end{cases}$$

$fX = \{\frac{1}{15}, \frac{1}{2}\}, gX = \{\frac{1}{10}, \frac{1}{2}\}, hX = \{\frac{1}{20}, \frac{1}{15}, \frac{1}{2}\}, pX = \{\frac{1}{10}, \frac{1}{2}, \frac{15}{20}\}$ showing that $fX \subseteq hX, gX \subseteq pX$

Clearly fX, gX, hX and QX are closed subsets of X

As $f(\frac{1}{2}) = p(\frac{1}{2})$ we have $pf(\frac{1}{2}) = fp(\frac{1}{2})$, showing that (f, p) is weakly compatible.

And $h(\frac{1}{2}) = g(\frac{1}{2})$ we have $gh(\frac{1}{2}) = hg(\frac{1}{2})$, showing that (g, h) is weakly compatible.

Consider the function $\phi(t) = \dfrac{100t}{101}$

Now we prove the condition (3.1.2) of the theorem 3.1.

case(i). If $x, y \in [0, \frac{1}{2})$

$$G(fx, gy, gy) = \tfrac{1}{15}, G(px, hy, hy) = \tfrac{9}{5} \ G(fx, hy, hy) = \tfrac{1}{30}, G(hy, gy, gy) = \tfrac{1}{10}$$

$$\lambda(x, y) = \max\left\{G(px, hy, hy), G(fx, hy, hy), G(hy, gy, gy)\right\} = \max\left\{\tfrac{9}{5}, \tfrac{1}{30}, \tfrac{1}{10}\right\} = \tfrac{9}{5}$$

$$\phi(\lambda(x, y) = \phi(\tfrac{9}{5}) = \tfrac{900}{505}$$

$$G(fx, gy, gy) = \tfrac{1}{15} \le \tfrac{900}{505} = \phi(\lambda(x, y)$$

Proving that the condition (3.1.2) of the Theorem3.1 satisfied in this case.

case(ii). if $x, y \in (\frac{1}{2}, 1]$

$$G(fx, gy, gy) = 0 \text{ , } G(px, hy, hy) = \frac{1}{15}, G(fx, hy, hy) = \frac{13}{15}, G(hy, gy, gy) = \frac{13}{15}$$

$$\lambda(x, y) = \max\left\{G(px, hy, hy), G(fx, hy, hy), G(hy, gy, gy)\right\} = \max\left\{\frac{1}{15}, \frac{13}{15}, \frac{13}{15}\right\} = \frac{13}{15}$$

$$\phi(\lambda(x, y) = \phi(\tfrac{13}{15}) = \frac{1300}{1515}$$

$$G(fx, gy, gy) = 0 \leq \frac{1300}{1515} = \phi(\lambda(x, y))$$

Proving that the condition (3.1.2) of the Theorem3.1 satisfied in this case.

case(iii). if $x \in [0, \frac{1}{2}), y \in (\frac{1}{2}, 1]$

$$G(fx, gy, gy) = \frac{13}{15}, G(px, hy, hy) = \frac{53}{30} \text{ , } G(fx, hy, hy) = 0, \; G(hy, gy, gy) = \frac{13}{15}$$

$$\lambda(x, y) = \max\left\{G(px, hy, hy), G(fx, hy, hy), G(hy, gy, gy)\right\} = \max\left\{\frac{53}{30}, 0, \frac{13}{15}\right\} = \frac{53}{30}$$

$$\phi(\lambda(x, y) = \phi(\tfrac{53}{30}) = \frac{5300}{3030}$$

$$G(fx, gy, gy) = \frac{13}{15} \leq \frac{5300}{3030} = \phi(\lambda(x, y))$$

Proving that the condition (3.1.2) of the Theorem3.1 satisfied in this case.

case(iv). if $y \in [0, \frac{1}{2}), x \in (\frac{1}{2}, 1]$

$$G(fx, gy, gy) = \frac{4}{5}, G(px, hy, hy) = \frac{1}{10} \text{ , } G(fx, hy, hy) = \frac{9}{10}, G(hy, gy, gy) = \frac{1}{10}$$

$$\lambda(x, y) = \max\left\{G(px, hy, hy), G(fx, hy, hy), G(hy, gy, gy)\right\} = \max\left\{\frac{1}{10}, \frac{9}{10}, \frac{1}{10}\right\} = \frac{9}{10}$$

$$\phi(\lambda(x, y) = \phi(\tfrac{9}{10}) = \frac{900}{1010}$$

$$G(fx, gy, gy) = \frac{4}{5} \leq \frac{900}{1010} = \phi(\lambda(x, y))$$

Proving that the condition (3.1.2) of the Theorem3.1 is true in this case.

case(v). if $x = \frac{1}{2}, y \in [0, \frac{1}{2})$

$$G(fx, gy, gy) = \frac{4}{5}, G(px, hy, hy) = \frac{9}{10}, \; G(fx, hy, hy) = \frac{9}{10}, G(hy, gy, gy) = \frac{1}{10}$$

$$\lambda(x,y)=\max\left\{G(px,hy,hy),G(fx,hy,hy),G(hy,gy,gy)\right\}=\max\left\{\tfrac{9}{10},\tfrac{9}{10},\tfrac{1}{10}\right\}=\tfrac{9}{10}$$

$$\phi(\lambda(x,y)=\phi(\tfrac{9}{10})=\tfrac{900}{1010}$$

$$G(fx,gy,gy)=\tfrac{4}{5}\leq\tfrac{900}{1010}=\phi(\lambda(x,y))$$

Proving that the condition (3.1.2) of the Theorem 3.1 is true in this case.

case(vi). if $x=\frac{1}{2}, y\in(\frac{1}{2},1]$

$$G(fx,gy,gy)=0,\ G(px,hy,hy)=\tfrac{13}{15},\ G(fx,hy,hy)=\tfrac{13}{15}, G(hy,gy,gy)=\tfrac{13}{15}$$

$$\lambda(x,y)=\max\left\{G(px,hy,hy),G(fx,hy,hy),G(hy,gy,gy)\right\}=\max\left\{\tfrac{13}{15},\tfrac{13}{15},\tfrac{13}{15}\right\}=\tfrac{13}{15}$$

$$G(fx,gy,gy)=0\leq\tfrac{1300}{1515}=\phi(\lambda(x,y)$$

Proving that the condition (3.1.2) of the Theorem3.1 is true in this case.

case(vii). if $y=\frac{1}{2}, x\in[0,\frac{1}{2})$

$$G(fx,gy,gy)=\tfrac{13}{15}, G(px,hy,hy)=\tfrac{9}{10},\ G(fx,hy,hy)=\tfrac{13}{15}, G(hy,gy,gy)=0$$

$$\lambda(x,y)=\max\left\{G(px,hy,hy),G(fx,hy,hy),G(hy,gy,gy)\right\}=\max\left\{\tfrac{9}{10},\tfrac{13}{15},0\right\}=\tfrac{9}{10}$$

$$G(fx,gy,gy)=\tfrac{13}{15}\leq\tfrac{900}{1010}=\phi(\lambda(x,y)$$

Proving that the condition (3.1.2) of the Theorem3.1 is true in this case.

case(viii). if $y=\frac{1}{2}, x\in(\frac{1}{2},1]$

$$G(fx,gy,gy)=0,\ G(px,hy,hy)=\tfrac{4}{5},\ G(fx,hy,hy)=0,\ G(hy,gy,gy)=0$$

$$\lambda(x,y)=\max\left\{G(px,hy,hy),G(fx,hy,hy),G(hy,gy,gy)\right\}=\max\left\{\tfrac{4}{5},0,0\right\}=\tfrac{4}{5}$$

$$G(fx,gy,gy)=0\leq\tfrac{400}{505}=\phi(\lambda(x,y)$$

Proving that the condition (3.1.2) of the Theorem3.1 is true in this case.

Hence the condition (3.1.2) in all cases.

Therefore, all the conditions of the Theorem 3.1 satisfied

Clearly $\frac{1}{2}$ is a unique common fixed point of f,g,h and p

Corollary 3.3: *suppose* f, g, h *and* P *are four selfmaps of a complete* G*-metric space* (X, G) *satisfying the following conditions*

(3.3.1) $f(X) \subseteq h(X)$ *and* $g(X) \subseteq p(X)$

(3.3.2) $G(fx, gy, gy) \leq \phi(\lambda(x, y)$ *where* ϕ *is an upper semicontinuous contractive modulus and* $\lambda(x, y) = \max\{G(px, hy, hy), G(fx, hy, hy), G(hy, gy, gy)\}$

(3.3.3) one of $f(X), g(X), h(X), P(X)$ *is closed sub subset of* X

(3.3.4) The pairs (f, p) *and* (g, h) *are commuting*

Then f, g, h *and* p *have a unique common fixed point in* X

Proof: From the fact that commutativity implies weakly compatibility, the proof of the Corollary follows from the Theorem3.1

Corollary 3.4: *suppose* f, g *and* P *are three selfmaps of a complete* G*-metric space* (X, G) *satisfying the following conditions*

(3.4.1) $f(X) \subseteq p(X)$ *and* $g(X) \subseteq p(X)$

(3.4.2) $G(fx, gy, gy) \leq \phi(\lambda(x, y)$ *where* ϕ *is an upper semi continuous contractive modulus and* $\lambda(x, y) = \max\{G(px, py, py), G(fx, py, py), G(py, gy, gy)\}$

(3.4.3) one of $f(X), g(X), P(X)$ *is closed sub subset of* X

(3.4.4) (f, p) *and* (g, p) *are weakly compatible pairs*

Then f, g *and* p *have a unique common fixed point in* X

Proof: By taking $h = p$ in Theorem 3.1.

Corollary 3.5: *suppose f and g are two selfmaps of a complete G-metric space (X,G) satisfying the following conditions*

(3.5.1) $f(X) \subseteq p(X)$

(3.5.2) $G(fx, fy, fy) \leq \phi(\lambda(x,y))$ *where* ϕ *is an upper semi continuous contractive modulus and* $\lambda(x,y) = \max\{G(px, py, py), G(fx, py, py), G(py, fy, fy)\}$

(3.5.3) one of $f(X), P(X)$ *is closed sub subset of* X

(3.5.4) (f, p) *is weakly compatible pair*

Then f *and* p *have a unique common fixed point in* X

Proof: By taking $h = p$ and $f = g$ in Theorem 3.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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