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COUPLED FIXED POINT RESULTS FOR RATIONAL TYPE CONTRACTIONS INVOLVING GENERALIZED ALTERING DISTANCE FUNCTION IN METRIC SPACES

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Abstract. In this paper, we study some unique coupled coincidence points for rational type contractions involving generalized altering distance functions in metric spaces. Our results unify and generalize various known comparable results from the current literature, Gupta et. al. [15], Nashine and Aydi [21], Rashwan and Saleh [25] and many known results. An example is also given to support our main results.

Keywords: coupled coincidence point; coupled fixed point; generalized altering distance function; rational contractions; mixed monotone property; metric and partially ordered metric spaces.

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1. Introduction

In 1922, The Banach contraction mapping theorem [5] was introduced and it remains a powerful tool in nonlinear analysis. Generalizations of this theorem have been obtained in several branches of mathematics. In 1984 Khan et al. [16] initiated the use of a control function that

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alters distance between two points in a metric space. Such mappings are called an altering distances.

In 2006, Bhaskar and Lakshmikantham [6] introduced the concept of coupled fixed point and proved some coupled fixed point results under certain conditions, in ordered metric spaces and applied their results to prove the existence and uniqueness of a solution for a periodic boundary value problem. Many researchers have obtained coupled fixed point results for mappings under various contractive conditions in the framework of partial metric spaces [1, 2, 3, 4, 10, 11, 18, 26].

2. Preliminaries

At first we state the following definitions and results.

Definition 1.1. [27] An altering distance function is a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfying

(i): ϕ is continuous and non-decreasing,

(ii): $\phi(t) = 0 \Leftrightarrow t = 0$.

Altering distances have been generalized to a two-variable function by Choudhury and Dutta [9] and to a three-variable function by Choudhury [8] and was applied for obtaining fixed point results in metric spaces. In [23], Rao et al. introduced the generalized altering distance function in five variables as a generalization of three variables.

Definition 1.2. [9] Let Ψ_2 be the set of all functions $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ that satisfy,

(i): ψ is continuous and non-decreasing in its two variables,

(ii): $\psi(x,y) = 0 \Leftrightarrow x = y = 0$.

Definition 1.3. [23] Let Ψ_5 denote the set of all functions $\psi : [0,\infty)^5 \to [0,\infty)$. Then ψ is said to be a generalized altering distance function iff,

(i): ψ is continuous and non-decreasing in all five variables,

(ii): $\psi(t_1,t_2,t_3,t_4,t_5) = 0 \Leftrightarrow t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

Definition 1.4. [17] Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say that F has the mixed g-monotone property if $F(x,y)$ is monotone g-non-decreasing in its first argument and monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X,\ \ g(x_1) \preceq g(x_2)\ \text{ implies }\ F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X,\ \ g(y_1) \preceq g(y_2)\ \text{ implies }\ F(x, y_1) \succeq F(x, y_2).$$

Definition 1.5. [17] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the two mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), F(y, x) = g(y).$$

If g is the identity mapping, then (x, y) is called a coupled fixed point of the mapping F.

The aim of this paper is to establish some unique coupled fixed point results for two self-mappings $F : X \times X \to X$ and $g : X \to X$ which satisfy rational type contractive condition involving a generalized altering distance function. In order to validate our established theorems and corollaries, an example has been given.

3. Main results

Let Φ denote all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy

(i): ϕ is continuous and non-decreasing,

(ii): $\phi(t) = 0$ if and only if $t = 0$,

(iii): $\phi(t + s) \leq \phi(t) + \phi(s)$, $\forall t, s \in [0, \infty)$,

(iv): $\phi(\frac{t}{2}) \leq \frac{1}{2}\phi(t)$, $\forall t \in [0, \infty)$.

Theorem 3.1. *Let (X, d) be a metric space and $F : X \times X \to X$, $g : X \to X$ be two self mappings such that $F(X, X) \subseteq g(X)$ and $g(X)$ is complete subspace of X. Suppose that there exist $\phi \in \Phi$ and $\psi_1, \psi_2 \in \Psi_2$ for which $\psi_1(x, x) \leq \phi(x)$ and*

$$\phi\big(d(F(x, y), F(u, v))\big) \leq \psi_1\left(\frac{K(x, u) + K(y, v)}{2}\right) - \psi_2\left(\frac{K(x, u) + K(y, v)}{2}\right), \tag{1}$$

where

$$K(x, u) = \left(\frac{d(gu, F(u, v))[1 + d(gx, F(x, y))]}{1 + d(gx, gu)}, d(gx, gu)\right)$$

and

$$K(y,v)=\left(\frac{d(gv,F(v,u))[1+d(gy,F(y,x))]}{1+d(gy,gv)},d(gy,gv)\right),$$

for all $x,y,u,v\in X$. Then there exist $x,y\in X$ such that $gx=F(x,y)$ and $gy=F(y,x)$. That is, F and g have a coupled coincidence point.

Proof. Choose x_0 and y_0 in X and set $gx_1=F(x_0,y_0)$ and $gy_1=F(y_0,x_0)$. Repeating this process, set

$$\begin{cases} gx_{n+1} &= F(x_n,y_n),\\ gy_{n+1} &= F(y_n,x_n).\end{cases} \tag{2}$$

This can be done because $F(X,X)\subseteq g(X)$. If for some $n\in N$,

$$gx_n=gx_{n+1} \texttt{ and } gy_n=gy_{n+1}$$

then, by (2) (x_n,y_n) is a coupled coincidence point of F and g. Hence we can assume that at least

$$gx_n\neq gx_{n+1} \texttt{ or } gy_n\neq gy_{n+1}.$$

From (1), we have

$$\begin{aligned}\phi\big(d(gx_n,gx_{n+1})\big)&=\phi\big(d(F(x_{n-1},y_{n-1}),F(x_n,y_n))\big)\\ &\leq\psi_1\left(\frac{K(x_{n-1},x_n)+K(y_{n-1},y_n)}{2}\right)-\psi_2\left(\frac{K(x_{n-1},x_n)+K(y_{n-1},y_n)}{2}\right),\end{aligned} \tag{3}$$

where

$$\begin{aligned}K(x_{n-1},x_n)&=\left(\frac{d(gx_n,F(x_n,y_n))[1+d(gx_{n-1},F(x_{n-1},y_{n-1}))]}{1+d(gx_{n-1},gx_n)},d(gx_{n-1},gx_n)\right)\\ &=\left(\frac{d(gx_n,gx_{n+1})[1+d(gx_{n-1},gx_n)]}{1+d(gx_{n-1},gx_n)},d(gx_{n-1},gx_n)\right)\\ &=\Big(d(gx_n,gx_{n+1}),d(gx_{n-1},gx_n)\Big)\end{aligned}$$

and

$$K(y_{n-1},y_n)=\Big(d(gy_n,gy_{n+1}),d(gy_{n-1},gy_n)\Big).$$

Hence

$$\begin{aligned}\frac{K(x_{n-1},x_n)+K(y_{n-1},y_n)}{2} &= \left(\frac{d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})}{2},\frac{d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)}{2}\right)\\ &= \left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right)\end{aligned}$$

where $w_n = d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})$. Similarly we get

$$\begin{aligned}\phi\big(d(gy_n,gy_{n+1})\big) &\leq \psi_1\left(\frac{K(y_{n-1},y_n)+K(x_{n-1},x_n)}{2}\right)-\psi_2\left(\frac{K(y_{n-1},y_n)+K(x_{n-1},x_n)}{2}\right)\\ &\leq \psi_1\left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right)-\psi_2\left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right).\end{aligned} \tag{4}$$

By adding (3) and (4) and using properties (iii) and (iv) of ϕ, we get

$$\begin{aligned}\phi(w_n) &\leq \phi\big(d(gx_n,gx_{n+1})\big)+\phi\big(d(gy_n,gy_{n+1})\big)\\ &\leq 2\psi_1\left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right)-2\psi_2\left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right)\end{aligned}$$

$$\phi(\frac{w_n}{2}) \leq \psi_1\left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right)-\psi_2\left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right). \tag{5}$$

Under the assumption that at least one of $gx_n \neq gx_{n+1}$ and $gy_n \neq gy_{n+1}$ holds for all n, then $w_n \neq 0,\ \forall n$ and $\psi_2\left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right) > 0$. Also, if $w_{n-1} < w_n$ then

$$\begin{aligned}\phi(\frac{w_n}{2}) &< \psi_1\left(\frac{w_n}{2},\frac{w_{n-1}}{2}\right)\\ &< \psi_1\left(\frac{w_n}{2},\frac{w_n}{2}\right) = \phi(\frac{w_n}{2}),\end{aligned}$$

which is a contradiction. Then $w_n \leq w_{n-1}\ \ \forall n$, that is the sequence $\{w_n\}$ is non-increasing. Therefore there is some $w \geq 0$ such that

$$\lim_{n\to\infty} w_n = w.$$

We shall show that $w = 0$. Suppose the contrary, that $w > 0$. Taking the limit as n tends to ∞ of both sides of (5) and using the continuity of ϕ, ψ_1 and ψ_2 imply that

$$\begin{aligned}\phi(\frac{w}{2}) &\leq \psi_1\left(\frac{w}{2},\frac{w}{2}\right)-\psi_2\left(\frac{w}{2},\frac{w}{2}\right)\\ &< \psi_1\left(\frac{w}{2},\frac{w}{2}\right) = \phi(\frac{w}{2}),\end{aligned}$$

a contradiction. Thus $w = 0$ and

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0. \tag{6}$$

In what follows, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. Suppose the contrary, that at least one of them is not Cauchy. Then there exists an $\varepsilon > 0$ for which we can find two sequences $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) \geq \varepsilon \tag{7}$$

and

$$d(gx_{n_k-1}, gx_{m_k}) + d(gy_{n_k-1}, gy_{m_k}) < \varepsilon. \tag{8}$$

By using triangle inequality, (7) and (8), one gets

$$\begin{aligned}
\varepsilon &\leq d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) \\
&\leq d(gx_{n_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{m_k}) \\
&\quad + d(gy_{n_k}, gy_{n_k-1}) + d(gy_{n_k-1}, gy_{m_k}) \\
&< d(gx_{n_k}, gx_{n_k-1}) + d(gy_{n_k}, gy_{n_k-1}) + \varepsilon.
\end{aligned} \tag{9}$$

Letting $k \to \infty$ in (9) and using (6) imply

$$\lim_{k\to\infty} d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) = \varepsilon. \tag{10}$$

Again by triangle inequality,

$$\begin{aligned}
d(gx_{n_k+1}, gx_{m_k+1}) + d(gy_{n_k+1}, gy_{m_k+1}) &\leq d(gx_{n_k+1}, gx_{n_k}) + d(gx_{n_k}, gx_{m_k}) + d(gx_{m_k}, gx_{m_k+1}) \\
&\quad + d(gy_{n_k+1}, gy_{n_k}) + d(gy_{n_k}, gy_{m_k}) + d(gy_{m_k}, gy_{m_k+1}) \\
&\leq w_{n_k} + w_{m_k} + d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k})
\end{aligned}$$

and

$$\begin{aligned}
d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) &\leq d(gx_{n_k}, gx_{n_k+1}) + d(gx_{n_k+1}, gx_{m_k+1}) + d(gx_{m_k+1}, gx_{m_k}) \\
&\quad + d(gy_{n_k}, gy_{n_k+1}) + d(gy_{n_k+1}, gy_{m_k+1}) + d(gy_{m_k+1}, gy_{m_k}) \\
&\leq w_{n_k} + w_{m_k} + d(gx_{n_k+1}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k+1}).
\end{aligned}$$

Hence

$$\lim_{k\to\infty} d(gx_{n_k+1}, gx_{m_k+1}) + d(gy_{n_k+1}, gy_{m_k+1}) = \varepsilon \tag{11}$$

Now we apply inequality (1) and use (10) and (11) to get a contradiction.

(12)

$$\begin{aligned}\phi\left(d(gx_{n_k+1}, gx_{m_k+1})\right) &= \phi\left(d(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k}))\right)\\ &\leq \psi_1\left(\frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2}\right) - \psi_2\left(\frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2}\right),\end{aligned}$$

where

$$K(x_{n_k}, x_{m_k}) = \left(\frac{d(gx_{m_k}, gx_{m_k+1})[1 + d(gx_{n_k}, gx_{n_k+1})]}{1 + d(gx_{n_k}, gx_{m_k})}, d(gx_{n_k}, gx_{m_k})\right)$$

and

$$K(y_{n_k}, y_{m_k}) = \left(\frac{d(gy_{m_k}, gy_{m_k+1})[1 + d(gy_{n_k}, gy_{n_k+1})]}{1 + d(gy_{n_k}, gy_{m_k})}, d(gy_{n_k}, gy_{m_k})\right).$$

By (6), we have

$$\frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2} \to (0, \varepsilon) \ \texttt{as} \ k \to \infty.$$

Similarly we get

(13)

$$\begin{aligned}\phi\left(d(gy_{n_k+1}, gy_{m_k+1})\right) &= \phi\left(d(F(y_{n_k}, x_{n_k}), F(y_{m_k}, x_{m_k}))\right)\\ &\leq \psi_1\left(\frac{K(y_{n_k}, y_{m_k}) + K(x_{n_k}, x_{m_k})}{2}\right) - \psi_2\left(\frac{K(y_{n_k}, y_{m_k}) + K(x_{n_k}, x_{m_k})}{2}\right).\end{aligned}$$

By adding (12) and (13) and from the properties of ϕ, we get

$$\begin{aligned}\phi(d(gx_{n_k+1}, gx_{m_k+1}) + d(gy_{n_k+1}, gy_{m_k+1})) &\leq 2\psi_1\left(\frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2}, \frac{w_{n-1}}{2}\right)\\ &\quad - 2\psi_2\left(\frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2}, \frac{w_{n-1}}{2}\right)\\ \phi\left(\frac{d(gx_{n_k+1}, gx_{m_k+1}) + d(gy_{n_k+1}, gy_{m_k+1})}{2}\right) &\leq \psi_1\left(\frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2}\right)\\ &\quad - \psi_2\left(\frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2}\right).\end{aligned}$$

Taking limit as k tends to infinity implies

$$\phi\left(\frac{\varepsilon}{2}\right) \leq \psi_1\left(0, \frac{\varepsilon}{2}\right) - \psi_2\left(0, \frac{\varepsilon}{2}\right),$$

which is a contradiction. This shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete then there exist $gx, gy \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \texttt{ and } \lim_{n\to\infty} gy_n = gy. \tag{14}$$

Now we show that (x,y) is coupled coincidence point for F and g. For this purpose we shall use (1) with $u = x_n$ and $v = y_n$, then take limit on both sides as n tends to infinity and use Equation (6).

$$\phi\big(d(F(x,y),F(x_n,y_n))\big) \le \psi_1\left(\frac{K(x,x_n)+K(y,y_n)}{2}\right) - \psi_2\left(\frac{K(x,x_n)+K(y,y_n)}{2}\right),$$

where

$$K(x,x_n) = \left(\frac{d(gx_n,F(x_n,y_n))[1+d(gx,F(x,y))]}{1+d(gx,gx_n)}, d(gx,gx_n)\right)$$

and

$$K(y,y_n) = \left(\frac{d(gy_n,F(y_n,x_n))[1+d(gy,F(y,x))]}{1+d(gy,gy_n)}, d(gy,gy_n)\right).$$

As $n\to\infty$, implies that $gx = F(x,y)$. A similar argument can be derived to show that $F(y,x) = gy$. This completes the proof and (x,y) is a coupled coincidence point of F and g.

For uniqueness, let (x,y) and (x^*,y^*) be two coupled coincidence points of F and g in X. Then by (1) we have

$$\phi\big(d(gx,gx^*)\big) = \phi\big(d(F(x,y),F(x^*,y^*))\big) \le \psi_1\left(\frac{K(x,x^*)+K(y,y^*)}{2}\right) - \psi_2\left(\frac{K(x,x^*)+K(y,y^*)}{2}\right),$$

where

$$\begin{aligned} K(x,x^*) &= \left(\frac{d(gx^*,F(x^*,y^*))[1+d(gx,F(x,y))]}{1+d(gx,gx^*)}, d(gx,gx^*)\right) \\ &= \big(0, d(gx,gx^*)\big) \end{aligned}$$

and

$$\begin{aligned} K(y,y^*) &= \left(\frac{d(gy^*,F(y^*,x^*))[1+d(gy,F(y,x))]}{1+d(gy,gy^*)}, d(gy,gy^*)\right) \\ &= \big(0, d(gy,gy^*)\big). \end{aligned}$$

That is

$$\frac{K(x,x^*)+K(y,y^*)}{2} = \left(0, \frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right).$$

By a similar way

$$\phi\big(d(gy,gy^*)\big)=\phi\big(d(F(y,x),F(y^*,x^*))\big)\leq\psi_1\left(\frac{K(y,y^*)+K(x,x^*)}{2}\right)-\psi_2\left(\frac{K(y,y^*)+K(x,x^*)}{2}\right),$$

Then,

$$\phi\big(d(gx,gx^*)\big)+\phi\big(d(gy,gy^*)\big)\leq 2\psi_1\left(0,\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right)-2\psi_2\left(0,\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right)$$

$$\phi\big(\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\big)\leq\psi_1\left(0,\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right)-\psi_2\left(0,\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right).$$

Since ψ_1 is increasing and $\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\geq 0$ then,

$$\begin{aligned}\phi\big(\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\big)&\leq\psi_1\left(\frac{d(gx,gx^*)+d(gy,gy^*)}{2},\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right)\\&\quad-\psi_2\left(0,\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right)\\ \phi\big(\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\big)&\leq\phi\left(\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right)\\&\quad-\psi_2\left(0,\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right).\end{aligned}$$

Therefore, $\psi_2\left(\frac{d(gx,gx^*)+d(gy,gy^*)}{2},\frac{d(gx,gx^*)+d(gy,gy^*)}{2}\right)=0\Rightarrow\frac{d(gx,gx^*)+d(gy,gy^*)}{2}=0$, i.e., $gx=gx^*$ and $gy=gy^*$. That is F and g have unique point of coincidence.

If we take $g=I$ (identity mapping) in Theorem ,we obtain the following corollary.

Corollary 3.2. *Let (X,d) be a complete metric space and $F:X\times X\to X$. Suppose that there exist $\phi\in\Phi$ and $\psi_1,\psi_2\in\Psi$ for which $\psi_1(x,x)\leq\phi(x)$ and*

$$\phi\big(d(F(x,y),F(u,v))\big)\leq\psi_1\left(\frac{K(x,u)+K(y,v)}{2}\right)-\psi_2\left(\frac{K(x,u)+K(y,v)}{2}\right),\tag{15}$$

where

$$K(x,u)=\left(\frac{d(u,F(u,v))[1+d(x,F(x,y))]}{1+d(x,u)},d(x,u)\right)$$

and

$$K(y,v)=\left(\frac{d(v,F(v,u))[1+d(y,F(y,x))]}{1+d(y,v)},d(y,v)\right),$$

for all $x,y,u,v\in X$. Then F has a unique coupled fixed point.

In the case of partially ordered metric space, we obtain the following theorem.

Theorem 3.3. *Let $(X,\preceq,d)$ be a partially ordered metric space. Suppose that $F: X\times X\to X$ and $g: X\to X$ are two self mappings on X, F has the g-mixed monotone property, $F(X,X)\subseteq g(X)$ and $g(X)$ is complete subspace of X. Suppose that there exist $\phi\in\Psi_1$ and $\psi_1,\psi_2\in\Psi_5$ such that $\psi_1(x,x,x,x,x)\le\phi(x)$ and*

$$\phi\big(d(F(x,y),F(u,v))\big)\le\psi_1\big(M(x,u)\big)-\psi_2\big(M(x,u)\big), \tag{16}$$

where

$$M(x,u)=\Bigg(\frac{d(gx,F(x,y))d(gu,F(u,v))}{d(gx,gu)},d(gx,gu),d(gx,F(x,y)),d(gu,F(u,v)),\\ \frac{1}{2}\big[d(gx,F(u,v))+d(gu,F(x,y))\big]\Bigg),$$

for all $x,y,u,v\in X$ with $gx\preceq gu$ and $gy\succeq gv$. Assume that X has the following properties:

- **(a):** *if a non-decreasing sequence $x_n\to x$, then $x_n\preceq x$ for all n,*
- **(b):** *if a non-increasing sequence $y_n\to y$, then $y_n\succeq y$ for all n.*

Moreover, if there exist $x_0,y_0\in X$ such that $x_0\preceq F(x_0,y_0)$ and $y_0\succeq F(y_0,x_0)$. Then F and g have a coupled coincidence point. This coupled coincidence point is unique if the set of all points of coincidence is totaly ordered subset of X, that is, if (x,y) and (x^*,y^*) are two coincidence points of F and g, then $gx\preceq gx^*$ and $gy\succeq gy^*$.

Proof.

Let $x_0,y_0\in X$ be such that $gx_0\preceq F(x_0,y_0)$ and $gy_0\succeq F(y_0,x_0)$. Since $F(X,X)\subseteq g(X)$, we can find $x_1,y_1\in X$ with $gx_1=F(x_0,y_0)$ and $gy_1=F(y_0,x_0)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1}=F(x_n,y_n)\ \text{and}\ gy_{n+1}=F(y_n,x_n). \tag{17}$$

By induction, we will prove that

$$gx_n\preceq gx_{n+1}\ \text{and}\ gy_n\succeq gy_{n+1}. \tag{18}$$

Since

$$gx_0\preceq F(x_0,y_0)=gx_1\ \text{and}\ gy_0\succeq F(y_0,x_0)=gy_1.$$

Thus (18) is true for $n = 0$. We suppose that (18) is true for some $n > 0$. Since F has the $g-$mixed monotone property, by (17) we have that

$$gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_{n+1}) = gx_{n+2} \text{ and } gy_{n+1} = F(y_n, x_n) \succeq F(y_{n+1}, x_{n+1}) = gy_{n+2},$$

that is (18)is true for all natural number n. If for some $n \in N$,

$$gx_n = gx_{n+1} \text{ and } gy_n = gy_{n+1}.$$

Then, by (17) (x_n, y_n) is a coupled coincidence point of F and g. Hence from now on we assume that at least

$$gx_n \neq gx_{n+1} \text{ or } gy_n \neq gy_{n+1}.$$

From (16) and (18), we get

$$\begin{aligned} \phi\big(d(gx_n, gx_{n+1})\big) &= \phi\big(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))\big) \\ &\leq \psi_1\big(M(x_{n-1}, x_n)\big) - \psi_2\big(M(x_{n-1}, x_n)\big) \qquad (19) \\ &\leq \psi_1\big(M(x_{n-1}, x_n)\big) \end{aligned}$$

and

$$\begin{aligned} M(x_{n-1}, x_n) = \Big(& \frac{d(gx_{n-1}, F(x_{n-1}, y_{n-1}))d(gx_n, F(x_n, y_n))}{d(gx_{n-1}, gx_n)}, d(gx_{n-1}, gx_n), d(gx_{n-1}, F(x_{n-1}, y_{n-1})), \\ & d(gx_n, F(x_n, y_n)), \frac{1}{2}\big[d(gx_{n-1}, F(x_n, y_n)) + d(gx_n, F(x_{n-1}, y_{n-1}))\big]\Big) \\ = & \Big(d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{1}{2}d(gx_{n-1}, gx_{n+1})\Big) \end{aligned}$$

By similar way, we have

$$\begin{aligned} \phi\big(d(gy_{n+1}, gy_n)\big) &= \phi\big(d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))\big) \\ &\leq \psi_1\big(M(y_n, y_{n-1})\big) - \psi_2\big(M(y_n, y_{n-1})\big) \\ &\leq \psi_1\big(M(y_n, y_{n-1})\big) \end{aligned}$$

and

$$M(y_n, y_{n-1}) = \Big(d(gy_n, gy_{n+1}), d(gy_{n-1}, gy_n), d(gy_n, gy_{n+1}), d(gy_{n-1}, gy_n), \frac{1}{2}d(gy_{n-1}, gy_{n+1})\Big)$$

Since ψ_1 is increasing with respect to the second argument, then we have for all $n \geq 1$

$$d(gx_n, gx_{n+1}) \leq d(gx_{n-1}, gx_n) \text{ and } d(gy_n, gy_{n+1}) \leq d(gy_{n-1}, gy_n).$$

With respect to the first inequality we conclude that the sequence $\{d(gx_n, gx_{n+1})\}$ is non-increasing sequence of positive real numbers, then there exist an $\varepsilon \geq 0$ such that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \varepsilon$$

We shall show that $\varepsilon = 0$. Taking the limit as $n \to \infty$ in (19) and using the continuity of ϕ, ψ_1 and ψ_2, we get.

$$\begin{aligned}\phi(\varepsilon) &\leq \psi_1(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon) - \psi_2(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon)\\ &\leq \phi(\varepsilon) - \psi_2(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon),\end{aligned}$$

which holds unless $\psi_2(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon) = 0$. Thus $\varepsilon = 0$ and

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{20}$$

Using the same idea, we deduce that

$$\lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0. \tag{21}$$

Now we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. For $\{gx_n\}$, suppose that it is not Cauchy. Then there exists an $\varepsilon > 0$ for which we can find two sequences $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$d(gx_{n_k}, gx_{m_k}) \geq \varepsilon \tag{22}$$

and

$$d(gx_{n_k-1}, gx_{m_k}) < \varepsilon. \tag{23}$$

By using triangle inequality and having in mind (20) to (23), we have

$$\begin{aligned}&\lim_{k\to\infty} d(gx_{n_k}, gx_{m_k}), \lim_{k\to\infty} d(gx_{n_k+1}, gx_{m_k+1}), \lim_{k\to\infty} d(gx_{n_k}, gx_{m_k+1}),\\ &\lim_{k\to\infty} d(gx_{n_k+1}, gx_{m_k}) \to \varepsilon \texttt{ as } k \to \infty.\end{aligned} \tag{24}$$

Since $n_k > m_k$, so from (18), $gx_{nk} \succeq gx_{m_k}$ and $gy_{nk} \preceq gy_{m_k}$. By (16) we have

$$\begin{aligned}\phi\big(d(gx_{m_k+1}, gx_{n_k+1})\big) &= \phi\big(d(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k}))\big)\\ &\leq \psi_1\big(M(x_{m_k}, x_{n_k})\big) - \psi_2\big(M(x_{m_k}, x_{n_k})\big)\\ &\leq \psi_1\big(M(x_{m_k}, x_{n_k})\big)\end{aligned}$$

and

$$M(x_{m_k},x_{n_k}) = \Big(\frac{d(gx_{m_k},gx_{m_k+1})d(gx_{n_k},gx_{n_k+1})}{d(gx_{m_k},gx_{n_k})}, d(gx_{m_k},gx_{n_k}),$$
$$d(gx_{m_k},gx_{m_k+1})), d(gx_{n_k},gx_{n_k+1})), \frac{1}{2}\big[d(gx_{m_k},gx_{n_k+1}) + d(gx_{n_k},gx_{m_k+1})\big]\Big)$$

Taking limit as k tends to infinity, using (20, 24) and all properties on ϕ, ψ_1, ψ_2 imply

$$\begin{aligned}\phi(\varepsilon) &\le \psi_1(0,\varepsilon,0,0,\varepsilon) - \psi_2(0,\varepsilon,0,0,\varepsilon)\\ &\le \psi_1(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon) - \psi_2(0,\varepsilon,0,0,\varepsilon)\\ &\le \phi(\varepsilon) - \psi_2(0,\varepsilon,0,0,\varepsilon)\\ &\Rightarrow \psi_2(0,\varepsilon,0,0,\varepsilon) = 0 \Rightarrow \varepsilon = 0,\end{aligned}$$

which is a contradiction. This shows that $\{gx_n\}$ is Cauchy sequence in $g(X)$. By a similar way one can deduce that also $\{gy_n\}$ is Cauchy sequence. Since $g(X)$ is complete then there exist $gx, gy \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \ \texttt{and} \ \lim_{n\to\infty} gy_n = gy. \tag{25}$$

Now we show that (x,y) is coupled coincidence point for F and g. For this purpose we shall use the properties on X and apply (16) with $x = x_n$, $y = y_n$, $u = x$ and $v = y$, then take limit on both sides as n tends to infinity and use Equation (25).

$$\phi\big(d(F(x_n,y_n),F(x,y))\big) \le \psi_1\big(M(x_n,x)\big) - \psi_2\big(M(x_n,x)\big),$$

where

$$M(x_n,x) = \Big(\frac{d(gx_n,gx_{n+1})d(gx,F(x,y))}{d(gx_n,gx)}, d(gx_n,gx), d(gx_n,gx_{n+1}), d(gx,F(x,y)),$$
$$\frac{1}{2}\big[d(gx_n,F(x,y)) + d(gx,gx_{n+1})\big]\Big) \to (0,0,0,d(gx,F(x,y)),\frac{d(gx,F(x,y))}{2})$$

Taking limit as n tends to infinity implies that $gx = F(x,y)$. A similar argument can be derived to show that $F(y,x) = gy$. This completes the proof and (x,y) is a coupled coincidence point of the mappings F and g.

For uniqueness, let (x,y) and (x^*,y^*) be two distinct coupled coincidence points of F and g in X. Then by (16) we have

$$\phi\big(d(F(x,y),F(x^*,y^*))\big) \leq \psi_1\big(M(x,x^*)\big) - \psi_2\big(M(x,x^*)\big),$$

where

$$\begin{aligned} M(x,x^*) = &\bigg(\frac{d(gx,F(x,y))d(gx^*,F(x^*,y^*))}{d(gx,gx^*)}, d(gx,gx^*), d(gx,F(x,y)), d(gx^*,F(x^*,y^*)), \\ &\frac{1}{2}\big[d(gx,F(x^*,y^*)) + d(gx^*,F(x,y))\big]\bigg) \\ &= (0, d(gx,gx^*), 0, 0, d(gx,gx^*)) \end{aligned}$$

By a similar way

$$\phi\big(d(gy,gy^*)\big) = \phi\big(d(F(y,x),F(y^*,x^*))\big) \leq \psi_1\big(M(y,y^*)\big) - \psi_2\big(M(y,y^*)\big),$$

$$\begin{aligned} M(y,y^*) = &\bigg(\frac{d(gy,F(y,x))d(gy^*,F(y^*,x^*))}{d(gy,gy^*)}, d(gy,gy^*), d(gy,F(y,x)), d(gy^*,F(y^*,x^*)), \\ &\frac{1}{2}\big[d(gy,F(y^*,x^*)) + d(gy^*,F(y,x))\big]\bigg) \\ &= (0, d(gy,gy^*), 0, 0, d(gy,gy^*)) \end{aligned}$$

Therefore, $gx = gx^*$ and $gy = gy^*$. That is F and g have unique point of coincidence.

Here, we derive the following corollaries from our Theorem . Also we illustrate our results by an example.

Corollary 3.4. *Let $(X,\preceq,d)$ be a complete partially ordered metric space. Suppose that $F : X \times X \to X$ is a self mapping on X such that F has the mixed monotone property. Suppose that there exist $\alpha,\beta,\gamma,\delta \in [0,1)$ with $\alpha+\beta+2\gamma+2\delta < 1$ such that*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq \alpha&\left(\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)}\right) + \beta d(x,u) + \\ &\gamma\big[d(x,F(x,y)) + d(u,F(u,v))\big] + \delta\big[d(x,F(u,v)) + d(u,F(x,y))\big] \end{aligned} \tag{26}$$

for all $x,y,u,v \in X$ with $x \preceq u$ and $y \succeq v$. Assume that either

- **1.:** *F is continuous or*
- **2.:** *X has the following properties:*
  - **(a):** *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*

(b): *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n.*

Moreover, if there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Then F has a coupled fixed point $(x, y) \in X \times X$. [Theorem 2.1 in [7]]

Proof. This corollary is a consequence of Theorem by taking

- $g(x) = x, \ \ \forall x \in X$
- $\phi(t) = t \ \ \forall t \geq 0$
- $\psi_1(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma[t_3 + t_4] + 2\delta t_5$
- $\psi_2(t_1, t_2, t_3, t_4, t_5) = 0.$

Corollary 3.5. *Let $(X, \preceq, d)$ be a complete partially ordered metric space and $F : X \times X \to X$ be a given mapping having the mixed monotone property such that there exists $k \in [0, 1)$ with*

(27)
$$d(F(x,y),F(u,v)) \leq k \max\left\{\frac{d(gx,F(x,y))d(gu,F(u,v))}{d(gx,gu)}, d(gx,gu), d(gx,F(x,y)), d(gu,F(u,v)),\right.$$
$$\left.\frac{1}{2}\big[d(gx,F(u,v)) + d(gu,F(x,y))\big]\right\}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$. Assume either

1.: *F is continuous or*

2.: *X has the following properties:*

(a): *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*

(b): *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Then F has a coupled fixed point.

Proof. It suffices to take

- $g(x) = x, \ \ \forall x \in X$
- $\phi(t) = t \ \ \forall t \geq 0$
- $\psi_1(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\}$
- $\psi_2(t_1, t_2, t_3, t_4, t_5) = (1 - k) \max\{t_1, t_2, t_3, t_4, t_5\}.$

Corollary 3.6. *Let $(X,\preceq,d)$ be a complete partially ordered metric space and $F : X\times X \to X$ be a given mapping having the mixed monotone property such that there exists $k \in [0,1)$ with*

(28)
$$d(F(x,y),F(u,v)) \leq \frac{k}{5}\Bigg[\frac{d(gx,F(x,y))d(gu,F(u,v))}{d(gx,gu)}, d(gx,gu), d(gx,F(x,y)), d(gu,F(u,v)), \frac{1}{2}\big[d(gx,F(u,v)) + d(gu,F(x,y))\big]\Bigg]$$

for all $x,y,u,v \in X$ with $x \preceq u$ and $y \succeq v$. Assume either F is continuous, or X has the following properties:

(a): *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*

(b): *if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all n.*

If there exist $x_0,y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. Then F has a coupled fixed point.

Proof. Here we take

- $g(x) = x, \ \forall x \in X$
- $\phi(t) = t \ \forall t \geq 0$
- $\psi_1(t_1,t_2,t_3,t_4,t_5) = \frac{1}{5}[t_1+t_2+t_3+t_4+t_5]$
- $\psi_2(t_1,t_2,t_3,t_4,t_5) = \frac{1-k}{5}[t_1+t_2+t_3+t_4+t_5]$.

Finally, we give the following example to verify Theorem as follows.

Example 3.1. *Let $X = [0,\infty)$ be endowed with its Euclidian metric $d(x,y) = |x-y|$ and its usual ordering $\leq$. Take $F : X \to X$ defined by*

$$F(x,y) = \begin{cases} \frac{x-3y}{5} & x \geq 3y; \\ 0 & otherwise. \end{cases}$$

and ϕ,ψ_1 and ψ_2 as in Corollary . Take $k = \frac{5}{6}$.

We claim that (28) holds for each $x \leq u$ and $y \geq v$. We divide the proof into the following four cases

Case I. If $x \geq 3y$ and $u \geq 3v$, here we have $F(x,y) = \frac{x-3y}{5}$ and $F(u,v) = \frac{u-3v}{5}$,

$$
\begin{aligned}
d\big(F(x,y),F(u,v)\big) &= \big|\frac{x-3y}{5} - \frac{u-3v}{5}\big| = \big|\frac{x-u}{5} + \frac{3}{5}(v-y)\big| \\
&= \frac{u-x}{5} + \frac{3}{5}(y-v) \\
&= \frac{u-x}{6} + \frac{u-x}{30} + (\frac{9+3+6}{30})(y-v) \\
&\leq \frac{u-x}{6} + \frac{u}{30} + (\frac{9+3+6}{30})y \\
&\leq \frac{u-x}{6} + \frac{u}{30} + \frac{3u}{30} + \frac{3y}{30} + \frac{4x}{30} \\
&\leq \frac{1}{6}\big[(u-x) + \frac{4u+3v}{30} + \frac{4x+3y}{30}\big] \\
&\leq \frac{1}{6}\big[d(x,u) + d(u,F(u,v)) + d(x,F(x,y))\big] \\
&\leq \frac{k}{5}\Big[\frac{d(gx,F(x,y))d(gu,F(u,v))}{d(gx,gu)}, d(gx,gu), d(gx,F(x,y)), d(gu,F(u,v)), \\
&\frac{1}{2}\big[d(gx,F(u,v)) + d(gu,F(x,y))\big]\Big]
\end{aligned}
$$

Note that, since $3y \leq x \leq u \Rightarrow 9y \leq 3u$.

Case II. If $x \geq 3y$ and $u < 3v$, here we have $F(x,y) = \frac{x-3y}{5}$ and $F(u,v) = 0$,

$$
\begin{aligned}
d\big(F(x,y),F(u,v)\big) &= \big|\frac{x-3y}{5}\big| \leq \frac{x}{5} \\
&\leq \frac{x-u}{6} + \frac{u}{6} + \frac{x}{30} \\
&\leq \frac{x-u}{6} + \frac{u}{6} + \frac{4x+3y}{30} \\
&\leq \frac{1}{6}\big[(x-u) + u + \frac{4x+3y}{30}\big] \\
&\leq \frac{1}{6}\big[d(x,u) + d(u,F(u,v)) + d(x,F(x,y))\big] \\
&\leq \frac{k}{5}\Big[\frac{d(gx,F(x,y))d(gu,F(u,v))}{d(gx,gu)}, d(gx,gu), d(gx,F(x,y)), d(gu,F(u,v)), \\
&\frac{1}{2}\big[d(gx,F(u,v)) + d(gu,F(x,y))\big]\Big]
\end{aligned}
$$

Case III. If $x < 3y$ and $u \geq 3v$, here we have $F(x,y) = 0$ and $F(u,v) = \frac{u-3v}{5}$,

$$\begin{aligned}
d\big(F(x,y),F(u,v)\big) &= \Big|\frac{u-3v}{5}\Big| \leq \frac{u}{5} \\
&\leq \frac{u-x}{6} + \frac{x}{6} + \frac{u}{30} \\
&\leq \frac{u-x}{6} + \frac{x}{6} + \frac{4u+3v}{30} \\
&\leq \frac{1}{6}\big[(u-x) + x + \frac{4u+3v}{30}\big] \\
&\leq \frac{1}{6}\big[d(x,u) + d(x,F(x,y)) + d(u,F(u,v))\big] \\
&\leq \frac{k}{5}\Bigg[\frac{d(gx,F(x,y))d(gu,F(u,v))}{d(gx,gu)}, d(gx,gu), d(gx,F(x,y)), d(gu,F(u,v)), \\
&\quad \frac{1}{2}\big[d(gx,F(u,v)) + d(gu,F(x,y))\big]\Bigg]
\end{aligned}$$

Case IV. If $x < 3y$ and $u < 3v$, here we have $F(x,y) = 0$ and $F(u,v) = 0$,

$$\begin{aligned}
d\big(F(x,y),F(u,v)\big) &= 0 \\
&\leq \frac{k}{5}\Bigg[\frac{d(gx,F(x,y))d(gu,F(u,v))}{d(gx,gu)}, d(gx,gu), d(gx,F(x,y)), d(gu,F(u,v)), \\
&\quad \frac{1}{2}\big[d(gx,F(u,v)) + d(gu,F(x,y))\big]\Bigg]
\end{aligned}$$

Moreover, it is easy to see that all other hypotheses of Corollary are verified . So F has a coupled fixed point $(0,0) \in X^2$.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] H. Aydi, Some coupled fixed point results on partial metric spaces, Int. J. Math. Math. Sci., 2011 (2011), Article ID 647091, 11 pages.

[2] H. Aydi, M. Abbas and M. Postolache, Coupled coincidence points for hybrid pair of mappings via mixed monotone property, J. Adv. Math. Stud. 5 (1) (2012), 118-126.

[3] H. Aydi, B. Damjanović, B. Samet and W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, Math. Comput. Model. 45 (2011), 2443-2450.

[4] H. Aydi, B. Samet and C. Vetro, Coupled fixed point results in cone metric spaces for w̃-compatible mappings, Fixed Point Theory and Appl. 2011 (2011), Article ID 27.

[5] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fund. Math. 3 (1922), 133-181.

[6] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006), 1379-1393.

[7] S. Chandok, T. D. Narang and M. A. Taoudi, Some coupled fixed point theorems for mappings satisfying a generalized contractive condition of rational type, Palestine J. Math. 4 (2) (2015), 360-366.

[8] B. S. Choudhury, A common unique fixed point result in metric spaces involving generalized altering distances, Math. Commun. 10 (2005), 105-110.

[9] B. S. Choudhury and P. N. Dutta, A unified fixed point result in metric spaces involving a two variable function, Filomat 14 (2000) , 43-48.

[10] B. S. Choudhury, N. Metiya and A. Kundu, Coupled coincidence point theorems in ordered metric spaces, Ann. Univ. Ferrara 57 (2011), 1-16.

[11] B. S. Choudhury and A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal. 73 (2010), 2524-2531.

[12] D. Delbosco, Unestensione di un teorema sul punto di S. Reich, Rendiconti del Seminario Matematico Universitie Politecnico di Torino 35 (1967), 233-238.

[13] V. Gupta and R. Deep, Some coupled fixed point theorems in partially ordered S-metric spaces, Miskolc Math. Notes 16 (1) (2015), 181-194.

[14] A. Gupta and R. Kushwaha, Tripled common fixed points for weak (μ, φ, ψ)- contractions in partially ordered metric space, Math. Theory Model. 3 (6) (2013), 46-53.

[15] V. Gupta, Ramandeep, N. Mani and A. K. Tripathi, Some fixed point result involving generalized altering distance function, Procedia Comput. Sci. 79 (2016), 112-117.

[16] M. S. Khan, M. Swalesh and S. Sessa, Fixed point theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1) (1984), 1- 9.

[17] V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341-4349.

[18] N. V. Luong and N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011), 983-992.

[19] R. D. Marr, Partially ordered space and metric spaces, Amer. Math. Monthly 72 (6) (1965), 628-631.

[20] Mujeeb Ur Rahman and M. Sawar, Coupled fixed point theorems for rational contraction conditions in dislocated quasi-metric space, Palestine J. Math. 5 (2) (2016), 6-11.

[21] H. K. Nashine and H. Aydi, Coupled fixed point theorems for contractions involving altering distances in ordered metric spaces, Math. Sci. 7 (2013), Article ID 20.

[22] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (5) (2003), 1435-1443.

[23] K. P. R. Rao, A. S. Babu and D. V. Babu, Common fixed point theorems through generalized altering distance functions, Int. Math. Forum 2 (65) (2007), 3233-3239.

[24] R. A. Rashwan and H. A. Hammad, On random coincidence point and random coupled fixed point theorems in ordered metric spaces, JP J. Fixed Point Theory Appl. 11 (2) (2016), 125-160.

[25] R. A. Rashwan and S. M. Saleh, Some common fixed point theorems for four (ψ,φ)-weakly contractive mappings satisfying rational expressions in ordered partial metric spaces, Int. J. Nonlinear Anal. Appl. 7 (1) (2016), 111-130.

[26] B. Samet and C. Vetro, Coupled fixed point, f-invariant set and fixed point of N-order, Ann. Funct. Anal. 1 (2) (2010), 46-56.

[27] F. Skof, Teorema di punti fisso per applicazioni negli spazi metrici, Atti. Aooad. Soi. Torino 111 (1977), 323-329.