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FIXED POINT THEOREM OF NONLINEAR CONTRACTION IN METRIC SPACE

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Abstract: In the present paper, I present a shorter proof by generalizing the main results of Sayyed et al. by replacing the containment condition in the context of metric space.

Keywords: common fixed point; compatible mapping property (E.A.); common property (E.A.); occasionally weakly compatible maps; coincidence point.

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1. Introduction

Aamri *et al.* [1] introduced the concept of property (E.A.) which was perhaps inspired by the condition of compatibility introduced by Jungck [11] and further Imdad *et al.* [10] extended this result. Babu *et al.* [7, 8, 9] proved common fixed point theorem for occasionally weakly compatible maps satisfying property (E.A.) using an inequality involving quadratic terms. Aliouche [4] proved a common fixed point theorem of Gregus type weakly compatible mappings satisfying generalized contractive conditions.

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Abbas [2] established a common fixed point for Lipschitzian mapping satisfying rational contractive conditions. Murty *et.al*. [15] proved fixed points of nonlinear contraction in metric space.

2. Preliminaries

Throughout this paper (X, d) is a metric space which is denoted by X.

Definition 2.1: Jungck and Rhoades [13]. Let A and S be selfmaps of a set X. If $Au = Su = \omega$ (say), $\omega \in X$, for some u in X, then u is called a coincidence point of A and S and the set of coincidence points of A and S is denoted by $C(A, S)$, and ω is called a point of coincidence of A and S.

Definition 2.2: Let A, B, S and T be self maps of a set X. If $u \in C(A,S)$ and $v \in C(B,T)$ for some $u, v \in X$ and $Au = Su = Bv = Tv = z$ (say), then z is called a common point of coincidence of the pairs $(A. S)$ and $(B. T)$.

Definition 2.3: The pair (A, S) is said to:

(I) Satisfy property $(E.A.)$ [1] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

(II) Compatible [11] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, for some t in X whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$.

(III) Weakly compatible [12], if they commute at their coincidence point.

(IV) Occasionally weakly compatible (owc) [3, 5, 6] if $ASx = SAx$ for some $x \in C(A,S)$.

Remark 2.4

(I) [12] Every compatible pair is weakly compatible but its converse need not be true.

(II) [16] Weak compatibility and property (*E. A.*) are independent of each other.

(III) [11] Every weakly compatible pair is occasionally weakly compatible but its converse need not be true.

(IV) [8] Occasionally weakly compatible and property (*E.A.*) are independent of each other.

Definition 2.5: [14] Let (*X*, *d*) be a metric space and *A, B, S* and *T* be four selfmaps on X. The pairs (A,S) and (B,T) are said to satisfy common property (*E.A.*) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n$ for some t in X.

Remark 2.6: Let A,B,S and *T* be self maps of a set X. If the pairs (*A, S*) and (*B, T*) have common point of coincidence in *X* then *C*(*A, S*) ≠ ϕ and *C* (*B, T*) ≠ ϕ. But converse is not true.

Example 2.7: Let *X* = [0, ∞) with usual metric and *A, B, S* and *T* self maps on *x* and defined by $Ax = 1 - x^2; sx = 1 - x; Bx = \frac{1}{2} + x^2; Tx = \frac{1+x}{2}$ for all $x \in X$.

It is easy to observe that $C(A,S) = \{0,1\}$ and $C(B,T) = \left\{0, \frac{1}{2}\right\}$ but the pairs (*A, S*) and (*B, T*) not having common point of coincidence.

Remark 2.8: The converse of the remark 2.6 is true, provided it satisfies inequality (3.1). This is given as proposition (3.1).

Preposition 2.9: [2] Let *A* and *S* be two self maps of a set *X* and the pair *(A, S)* satisfies occasionally weakly compatible (owc) condition. If the pairs (*A, S*) have unique point of coincidence *Ax* = *Sx* = *z* then *z* is the unique common fixed point of *A* and *S*.

Proof: To be given *Ax* = *Sx* = {*z*} (say) for any $x \in C(A,S)$. (2.1)

Since the pair (*A*, *S*) satisfies the property owc, therefore

$$Az = ASx = SAx = Sz \text{ implies that } \quad z \in C(A,S).$$

From (2.1), $Az = Sz = z$. Hence proposition follows.

In 1996, Tas et al. [18] proved the following.

Theorem 2.10: *Let A, B, S and T be selfmaps of a complete metric space* (X,d) *such that* $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$ *and satisfying the inequality.*

$$[d(Ax,By)]^2 \le C_1 \max\left\{[d(Sx,Ax)]^2, [d(Ty,By)]^2, [d(Sx,Ty)]^2\right\}$$

$$+C_2 \max\{d(Sx,Ax)d(Sx,By), d(Ty,Ax)d(Ty,By)\}$$

$$+C_3 d(Sx,By)d(Ty,Ax)$$

for all $x, y \in X$, *where* $C_1 + C_3, C_2, C_3 \ge 0, C_1 + 2C_2 < 1, C_1 + C_3 < 1$. *Further, assume that the pairs* (A,S) *and* (B,T) *are compatible on X. If one of the mappings* A, B, S *and T is continuous then* A, B, S *and T have a unique common fixed point in X.*

3. Main results

Proposition 3.1. Let *A*, *B*, *S* and *T* be self maps of a metric space (*X*, *d*) and satisfying the inequality.

$$\text{d(Ax,By)} \le \text{k max} \left\{ \frac{d(Sx,Ax)[1+d(Sx,Ax)]}{1+d(Sx,ty)}, \text{d(Sx,Ty)}, \frac{d(Ty,By)[1+d(Sx,Ty)]}{1+d(Ax,Ty)} \right\} \quad (3.1)$$

for all $x, y \in X$, where k ≥ 0 and k < 1. Then the pairs (A,S) and (B,T) have common point of coincidence in X if and only if $C(A,S) \ne \phi$ and $C(B,T) \ne \phi$.

Proof: If part: It is trivial

Only if part: Assume $C(A,S) \ne \phi$ and $C(B,T) \ne \phi$.

Then there is a $u \in$ C (A, S) and $v \in$ C (B, T) such that

$$Au = Su = p \quad \text{(say)} \qquad (3.2)$$

$$Bv = Tv = q \quad \text{(say)} \qquad (3.3)$$

on taking $x = u$ and $y = v$ in (3.1), we get

$$d(Au,Bv) \leq k \max \left\{ \frac{d(su,Au)[1+d(Su,Au)]}{1+d(Su,tv)},\ d(Su,Tv), \frac{d(Tv,Bv)[1+d(Su,Tv)]}{1+d(Au,Tv)} \right\}.$$

Using (3.2) and (3.3), we get

d (p,q) $\leq$ k d(p,q), a contradiction. Thus $p = q$.

Therefore A, B, S and T have common point of coincidence in X.

In the proposition (2.1) of Babu et al. [9], we can obtain some more conclusions from his paper. Therefore our result improves and strengthens proposition (3.1) and subsequent theorems in metric spaces.

Proposition 3.2: Let *A*, *B*, *S* and *T* be four self maps of a metric space (X,d) satisfying the inequality (3.1). Suppose that either

(i) $B(X) \subseteq S(X)$, the pair (B,T) satisfies property (E.A.) and $T(X)$ is a closed subspace of *X*; or

(ii) $A(X) \subseteq T(X)$, the pair (A,S) satisfies property $(E.A)$ and $S(X)$ is a closed subspace of X holds.

Then the pair (A,S) and (B,T) satisfies the common property (*E*.A), also both the pairs (A,S) and (B,T) have common point of coincidence in *X*.

I have shortened the proof of theorem 2.2 of [9] by relaxing many lines:

Theorem 3.3: *(Improved version of theorem (2.2) of [9])*
Let A, B, S *and T are satisfying all the conditions given in proposition (3.2) with the following additional assumption.*

The pairs (A, S) *and* (B, T) *are on X.*

Then A, B, S *and* T *have a unique common fixed point in X.*

Proof: By proposition (3.2) the pairs (A, S) and (B, T) have common point of coincidence.

Therefore there is $u \in C(A, S)$ and $v \in C(B, T)$ such that

$$Au = Su = z \quad \text{(say)} = Bv = Tv \qquad (3.4)$$

Now, we show that z is unique common point of coincidence of the pairs (A, S) and (B, T).

Let if possible z' is another point of coincidence of A, B, S and T. Then there is $u' \in$ C (A,S) and $v' \in$ C (B,T) such that

$$Au' = Su' = z' \quad \text{(say)} = Bv' = Tv' \qquad (3.5)$$

Putting $x = u$ and $y = v'$ in inequality (3.1), we have

$$d(Au,Bv') \leq k \max \left\{ \frac{d(Su,Au)[1+d(Su,Au)]}{1+d(Su,Tv')}, d(Su,Tv'), \frac{d(Tv',Bv')[1+d(Su,Tv')]}{1+d(Au,Tv')} \right\}$$

Now using (3.4) and (3.5), we get
$d(z,z') \leq k\, d(z,z')$, and arrive at a contradiction. Hence $z = z'$ and we have

$C(A, S) = \{z\} = C(B, T)$. By proposition (2.9), z is the unique common fixed point of A, B, S and T in X.

Remark 3.4: Proposition (2.5) of [9] and theorem (2.6) of [9] remain true, if we replace completeness of $S(X)$ and $T(X)$ by the completeness of $S(X) \cap T(X)$ in X. For this we have given an example 2.7 in the following manner without proof.

Now we rewriting the proposition (2.5) and theorem 2.6 of [9].

Proposition 3.5: Let *A*, *B*, *S* and *T* be four self maps of a metric space (*X*, *d*) satisfying the inequality (3.1) of proposition (3.1). Suppose that (A,S) and (B,T) satisfy a common property $(E.A)$ and $S(X)\cap T(X)$ are closed subset of *X*, then *A*, *B*, *S* and *T* have a unique common point of coincidence.

Therefore theorem (3.3) in addition to the above proposition (3.5) on *A*, *B*, *S* and *T*, if both the pairs (*A*, *S*) and (*B*, *T*) are owc maps on *X*, then the point of coincidence is a unique common fixed point of A,B,S and *T*.

Conflict of Interests

The authors declare that there is no conflict of interests.

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