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FIXED POINT THEOREMS FOR WEAK S-CONTRACTIONS IN PARTIALLY ORDERED 2-METRIC SPACES

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Abstract. Fixed point results for weak S-contractions on partially ordered 2-metric spaces are developed.

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1. Introduction

Banach's contraction theorem is one of the significant results of nonlinear analysis, which also became the origin of understanding iterative and dynamical processes. Some methods, such as Picard and Newton iterative methods, are based on this theorem. A mapping $T : X \to X$ where (X,d) is a metric space, is said to be a contraction if there exists $k \in [0,1)$ such that for

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all $x, y \in X$,

$$d(Tx, Ty) \le kd(x, y). \tag{1}$$

The mapping satisfying (1) has a unique fixed point, provided that the metric space (X, d) is complete. Also, inequality (1) implies continuity of T. In [1, Theorem 3.5], Banach's contraction theorem has been generalized for two self-maps $T : X \to X$ and $S : X \to X$. These contractions can be employed in metric spaces, 2-metric spaces and b_2-metric spaces;see[2, Lemma 1.6].

In [3], Kannan established contraction conditions which imply existence of fixed point in complete metric space but do not imply continuity. Based on the result, if $T : X \to X$ where (X, d) is a complete metric space, satisfies the inequality

$$d(Tx, Ty) \le k[d(x, Tx) + d(y, Ty)], \tag{2}$$

where $k \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$, then T has a unique fixed point. A similar contractive condition has been introduced by Shukla [4].

2. Preliminaries

Definition 1 (Weak Contraction [5])**.** *A mapping $T : X \to X$ on a complete metric space (X, d) is said to be a weakly contractive mapping if*

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)), \tag{3}$$

where $x, y \in X$, $\psi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing such that $\psi(x) = 0$ iff $x = 0$ and $\lim_{\mathrm{x} \to \infty} \psi(\mathrm{x}) = \infty$.

By setting $\psi(x) = kx$, where $k \in [0, 1)$, then (3) reduces to (1).

Definition 2 (S-Contraction [4])**.** *A mapping $T : X \to X$ where (X, d) is a complete metric space, is said to be a S-contraction if there exists $k \in \left[0, \frac{1}{3}\right)$ such that for all $x, y \in X$ the following inequality holds:*

$$d(Tx, Ty) \le k[d(x, Ty) + d(Tx, y) + d(x, y)]. \tag{4}$$

In [6], a weaker contraction has been introduced in Hilbert Spaces. The above notion was generalized to a weakly S-contraction by Shukla in [4].

Definition 3 (Weak S-Contraction [4])**.** *A mapping $T : X \to X$ on a complete metric space (X,d) is said to be weak S-contractive mapping or weak S-contractions if the following inequality holds:*

$$
\begin{aligned}
d(Tx,Ty) &\leq \frac{1}{3}\Big[d(x,Ty)+d(Tx,y)+d(x,y)\Big] \\
&\quad -\psi\left(d(x,Ty),d(Tx,y),d(x,y)\right)
\end{aligned}
\tag{5}
$$

for all $x,y \in X$, where $\psi : [0,\infty)^3 \longrightarrow [0,\infty)$ is a continuous mapping with $\psi(x,y,z) = 0$ iff $x = y = z = 0$ and $\lim_{x\to\infty} \psi(x) = \infty$.

By setting $\psi(x,y,z) = k(x+y+z)$, where $k \in \left[0,\frac{1}{3}\right)$, the generalized mapping (5) reduces to (4).

Definition 4 (Partially ordered set)**.** *If $(X,\preceq)$ is a partially ordered set and $T : X \to X$, we say that T is monotone non-decreasing if $x,y \in X$, $x \preceq y$, then $Tx \preceq Ty$.*

This definition coincides with the notion of a non-decreasing map in the case where $X = R$ and $\preceq$ represents the usual total order in R.

Definition 5 (2-metric space [7])**.** *Let X be a non-empty set and let $d : X \times X \times X \to R$ be a map satisfying the following conditions:*

1. *For every pair of distinct points $x,y \in X$, thee exists a point $z \in X$ such that $d(x,y,z) \neq 0$.*
2. *If at least two of three points x,y,z are the same, then $d(x,y,z) = 0$.*
3. *The symmetric property $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$ holds for all $x,y,z \in X$.*
4. *The rectangle inequality $d(x,y,z) \leq d(x,y,t)+d(y,z,t)+d(z,x,t)$ holds for all $x,y,z,t \in X$.*

Then d is called a 2-metric on X and (X,d) is called a 2-metric space.

Definition 6 ([7])**.** *Let (X,d) be a 2-metric space and $a,b \in X$, $r \geq 0$. The set $B(a,b,r)$ is called a 2-ball centered at a and b with radius r. The topology generated by the collection of all 2-balls as a sub-basis is called a 2-metric topology on X.*

Definition 7 ([8])**.** *Let $\{x_n\}$ be a sequence in a 2-metric space (X,d).*

1. *$\{x_n\}$ is said to be convergent to x in (X,d), written $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, we have $\lim_{n\to\infty} d(x_n,x,a) = 0$.*
2. *$\{x_n\}$ is said to be Cauchy in X if for all $a \in X$, $\lim_{n\to\infty} d(x_n,x_m,a) = 0$. That is, for each $\varepsilon > 0$, there exists n_l such that $d(x_n,x_m,a) < \varepsilon$ for all $n,m \geq n_l$.*
3. *(X,d) is said to be complete if every Cauchy sequence is a convergent sequence.*

Lemma 1 ([9], Lemma 3)**.** *Every 2-metric space is a T_1-space.*

Lemma 2 ([9], Lemma 4)**.** $\lim_{n\to\infty} x_n = x$ *in a 2-metric space (X,d), iff* $\lim_{n\to\infty} x_n = x$ *in the 2-metric topological space X.*

Lemma 3 ([9], Lemma 5)**.** *If $T : X \to Y$ is a continuous map from a 2-metric space X to a 2-metric space Y, then* $\lim_{n\to\infty} x_n = x$ *in X implies* $\lim_{n\to\infty} Tx_n = Tx$ *in Y.*

Remark 1. *(*[10]*)*

1. *Every 2-metric is non-negative and every 2-metric space contains at least three points.*
2. *A 2-metric $d(x,y,z)$ is sequentially continuous in one argument. Furthermore, if a 2-metric $d(x,y,z)$ is sequentially continuous in two arguments, then it is sequentially continuous in all three arguments.*
3. *A convergent sequence in a 2-metric space need not be a Cauchy sequence.*
4. *In a 2-metric space (X,d), every convergent sequence in a Cauchy sequence if d is continuous.*
5. *There exists a 2-metric space (X,d) such that every convergent sequence is a Cauchy sequence but d is not continuous.*

3. Main result

Our starting point is the definition of weak S-contraction on a partially ordered 2-metric space.

Definition 8. *Let $(X,\preceq,d)$ be a partially ordered 2-metric space and $T: X \to X$ be a map. Then T is called a weak S-contraction if there exists $\psi : [0,\infty)^3 \to [0,\infty)$ which is continuous and $\psi(w,s,t) = 0$ iff $s = w = t = 0$ such that:*

$$\begin{aligned} d(Tx,Ty,a) &\leq \frac{1}{3}\Big[d(x,Ty,a)+d(y,Tx,a)+d(x,y,a)\Big] \\ &\quad -\psi\left(d(x,Ty,a),d(y,Tx,a),d(x,y,a)\right) \end{aligned} \tag{6}$$

for all $x,y,a \in X$ and $x \preceq y$ or $y \preceq x$.

In what follows, we present our main fixed point theorems for weak S-contraction mappings on partially ordered 2-metric spaces.

Theorem 4. *Let $(X,\preceq,d)$ be a complete, partially ordered 2-metric space and $T: X \to X$ be a weak S-contraction such that:*

1. *T is continuous and non-decreasing.*
2. *There exists $x_0 \in X$ with $x_0 \preceq Tx_0$.*

Then T has a fixed point.

Proof. If $x_0 = Tx_0$, then the proof is finished. Suppose now that $x_0 \preceq Tx_0$. Since T is a non decreasing map, we have $x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \cdots \leq T^nx_0 \preceq \cdots$. Put $x_{n+1} = Tx_n$. Then, for all $n \geq 1$, form (6) and noting that x_{n-1} and x_n are comparable, we obtain:

$$\begin{aligned} d(x_{n+1},x_n,a) &= d(Tx_n,Tx_{n-1},a) \\ &\leq \frac{1}{3}\Big[d(x_n,Tx_{n-1},a)+d(x_{n-1},Tx_n,a)+d(x_n,x_{n-1},a)\Big] \\ &\quad -\psi\Big(dx_n,Tx_{n-1},a),d(x_{n-1},Tx_n,a),d(x_n,x_{n-1},a)\Big) \\ &= \frac{1}{3}\Big[d(x_n,x_n,a)+d(x_{n-1},x_{n+1},a)+d(x_n,x_{n-1},a)\Big] \end{aligned}$$

(7)

$$
\begin{aligned}
& -\psi\Big(d(x_n,x_n,a),d(x_{n-1},x_{n+1},a),d(x_n,x_{n-1},a)\Big)\\
= \;& \frac{1}{3}\Big[d(x_{n-1},x_{n+1},a)+d(x_n,x_{n-1},a)\Big]\\
& -\psi\Big(0,d(x_{n-1},x_{n+1},a),d(x_n,x_{n-1},a)\Big)\\
\leq \;& \frac{1}{3}\Big[d(x_{n-1},x_{n+1},a)+d(x_n,x_{n-1},a)\Big] \qquad (8)
\end{aligned}
$$

for all $a \in X$. By setting $a = x_{n-1}$ in (7), we obtain $d(x_{n+1},x_n,x_{n-1}) \leq 0$, that is

$$d(x_{n+1},x_n,x_{n-1}) = 0. \qquad (9)$$

It follow from (7) and (9):

$$
\begin{aligned}
d(x_{n+1},x_n,a) \;\leq\; & \frac{1}{3}\Big[d(x_{n-1},x_n,a)+d(x_n,x_{n+1},a)\\
& +d(x_{n-1},x_n,x_{n+1})+d(x_n,x_{n-1},a)\Big]\\
= \;& \frac{2}{3}d(x_{n-1},x_n,a)+\frac{1}{3}d(x_n,x_{n+1},a). \qquad (10)
\end{aligned}
$$

It implies that:

$$d(x_{n+1},x_n,a) \leq d(x_{n-1},x_n,a). \qquad (11)$$

Thus $\Big\{d(x_n,x_{n+1},a)\Big\}$ is a decreasing sequence of non-negative real numbers and hence it is convergent. Let

$$\lim_{n\to\infty} d(x_n,x_{n+1},a) = t. \qquad (12)$$

Taking the limit as $n \to \infty$ in (10) and using (12), we obtain:

$$t \;\leq\; \frac{1}{3}\Big[\lim_{n\to\infty} d(x_{n-1},x_{n+1},a)+t\Big] \leq \frac{1}{3}(t+t+t) = t.$$

That is,

$$\frac{2}{3}t \leq \frac{1}{3}\lim_{n\to\infty} d(x_{n-1},x_{n+1},a) \leq \frac{2}{3}t.$$

That is,

$$2t \leq \lim_{n\to\infty} d(x_{n-1},x_{n+1},a) \leq 2t.$$

Therefore,

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}, a) = 2t. \tag{13}$$

Taking the limit as $n \to \infty$ in (7) and using (12) and (13) we get:

$$t \leq \frac{1}{3}(2t+t) - \psi(0, 2t, t) \leq \frac{1}{3}(2t+t) = t. \tag{14}$$

It implies that $\psi(0, 2t, t) = 0$, that is, $t = 0$. Then (12) becomes:

$$\lim_{n\to\infty} d(x_{n+1}, x_n, a) = 0. \tag{15}$$

From (11), we have if $d(x_{n-1}, x_n, a) = 0$, then $d(x_n, x_{n+1}, a) = 0$. Since $d(x_0, x_1, x_0) = 0$, we have $d(x_n,\ x_{n+1}, x_0) = 0$ for all $n \in N$.
Since $d(x_{m-1}, x_m, x_m) = 0$, we have:

$$d(x_n, x_{n+1}, x_m) = 0 \tag{16}$$

for all $n \geq m-1$. For $0 \leq n < m-1$, noting that $m-1 \geq n+1$, from (16) we have $d(x_{m-1}, x_m, x_{n+1}) = d(x_{m-1}, x_m, x_n) = 0$. It implies that:

$$\begin{aligned} d(x_n, x_{n+1}, x_m) &\leq d(x_n, x_{n+1}, x_{m-1}) + d(x_{n+1}, x_m, x_{m-1}) + d(x_n, x_m, x_{m-1}) \\ &= d(x_n, x_{n+1}, x_{m-1}). \end{aligned} \tag{17}$$

Since $d(x_n, x_{n+1}, x_{n+1}) = 0$ from (17) we have:

$$d(x_n, x_{n+1}, x_m) = 0 \tag{18}$$

for $0 \leq n < m-1$.

From (16) and (18), we have $d(x_n, x_{n+1}, x_m) = 0$ for all $n, m \in N$. Now, for all $i, j, k \in N$ with $i > j$ we have $d(x_{i-1}, x_i, x_j) = d(x_{i-1}, x_i, x_k) = 0$. Therefore,

$$\begin{aligned} d(x_i, x_j, x_k) &\leq d(x_j, x_i, x_{i-1}) + d(x_i, x_k, x_{i-1}) + d(x_k, x_j, x_{i-1}) \\ &\leq d(x_j, x_{i-1}, x_k) \leq \cdots \leq d(x_i, x_j, x_k) = 0. \end{aligned} \tag{19}$$

This proves that for all $i, j, k \in N$, we have:

$$d(x_i, x_j, x_k) = 0. \tag{20}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find sub-sequence $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ where $n(k)$ is the smallest integer such that $n(k) > m(k) > k$ and $x_{n(k)-1}$, $x_{m(k)-1}$ are comparable, and

$$d(x_{n(k)}, x_{m(k)}, a) \geq \varepsilon \tag{21}$$

for all $k \in N$. Therefore,

$$d(x_{n(k)-1}, x_{m(k)}, a) < \varepsilon. \tag{22}$$

Then by using (20), (21) and (22), we have:

$$\begin{aligned}
\varepsilon &\leq d(x_{n(k)}, x_{m(x)}, a) = d(Tx_{n(k)-1}, Tx_{m(k)-1}, a) \\
&\leq \frac{1}{3}\Big[d(x_{n(k)-1}, Tx_{m(k)-1}, a) + d(x_{m(k)-1}, Tx_{n(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a) \\
&\quad -\psi\Big(d(x_{n(k)-1}, Tx_{m(k)-1}, a), d(x_{m(k)-1}, Tx_{n(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a)\Big) \\
&= \frac{1}{3}\Big[d(x_{n(k)-1}, x_{m(k)}, a) + d(x_{m(k)-1}, x_{n(k)}, a), d(x_{m(k)-1}, x_{n(k)-1}, a)\Big] \\
&\quad -\psi\Big(d(x_{n(k)-1}, x_{m(k)}, a), d(x_{m(k)-1}, x_{n(k)}, a), d(x_{m(k)-1}, x_{n(k)-1}, a)\Big).
\end{aligned} \tag{23}$$

Again, by using (20), (21) and (22), we have:

$$\begin{aligned}
\varepsilon &\leq d(x_{n(k)}, x_{m(k)}, a) \\
&\leq d(x_{n(k)}, x_{n(k)-1}, a) + d(x_{n(k)-1}, x_{m(k)}, a) + d(x_{n(k)}, x_{m(k)}, x_{n(k)-1}).
\end{aligned} \tag{24}$$

Taking the limit as $k \to \infty$ in the above inequality and using (15), we obtain:

$$\varepsilon \leq \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}, a) \leq \varepsilon \tag{25}$$

and

$$\varepsilon \leq \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}, a) + \lim_{k\to\infty} d(x_{m(k)-1}, x_{m(k)}, a) \tag{26}$$

$$+ \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}, x_{m(k)}) \leq \varepsilon. \tag{27}$$

Therefore, we have:

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}, a) = \varepsilon, \tag{28}$$

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}, a) = \varepsilon, \tag{29}$$

and

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}, a) = \varepsilon. \tag{30}$$

Taking the limit as $k \to \infty$ in (23) and using (28), (29) and (30), and considering the continuity of ψ, we have:

$$\varepsilon \leq \frac{1}{3}(\varepsilon + \varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon, \varepsilon) = \varepsilon - \psi(\varepsilon, \varepsilon, \varepsilon) \leq \varepsilon. \tag{31}$$

That is, $\psi(\varepsilon, \varepsilon, \varepsilon) \leq 0$, which proves that $\psi(\varepsilon, \varepsilon, \varepsilon) = 0$ is a contraction since $\varepsilon \geq 0$. Hence $\{x_n\}$ is a Cauchy sequence and since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Then it follows from the continuity of T that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz.$$

Therefore, z is a fixed point of T. $\square$

Now we prove that we can relax the continuity condition, i.e., T does not need to be continuous.

Theorem 5. *Let $(X, \preceq, d)$ be a complete, partially ordered $2-$metric space and $T : X \to X$ be a non-decreasing mapping such that:*

$$d(Tx, Ty) \leq \frac{1}{3}\Big[d(x, Ty) + d(y, Tx) + d(x, y)\Big] - \psi\Big(d(x, Ty), d(y, Tx), d(x, y)\Big)$$

for $x \succeq y$, where $\psi : [0,\infty)^3 \longrightarrow [0,\infty)$ is a continuous function such that $\psi(x, y, z) = 0$ iff $x = y = z = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ and if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ then $x_n \preceq x$ for all $n \in N$, then T has a fixed point.

Proof. Since $\{x_n\}$ is a non-decreasing Cauchy sequence and $\lim_{n\to\infty} x_n = z$ we have $x_n \preceq z$ for all $n \in N$. From (6), we have:

$$\begin{aligned} d(x_{n+1},z,a) = d(Tx_n,Tz,a) &\leq \frac{1}{3}\Big[d(x_n,Tz,a) + d(z,Tx_n,a) + d(x_n,z,a)\Big] \\ &\quad -\psi\Big(d(x_n,Tz,a), d(z,Tx_n,a), d(x_n,z,a)\Big) \\ &= \frac{1}{3}\Big[d(x_n,Tz,a) + d(z,x_{n+1},a) + d(x_n,z,a)\Big] \\ &\quad -\psi\Big(d(x_n,Tz,a), d(z,x_{n+1},a), d(x_n,z,a). \end{aligned} \tag{32}$$

Taking the limit as $n \to \infty$ in (32), we have:

$$\begin{aligned} d(z,Tz,a) &\leq \frac{1}{3}\Big[d(z,Tz,a) + d(z,z,a) + d(z,z,a)\Big] \\ &\quad -\psi\Big(d(z,Tz,a), d(z,z,a), d(z,z,a)\Big) \\ &\leq \frac{1}{3}\Big[d(z,Tz,a)\Big] - \psi\Big(d(z,Tz,a),0,0\Big) \\ &\leq \frac{1}{3} d(z,Tz,a). \end{aligned}$$

It implies that $d(z,Tz,a) = 0$ for all $a \in X$, that is, $Tz = z$. □

In what follows, we prove a sufficient condition for the uniqueness of the fixed point in the previous theorems.

Theorem 6. *Suppose that either hypotheses from two previous theorems hold and for each $x,y \in X$, there exists $z \in X$ that is comparable to x and y. Then T has a unique fixed point.*

Proof. In the previous theorems we proved that T has a fixed point. It remains to be proven that the fixed points are unique. Let to this end x,y be two fixed points of T. We consider the following two cases.

Case 1: If y is comparable to z, then $T^n y = y$ is comparable to $T^n z = z$ for all $n \in N$. Therefore, for all $a \in X$, we have:

$$\begin{aligned} d(y,z,a) &= d(T^n y, T^n z, a) \\ &\leq \frac{1}{3}\Big[d(T^{n-1}y, T^n z, a) + d(T^{n-1}z, T^n y, a) + d(y,z,a)\Big] \\ &\quad -\psi\Big(d(T^{n-1}y, T^n z, a), d(T^{n-1}z, T^n y, a), d(y,z,a)\Big) \\ &= \frac{1}{3}\Big[d(y,z,a) + d(z,y,a) + d(y,z,a)\Big] \\ &\quad -\psi\Big(d(y,z,a), d(z,y,a), d(z,y,a)\Big) \\ &\leq d(y,z,a). \qquad (33) \end{aligned}$$

As a results, we have $\psi\Big(d(y,z,a), d(z,y,a), d(z,y,a)\Big) = 0$. Therefore, taking into account the assumption about ψ, we get $d(y,z,a) = 0$, or equivalently, $y = z$.

Case 2: If y is not comparable to z, then exists $x \in X$ comparable to y and z. It implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$. Therefore, for all $n \in N$ and $a \in X$, we have:

$$\begin{aligned} d(z, T^n x, a) &= d(T^n z, T^n x, a) \\ &\leq \frac{1}{3}\Big[d(T^{n-1}z, T^n x, a) + d(T^{n-1}x, T^n z, a) + d(T^n z, T^n x, a)\Big] \\ &\quad -\psi\Big(d(T^{n-1}z, T^n x, a), d(T^{n-1}x, T^n z, a), d(T^n z, T^n x, a)\Big) \\ &= \frac{1}{3}\Big[d(z, T^n x, a) + d(T^{n-1}x, z, a) + d(z, T^n x, a)\Big] \\ &\quad -\psi\Big(d(z, T^n x, a), d(T^{n-1}x, z, a), d(z, T^n x, a)\Big) \\ &\leq \frac{2}{3}d(z, T^n x, a) + \frac{1}{3}d(z, T^{n-1}x, a). \qquad (34) \end{aligned}$$

It implies that

$$d(z, T^n x, a) \leq d(z, T^{n-1}x, a).$$

Then there exists $\lim_{n\to\infty} d(z, T^n x, a) = t$. Letting $n \to \infty$ in (34) and taking into account the continuity of ψ, we obtain:

$$t \leq \frac{1}{3}(t+t+t) - \psi(t,t,t) \leq t.$$

This gives us $\psi(t,t,t)=0$. Then $t=0$, that is, $\lim_{n\to\infty}T^nx=z$. Analogously, $\lim_{n\to\infty}T^nx=y$. Finally the uniqueness of the limit gives us $y=z$. □

Theorem 7. *Let $(X,\preceq,d)$ be a complete, partially ordered 2-metric space and $T:X\to X$ be a S-contraction such that:*

1): *For all $x,y\in X$, if $x\preceq y$, then $Tx\succeq Ty$.*

2): *For each $x,y\in X$, there exists $z\in X$ that is comparable to x and y.*

3): *There exists $x_0\in X$ with $x_0\preceq Tx_0$ or $x_0\succeq Tx_0$.*

Then, for all $a\in X$, $\inf\{d(x,Tx,a):x\in X\setminus\{a\}\}=0$. In particular $\inf\{d(x,Tx,a):x\in X\}=0$.

Proof. We consider the following two cases:

Case 1: $x_0\preceq Tx_0$, by the hypothesis (1), consecutive terms of the sequence $\{T^nx_0\}$ are comparable. It follows from (6) that for all $a\in X$:

$$\begin{aligned}
& d(T^{n+1}x_0,T^nx_0,a)\\
\leq\ & \frac{1}{3}\Big[d(T^nx_0,T^nx_0,a)+d(T^{n-1}x_0,T^{n+1}x_0,a)+d(T^{n+1}x_0,T^nx_0,a)\Big]\\
& -\psi\Big(d(T^nx_0,T^nx_0,a),d(T^{n-1}x_0,T^{n+1}x_0,a),d(T^{n+1}x_0,T^nx_0,a)\Big)\\
=\ & \frac{1}{3}\Big[d(T^{n-1}x_0,T^{n+1}x_0,a)+d(T^{n+1}x_0,T^nx_0,a)\Big]\\
& -\psi(0,d(T^{n-1}x_0,T^nx_0,a),d(T^{n+1}x_0,T^nx_0,a)\Big)\\
\leq\ & \frac{1}{3}\left[d(T^{n-1}x_0,T^{n+1}x_0,a)+d(T^{n+1}x_0,T^nx_0,a)\right]\\
\leq\ & \frac{1}{3}\Big[d(T^{n-1}x_0,T^nx_0,a)+d(T^nx_0,T^{n+1}x_0,a)\\
& \quad +d(T^{n-1}x_0,T^nx_0,T^{n+1}x_0)+d(T^{n+1}x_0,T^nx_0,a)\Big]. \qquad (35)
\end{aligned}$$

We have $d(x_i,x_j,x_k)=0$ for all $i,j,k\in N$. Then (35) implies:

$$d(T^{n+1}x_0,T^nx_0,a)\ \leq\ \frac{1}{3}d(T^{n-1}x_0,T^nx_0,a)+\frac{2}{3}d(T^nx_0,T^{n+1}x_0,a).$$

That is,

$$d(T^{n+1}x_0,T^nx_0,a)\leq d(T^{n-1}x_0,T^nx_0,a).$$

Then there exists $\lim_{n\to\infty} d(T^{n+1}x_0, T^n x_0, a) = t$. Therefore, according to the previous theorems, we get $t = 0$. Then

$$\lim_{n\to\infty} d(T^{n+1}x_0, T^n x_0, a) = 0.$$

That is, $\inf\Big\{d(x, Tx, a) : x \in X\Big\} = 0$.

Case 2: $x_0 \succ Tx_0$. The same as in Case 1. □

4. Conclusion

In this paper, we have proposed a novel contraction mapping and explored its properties. The contraction mapping theorems developed here are novel weak s-contraction mappings on partially ordered 2-metric spaces.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] P. Debnath, M. Neog, S. Radenovic, New extension of some fixed point results in complete metric spaces, Tbilisi Math. J. 10 (2) (2017) 201–210.

[2] Z. Fadail, A. Ahmad, V. Ozturk, S. Radenovic, Some remarks on fixed point results of $b2$-metric spaces, Far East J. Math. Sci. 97 (5) (2015), 533–548.

[3] R. Kannan, Some results on fixed points, Bull. Cal. Math. Soc. 60 (1968) 71–76.

[4] D. P. Shukla, S. K. Tiwari, Unique fixed point theorem for weakly S-contractive mappings, Gen. Math. Notes 4 (1) (2011) 28–34.

[5] B. S. Choudhury, Unique fixed point theorem for weakly C-contractive mappings, Kathmandu Univ. J. Sci. Eng. Technol. 5 (1) (2009) 6–13.

[6] A. Azam, M. Arshad, Kannan fixed point theorem on generalized metric spaces., J. Nonlinear Sci. Appl. 1 (1) (2008) 190 45–48.

[7] S. Gahler, 2-metrische Raume und ihre topologische Struktur, Math. Nachr. 26 (1-4) (1963) 115–148.

[8] K. Iseki, Fixed point theorems in 2-metric spaces, Math. Sem. Notes 3 (1975) 133–136.

[9] B. K. Lahiri, P. Das, L. K. Dey, Cantor's theorem in 2-metric spaces and its applications to fixed point problems, Taiwan. J. Math 15 (2011) 337–352.

[10] S. V. R. Naidu, J. R. Prasad, Fixed-point theorems in 2-metric spaces, Indian J. Pure Appl. Math. 17 (8) (1986) 974–993.