Available online at http://scik.org
Advances in Fixed Point Theory, 2 (2012), No. 2, 157-175
ISSN: 1927-6303

TRIPLED COMMON FIXED POINT THEOREMS FOR w-COMPATIBLE MAPPINGS IN ORDERED CONE METRIC SPACES

P. P. MURTHY* AND RASHMI

Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya(A Central University),

Koni, Bilaspur(Chhattisgarh), 495 009, India

Abstract. The purpose of this note is to establish a triplet coincidence point theorem in ordered cone metric spaces over solid cone. Our result extends coupled common fixed point theorems due to Nashine, Kadelburg and Radenovic [1].

Keywords: cone metric space; partially ordered set; tripled coincidence point; w-compatible maps; mixed g-monotone property.

2000 AMS Subject Classification: 46T99, 47H10, 54H25.

1. Introduction

The concept of cone metric spaces was introduced initially by Huang Lang and Zhang-Xian ([3]) which is the generalization of a metric space. In this space, they have replaced the set of real numbers by real Banach Space in the definition of metric space.

Very recently, in 2004 the concept of partially ordered metric space which was introduced by Ran and Reurings ([4])Guo and Lakshmikantham ([5]) studied the concept of coupled fixed points in partially ordered metric spaces. Bhaskar and Lakshmikantham ([8]) introduced monotone property in partially ordered metric spaces and given an application to the existence of periodic boundary value problem.

*Corresponding author

The first author would like to thanks Dr. Lakshman Chaturvedi, Vice-Chancellor of GGV, BSP for his inspiration.

Received April 25, 2012

Nieto and Lopez ([6] - [7]) rediscovered the partially ordered metric spaces and applied their problems to periodic boundary value problems.

Recently, Karapinar ([9]) proved coupled fixed point theorems for nonlinear contractions in ordered cone metric spaces over normal cones without regularity. He considered the continuity of communting mappings in a whole complete space. Shatanawi ([10]) proved coupled coincidence fixed point theorems in cone metric spaces over which were not necessarily normal. See also the results of Sabetghadam ([11]), Ding and Li ([12]), and Aydi, Samet and Vetro([13]).

In this paper we have studied unique common triple fixed point theorem for two maps by using g - monotone and w - compatible mappings satisfying more general contractive condition in ordered cone metric spaces over a cone that is only solid(i.e., has a nonempty interior). We furnish example to demonstrate the validity of the results.

2. Preliminaries

Now here we recall some definition.

Definition 2.1. Let E be a real Banach space with respect to a given norm $\| \, . \, \|_E$ and let 0_E be the zero vector of E. A non - empty subset P of E is called a cone if the following condition hold:

(1) P is closed, nonempty and $P \neq \{0_E\}$;

(2) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow (ax + by) \in P$;

(3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$ a partial ordering $\leq_P$ with respect to P is naturally defined by $x \leq_P y$ if and only if $y - x \in P$ for $x, y \in E$. We shall write $x <_P y$ to indicate that $x \leq_P y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$, intP denotes the interior of P .

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0_E \leq_P x \leq_P y$ implies $\| x \|_E \leq K \| y \|_E$.

In what follows we always suppose that E is a real Banach spaces with cone P satisfying $intP \neq \emptyset$(such cones are called solid).

Definition 2.2. Let X be a nonempty set. Suppose the mapping $d : X \times X \to P$ satisfies:

(1) $d(x, y) = 0_E$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,y) \leq_P d(x,z) + d(y,z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space. It is obvious that cone metric spaces generalize metric spaces.

Example 2.3. Let $E = R^2$, $P = \{(x,y) \in E : x, y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \leq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 2.4. Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(a) If for every $c \in E$ with $0_E \ll_P c$ there is $N \in \mathbf{N}$ such that $d(x_n, x) \ll_P c$, for all $n \geq N$, then $\{x_n\}$ is said to be convergent to x. This limit is denoted by $lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) If for every $c \in E$ with $0_E \ll_P c$ there is $N \in \mathbf{N}$ such that $d(x_n, x_m) \ll_P c$, for all $n, m \geq N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(c) If every Cauchy sequence in X is convergent in X,

Then (X,d) is called a complete cone metric spaces.

Let (X,d) be a cone metric space. Then the following properties are often used(particularly when dealing with cone metric spaces in which the cone need not be normal):

(P_1) If E is a real Banach space with a cone P and if $a \leq_P ha$ where $a \in P$ and $h \in [0,1)$, then $a = 0_E$;

(P_2) if $0_E \leq_P u \ll c$ for each $0_E \ll c$, then $u = 0_E$;

(P_3) if $u, v, w \in E, u \leq_P v$ and $v \ll w$, then $u \ll w$;

(P_4) if $c \in intP, 0 \leq_P a_n \in E$ and $a_n \to 0_E$, then there exists $k \in \mathbf{N}$ such that for all $n > k$ we have $a_n \ll c$.

Definition 2.5. Let X be nonempty set and $F : X \times X \times X \to X$, $g : X \to X$. An element $(x,y,z) \in X \times X \times X$ called:

(T_1) a tripled fixed point of the F if $F(x,y,z) = x$, $F(y,x,z) = y$ and $F(z,y,x) = z$;

(T_2) a tripled coincidence point of mappings F and g if $F(x,y,z) = g(x)$, $F(y,x,z) = g(y)$ and $F(z,y,x) = g(z)$ and in this case (gx, gy, gz) is called a triplet point of coincidence.

Definition 2.6. Let X be nonempty set. Mappings $F : X \times X \times X \to X$, $g : X \to X$ are called w - compatible if $gF(x, y, z) = F(gx, gy, gz), gF(y, z, x) = F(gy, gz, gx) \quad \text{and} \quad gF(z, x, y) = F(gz, gx, gy)$ whenever $gx = F(x, y, z), gy = F(y, x, z) \quad \text{and} \quad g(z) = F(z, y, x)$.

According to Borcut and Berinde [15], we give also the following concepts.

Consider on the product $X \times X \times X$ the following partial order:

for $(x, y, z), (u, v, w) \in X \times X \times X$, $(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w$.

Definition 2.7. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$ and $g : X \to X$. The mapping F is said to have mixed g - monotone property if F is monotone g - non-decreasing in x and z is monotone g - non-increasing in y that is, for any $x, y, z \in X$

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z) \leq F(x_2, y, z) \tag{2.1}$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z) \geq F(x, y_2, z) \tag{2.2}$$

$$z_1, z_2 \in X, g(z_1) \leq g(z_2) \Rightarrow F(x, y, z_1) \leq F(x, y, z_2) \tag{2.3}$$

hold.

Definition 2.8. [2] Let X be non - empty set. Then we say that the mappings $F : X \times X \times X \to X$ and $g : X \to X$ are w - compatible if $gF(x, y, z) = F(gx, gy, gz), gF(y, z, x) = F(gy, gz, gx) \quad \text{and} \quad gF(z, y, x) = F(gz, gy, gx)$ whenever $g(x) = F(x, y, z), g(y) = F(y, z, x) \quad \text{and} \quad g(z) = F(z, y, x)$.

3. Main results

Theorem 3.1. Let $(X, d, \preceq)$ be an ordered cone metric space over a solid cone P. Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g - monotone property on X and there exists three elements $x_0, y_0, z_0 \in X$ with $gx_0 \preceq F(x_0, y_0, z_0), gy_0 \succeq F(y_0, z_0, x_0)$ and $gz_0 \preceq F(z_0, x_0, y_0)$. Suppose further that F, g satisfy

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) \preceq_p \; & a_1 d(gx, gu) + a_2 d(F(x, y, z), gx) + a_3 d(gy, gv) \\ & + a_4 d(F(u, v, w), gu) + a_5 d(F(x, y, z), gu) + a_6 d(F(u, v, w), gx) + a_7 d(gz, gw), \end{aligned} \tag{3.1}$$

for all $(x, y, z), (u, v, w) \in X \times X \times X$ with $(gx \preceq gu, gy \succeq gv \quad \text{and} \quad gz \preceq gw)$, where $a_i \geq 0$, for $i = 1, 2, ..., 7$ and $\sum_{i=1}^{7} a_i < 1$. Further suppose

(1) $F(X \times X \times X) \subseteq g(X)$;

(2) $g(X)$ is a complete subspaces of X.

Also, suppose that X has the following properties:

(i) if a non - decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$, then $x_n \preceq x$ for all $n \in N$,

(ii) if a non - increasing sequence $\{y_n\}$ in X is such that $y_n \to y$, then $y_n \succeq y$ for all $n \in N$,

(iii) if a non - decreasing sequence $\{z_n\}$ in X is such that $z_n \to z$, then $z_n \preceq z$ for all $n \in N$.

Then there exists $x, y, z \in X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, z, x)$ and $g(z) = F(z, x, y)$, that is, F and g have tripled coincidence point in X.

Proof.

Let $x_0, y_0, z_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0, z_0)$, $g(y_0) \succeq F(y_0, z_0, x_0)$ and $g(z_0) \preceq F(z_0, x_0, y_0)$. Since $F(X \times X \times X) \subseteq g(X)$, we can define $x_1, y_1, z_1 \in X$ such that

$$g(x_1) = F(x_0, y_0, z_0), \quad g(y_1) = F(y_0, z_0, x_0) \text{ and } g(z_1) = F(z_0, x_0, y_0).$$

In the same way, we can choose $x_2, y_2, z_2 \in X$ such that $g(x_2) = F(x_1, y_1, z_1)$, $g(y_2) = F(y_1, z_1, x_1)$ and $g(z_2) = F(z_1, x_1, y_1)$. Continuing like this, we construct three sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that, for all $n \geq 0$, we get

$$g(x_{n+1}) = F(x_n, y_n, z_n), g(y_{n+1}) = F(y_n, z_n, x_n), g(z_{n+1}) = F(z_n, x_n, y_n). \tag{3.2}$$

Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}), g(y_n) \succeq g(y_{n+1}) \text{ and } g(z_n) \preceq g(z_{n+1}). \tag{3.3}$$

we shall use the mathematical induction. By contraction condition we have

$g(x_0) \preceq F(x_0, y_0, z_0) = gx_1$, $g(y_0) \succeq F(y_0, z_0, x_0) = gy_1$ and $g(z_0) \preceq F(z_0, x_0, y_0) = gz_1$.

i.e., (3.3) holds for $n = 0$. We assume that (3.3) holds for some $n > 0$. As F has the mixed g - monotone property and $gx_n \preceq gx_{n+1}, gy_n \succeq gy_{n+1}$ and $gz_n \preceq gz_{n+1}$, from (3.2) and (2.1) – (2.3) we get

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n) \preceq F(x_{n+1}, y_n, z_n), \\ gy_{n+1} &= F(y_n, z_n, x_n) \succeq F(y_{n+1}, z_n, x_n), \\ \text{and} \quad gz_{n+1} &= F(z_n, x_n, y_n) \preceq F(z_{n+1}, x_n, y_n). \end{aligned} \tag{3.4}$$

Also for the same reason we have,

$$\begin{aligned} gx_{n+2} &= F(x_{n+1}, y_{n+1}, z_{n+1}) \preceq F(x_{n+1}, y_n, z_n), \\ gy_{n+2} &= F(y_{n+1}, z_{n+1}, x_{n+1}) \succeq F(y_{n+1}, z_n, x_n), \\ \text{and} \quad gz_{n+2} &= F(z_{n+1}, x_{n+1}, y_{n+1}) \preceq F(z_{n+1}, x_n, y_n). \end{aligned} \tag{3.5}$$

Then from (3.4) and (3.5) we obtain

$g(x_{n+1}) \preceq g(x_{n+2}), g(y_{n+1}) \succeq g(y_{n+2})$ and $g(z_{n+2}) \preceq g(z_{n+2})$.

Thus by mathematical induction, we conclude that (3.3) holds for all $n \geq 0$.

On using condition (3.1) we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\
&\leq_P a_1 d(gx_{n-1}, gx_n) + a_2 d(F(x_{n-1}, y_{n-1}, z_{n-1}), gx_{n-1})) + a_3 d(gy_{n-1}, gy_n) \\
&\quad + a_4 d(F(x_n, y_n, z_n), gx_n)) + a_5 d(F(x_{n-1}, y_{n-1}, z_{n-1}), gx_n)) \\
&\quad + a_6 d(F(x_n, y_n, z_n), gx_{n-1})) + a_7 d(gz_{n-1}, gz_n) \\
&= a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) + a_4 d(gx_{n+1}, gx_n) \\
&\quad + a_5 d(gx_n, gx_n) + a_6 d(gx_{n+1}, gx_{n-1}) + a_7 d(gz_{n-1}, gz_n) \\
&\leq_P a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) + a_4 d(gx_{n+1}, gx_n) \\
&\quad + a_6 [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})] + a_7 d(gz_{n-1}, gz_n). \\
&= (a_1 + a_2 + a_6) d(gx_{n-1}, gx_n) + (a_4 + a_6) d(gx_n, gx_{n+1}) + a_3 d(gy_{n-1}, gy_n) \\
&\quad + a_7 d(gz_{n-1}, gz_n)
\end{aligned}$$

which implies that

$$\begin{aligned}
(1 - a_4 - a_6) d(gx_n, gx_{n+1}) \leq_P {} & (a_1 + a_2 + a_6) d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n) \\
& + a_7 d(gz_{n-1}, gz_n)
\end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned}
d(gy_n, gy_{n+1}) &= d(F(y_{n-1}, z_{n-1}, x_{n-1}, F(y_n, z_n, x_n)) \\
&\leq_P a_1 d(gy_{n-1}, gy_n) + a_2 d(F(y_{n-1}, z_{n-1}, x_{n-1}, gy_{n-1})) + a_3 d(gz_{n-1}, gz_n) \\
&\quad + a_4 d(F(y_n, z_n, x_n, gy_n)) + a_5 d(F(y_{n-1}, z_{n-1}, x_{n-1}, gy_n)) \\
&\quad + a_6 d(F(y_n, z_n, x_n, gy_{n-1})) + a_7 d(gx_{n-1}, gx_n) \\
&= a_1 d(gy_{n-1}, gy_n) + a_2 d(gy_n, gy_{n-1}) + a_3 d(gz_{n-1}, gz_n) + a_4 d(gy_{n+1}, gy_n) \\
&\quad + a_5 d(gy_n, gy_n) + a_6 d(gy_{n+1}, gy_{n-1}) + a_7 d(gx_{n-1}, gx_n) \\
&\leq_P a_1 d(gy_{n-1}, gy_n) + a_2 d(gy_n, gy_{n-1}) + a_3 d(gz_{n-1}, gz_n) + a_4 d(gy_{n+1}, gy_n) \\
&\quad + a_6 [d(gy_{n+1}, gy_n) + d(gy_n, gy_{n-1})] + a_7 d(gx_{n-1}, gx_n). \\
&= (a_1 + a_2 + a_6) d(gy_{n-1}, gy_n) + (a_4 + a_6) d(gy_n, gy_{n+1}) + a_3 d(gz_{n-1}, gz_n) \\
&\quad + a_7 d(gx_{n-1}, gx_n)
\end{aligned}$$

which implies that

$$(1-a_4-a_6)d(gy_n,gy_{n+1}) \leq_P (a_1+a_2+a_6)d(gy_{n-1},gy_n)+a_3d(gz_{n-1},gz_n) + a_7d(gx_{n-1},gx_n) \tag{3.7}$$

Also,

$$\begin{aligned}
d(gz_n,gz_{n+1}) &= d(F(z_{n-1},x_{n-1},y_{n-1}),F(z_n,x_n,y_n)) \\
&\leq_P a_1d(gz_{n-1},gz_n)+a_2d(F(z_{n-1},x_{n-1},y_{n-1},gz_{n-1}))+a_3d(gx_{n-1},gx_n) \\
&\quad + a_4d(F(z_n,x_n,y_n,gz_n))+a_5d(F(z_{n-1},x_{n-1},y_{n-1},gz_n)) \\
&\quad + a_6d(F(z_n,x_n,y_n,gz_{n-1}))+a_7d(gy_{n-1},gy_n) \\
&= a_1d(gz_{n-1},gz_n)+a_2d(gz_n,gz_{n-1})+a_3d(gx_{n-1},gx_n)+a_4d(gz_{n+1},gz_n) \\
&\quad + a_5d(gz_n,gz_n)+a_6d(gz_{n+1},gz_{n-1})+a_7d(gy_{n-1},gy_n) \\
&\leq_P a_1d(gz_{n-1},gz_n)+a_2d(gz_n,gz_{n-1})+a_3d(gx_{n-1},gx_n)+a_4d(gz_{n+1},gz_n) \\
&\quad + a_6[d(gz_{n+1},gz_n)+d(gz_n,gz_{n-1})]+a_7d(gy_{n-1},gy_n) \\
&= (a_1+a_2)d(gz_{n-1},gz_n)+a_3d(gx_{n-1},gx_n)+a_4d(gz_{n+1},gz_n) \\
&\quad a_6[d(gz_{n+1},gz_n)+d(gz_n,gz_{n-1})]+a_7d(gy_{n-1},gy_n) \\
&= (a_1+a_2+a_6)d(gz_{n-1},gz_n)+(a_4+a_6)d(gz_{n+1},gz_n)+a_3d(gx_{n-1},gx_n) \\
&\quad + a_7d(gy_{n-1},gy_n)
\end{aligned}$$

which implies that

$$(1-a_4-a_6)d(gz_n,gz_{n+1}) \leq_P (a_1+a_2+a_6)d(gz_{n-1},gz_n)+a_3d(gx_{n-1},gx_n)+a_7d(gy_{n-1},gy_n) \tag{3.8}$$

Adding (3.6), (3.7) and (3.8) we obtain that

$$\begin{aligned}
&(1-a_4-a_6)[d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})+d(gz_n,gz_{n+1})] \\
&\leq_P (a_1+a_2+a_6)[d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)+d(gz_{n-1},gz_n)] \\
&\quad +a_3[d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)+d(gz_{n-1},gz_n)] \\
&\quad +a_7[d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)+d(gz_{n-1},gz_n)] \\
&= (a_1+a_2+a_3+a_6+a_7[d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)+d(gz_{n-1},gz_n)
\end{aligned} \tag{3.9}$$

Now stating from $d(gx_{n+1},gx_n)=d(F(x_n,y_n,z_n),F(x_{n-1},y_{n-1},z_{n-1}))$ and using that $g(x_{n-1})\preceq g(x_n)$, $g(y_{n-1})\succeq g(y_n)$ and $g(z_{n-1})\preceq g(z_n)$, we get that

$$
\begin{aligned}
d(gx_{n+1}, gx_n) &= d(F(x_n, y_n, z_n, F(x_{n-1}, y_{n-1}, z_{n-1})) \\
&\leq_P a_1 d(gx_n, gx_{n-1}) + a_2 d(F(x_n, y_n, z_n, gx_n)) + a_3 d(gy_n, gy_{n-1}) \\
&\quad + a_4 d(F(x_{n-1}, y_{n-1}, z_{n-1}, gx_n)) + a_5 d(F(x_n, y_n, z_n, gx_{n-1})) \\
&\quad + a_6 d(F(x_{n-1}, y_{n-1}, z_{n-1}, gx_n)) + a_7 d(gz_n, gz_{n-1}) \\
&= a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_{n+1}, gx_n) + a_3 d(gy_n, gy_{n-1}) + a_4 d(gx_n, gx_{n-1}) \\
&\quad + a_5 d(gx_{n+1}, gx_{n-1}) + a_6 d(gx_n, gx_n)) + a_7 d(gz_n, gz_{n-1}) \\
&= a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_{n+1}, gx_n) + a_3 d(gy_n, gy_{n-1}) + a_4 d(gx_n, gx_{n-1}) \\
&\quad + a_5 [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})] + a_7 d(gz_n, gz_{n-1})
\end{aligned}
$$

$$
\begin{aligned}
d(gx_{n+1}, gx_n) = {} & (a_1 + a_4 + a_5) d(gx_n, gx_{n-1}) + (a_2 + a_5) d(gx_{n+1}, gx_n) + a_3 d(gy_n, gy_{n-1}) \\
& + a_7 d(gz_n, gz_{n-1})
\end{aligned}
\tag{3.10}
$$

Similarly,

$$
\begin{aligned}
d(gy_{n+1}, gy_n) &= d(F(y_n, z_n, x_n, F(y_{n-1}, z_{n-1}, x_{n-1})) \\
&\leq_P (a_1 + a_4 + a_5) d(gy_n, gy_{n-1}) + (a_2 + a_5) d(gy_{n+1}, gy_n) \\
&\quad + a_3 d(gz_n, gz_{n-1}) + a_7 d(gx_n, gx_{n-1})
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
d(gz_{n+1}, gz_n) &\leq_P (a_1 + a_4 + a_5) d(gz_n, gz_{n-1}) + (a_2 + a_5) d(gz_{n+1}, gz_n) \\
&\quad + a_3 d(gx_n, gx_{n-1}) + a_7 d(gy_n, gy_{n-1})
\end{aligned}
\tag{3.12}
$$

Adding (3.10), (3.11) and (3.12) we obtain that

$$
\begin{aligned}
& (1 - a_2 - a_5)[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)] \\
& \leq_P (a_1 + a_4 + a_5)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})] \\
& \qquad + a_3 [d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})] \\
& \qquad + a_7 [d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})] \\
& = (a_1 + a_3 + a_4 + a_5 + a_7)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})]
\end{aligned}
\tag{3.13}
$$

Now adding (3.9) and (3.13) we get

$$
\leq_P \left(\frac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6 + 2a_7}{2 - a_2 - a_4 - a_5 - a_6}\right)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})]
$$

with $0 \leq \lambda < 1$. where $\lambda = \frac{2a_1+a_2+2a_3+a_4+a_5+a_6+2a_7}{2-a_2-a_4-a_5-a_6}$

Since $\sum_{i=1}^{7} a_i < 1$. using relation (3.13) n - times, we obtain

$$\begin{aligned} d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n) &\leq_P \lambda[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)] \\ &\leq_P \lambda^2[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1}) + d(gz_{n-2}, gz_{n-1})] \\ &\vdots \\ &\leq_P \lambda^n[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)]. \end{aligned}$$

Then $d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n) \to 0_E$ as $n \to \infty$.

Thus $d(gx_{n+1}, gx_n) = d(gy_{n+1}, gy_n) = d(gz_{n+1}, gz_n) \to 0_E$ as $n \to \infty$.

Next we show that $\{gx_n\}, \{gy_n\}$and$\{gz_n\}$ are Cauchy sequences. For any $m > n \geq 1$, repeated use of triangle inequality gives

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m) &\leq_P d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ &\quad + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) + \cdots + d(gy_{m-1}, gy_m) \\ &\quad + d(gz_n, gz_{n+1}) + d(gz_{n+1}, gz_{n+2}) + \cdots + d(gz_{m-1}, gz_m) \\ &\leq_P [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}][d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \\ &\leq_P \frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \\ &\to 0_E \quad \text{as} \quad n \to \infty. \end{aligned}$$

from (P_4) it follows that for $0_E \ll c$, and large $n : \frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \ll c$, thus according to (P_3),

$[d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)] \ll c$.

Since,

$$d(gx_n, gx_m) \preceq_P [d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)],$$

$$d(gy_n, gy_m) \preceq_P [d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)],$$

and

$$d(gz_n, gz_m) \preceq_P [d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)]$$

then by $(P_3), d(gx_n, gx_m) \ll c, d(gy_n, gy_m) \ll c$ and $d(gz_n, gz_m) \ll c$. for n large enough and hence $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $g(X)$. By completeness of $g(X)$ there exists $gx, gy, gz \in g(X)$ such that $gx_n \to gx$, $gy_n \to gy$ and $gz_n \to gz$ as $n \to \infty$.

Since $\{gx_n\}$ and $\{gz_n\}$ is nondecreasing and $\{gy_n\}$ is non-increasing, using the conditions (i), (ii) and (iii), we have $gx_n \preceq gx$, $gy_n \succeq gy$ and $gz_n \preceq gz$ for all $n \geq 0$. If $gx_n = gx, gy_n = gy$ and $gz_n = gz$

for some $n \geq 0$, then $gx = gx_n \preceq gx_{n+1} \preceq gx = gx_n, gy \preceq gy_{n+1} \preceq gy_n = gy$ and $gz = gz_n \preceq gz_{n+1} \preceq gz = gz_n$ which implies that $gx = gx_n = F(x_n, y_n, z_n), gy = gy_n = F(y_n, z_n, x_n)$, and $gz = gz_n = F(z_n, x_n, y_n)$, that is, (x_n, y_n, z_n) is a triplet coincidence point of F and g. Then, we suppose that $(gx_n, gy_n, gz_n) \neq (gx, gy, gz)$ for all $n \geq 0$.

Now we prove that $F(x, y, z) = gx, F(y, z, x) = gy$ and $F(z, x, y) = gz$. For this, consider

$$\begin{aligned}
d(F(x,y,z),gx) &\leq_P d(F(x,y,z),gx_{n+1}) + d(gx_{n+1},gx) \\
&= d(F(x,y,z),F(x_n,y_n,z_n)) + d(gx_{n+1},gx) \\
&\leq_P a_1 d(gx,gx_n) + a_2 d(F(x,y,z),gx) + a_3 d(gy,gy_n) + a_4 d(F(x_n,y_n,z_n),gx_n) \\
&\quad + a_5 d(F(x,y,z),gx_n) + a_6 d(F(x_n,y_n,z_n),gx) + a_7 d(gz,gz_n) + d(gx_{n+1},gx) \\
&= a_1 d(gx,gx_n) + a_2 d(F(x,y,z),gx) + a_3 d(gy,gy_n) + a_4 d(F(gx_{n+1},gx_n) \\
&\quad + a_5 d(F(x,y,z),gx_n) + a_6 d(gx_{n+1},gx) + a_6 d(gx_n,gx) + a_7 d(gz,gz_n) \\
&\quad + d(gx_{n+1},gx_n) + d(gx_n,gx)
\end{aligned}$$

$$(1-a_2-a_5)d(F(x,y,z),gx) \leq_P (1+a_1+a_5+a_6)d(gx,gx_n) + (1+a_4+a_6)d(gx_{n+1},gx_n)$$
$$a_3 d(gy,gy_n) + a_7 d(gz,gz_n)$$

Which further implies that

$$\begin{aligned}
d(F(x,y,z),gx) \leq_P & \frac{1+a_1+a_5+a_6}{(1-a_2-a_5)} d(gx,gx_n) + \frac{a_3}{(1-a_2-a_5)} d(gy,gy_n) \\
& + \frac{1+a_4+a_6}{(1-a_2-a_5)} d(gx_{n+1},gx_n) + \frac{a_7}{(1-a_2-a_5)} d(gz,gz_n)
\end{aligned}$$

Since $gx_n \to gx$, $gy_n \to gy$ and $gz_n \to gz$, then for $0_E \ll c$ there exists $N \in \mathbf{N}$ such that $d(gx, gx_n) \ll \frac{(1+a_1+a_5+a_6)c}{(1-a_2-a_5)}$, $d(gy, gy_n) \ll \frac{(a_3)c}{(1-a_2-a_5)}$, $d(gx_{n+1}, gx_n) \ll \frac{(1+a_4+a_6)c}{(1-a_2-a_5)}$ and $d(gz, gz_n) \ll \frac{(a_7)c}{(1-a_2-a_5)}$, for all $n \geq N$. Thus,
$d(F(x, y, z), gx) \ll c$. Now according to (P_2) it follows that $d(F(x, y, z), gx) = 0_E$, and $F(x, y, z) = gx$. Similarly, we can get $F(y, z, x) = gy$ and $F(z, x, y) = gz$. Hence (gx, gy, gz) is tripled coincidence point of mappings F and g.

This completes the proof.

Theorem 3.2. In addition to the hypotheses of Theorem 3.1, Suppose that for every $(x, y, z), (x^*, y^*, z^*) \in X \times X \times X$ there exists $(u, v, w) \in X \times X \times X$ such that $(F(u, v, w), F(v, w, u), F(w, u, v))$ is comparable to $(F(x, y, z), F(y, z, x), F(z, x, y))$ and $(F(x^*, y^*, z^*), F(y^*, z^*, x^*), (z^*, x^*, y^*))$. Then F and g have a unique triple common fixed point, that is, there exists a unique $(u, v, w) \in X \times X \times X$ such that $u = g(u) = F(u, v, w)$, $g(v) = F(v, w, u)$ and $g(w) = F(w, u, v)$, provided F and g are w - compatible.

Proof.

From Theorem 3.1, the set of tripled coincidence points of F and g is non - empty. Suppose (x, y, z) and (x^*, y^*, z^*) are tripled coincidence points of F, that is $gx = F(x, y, z), g(y) = F(y, z, x), g(z) = F(z, x, y), g(x^*) = F(x^*, y^*, z^*), g(y^*) = F(y^*, z^*, x^*)$ and $g(z^*) = F(z^*, x^*, y^*)$. We will prove that

$$g(x) = g(x^*), g(y) = g(y^*) \quad \text{and} \quad g(z) = g(z^*).$$

By assumption, there exists $(u, v, w) \in X \times X \times X$ such that $(F(u, v, w), F(v, w, u), F(w, u, v))$ is comparable to $(F(x, y, z), F(y, z, x), F(z, x, y))$ and $(F(x^*, y^*, z^*), F(y^*, z^*, x^*), (z^*, x^*, y^*))$. Put $u_0 = u, v_0 = v, w_0 = w$ and choose $u_1, v_1, w_1 \in X$ so that $gu_1 = F(u_0, v_0, w_0), gv_1 = F(v_0, w_0, u_0)$ and $gw_1 = F(w_0, u_0, v_0)$. Then, similarly as in the proof of Theorem 3.1, we can inductively define sequences $\{gu_n\}, \{gv_n\}$ and $\{gw_n\}$ with

$$gu_{n+1} = F(u_n, v_n, w_n), gv_{n+1} = F(v_n, w_n, u_n) \quad \text{and} \quad gw_{n+1} = F(w_n, u_n, v_n) \qquad \forall \quad n.$$

Further, set $x_0 = x, y_0 = y, z_0 = z, x_0^* = x^*, y_0^* = y^*$ and $z_0^* = z^*$ and in a similar way, define the sequence $\{gx_n\}, \{gy_n\}, \{gz_n\}$ and $\{gx_n^*\}, \{y_n^*\}, \{z_n^*\}$. Then it is easy to show that

$$gx_n \to F(x, y, z), gy_n \to F(y, z, x) \quad \text{and} \quad gz_n \to F(z, x, y)$$

and

$$gx_n^* \to F(x^*, y^*, z^*), gy_n^* \to F(y^*, z^*, x^*) \quad \text{and} \quad gz_n^* \to F(z^*, x^*, y^*)$$

as $n \to \infty$. Since $(gx, gy, gz) = (F(x, y, z), F(y, z, x), F(z, x, y)) = (gx_1, gy_1, gz_1)$ and$(F(u, v, w), F(v, w, u), F(w, u, v)) = (gu_1, gv_1, gw_1)$ are comparable, then $gx \preceq gu_1, gy \succeq gv_1$ and $gz \preceq gw_1$. It is easy to show that, similarly, (gx, gy, gz) and (gu_n, gv_n, gw_n) are comparable for all $n \geq 1$, that is, $gx \preceq gu_n, gy \succeq gv_n$ and $gz \preceq gw_n$, or vice versa. Thus from (3.1), we have

$$\begin{aligned}
d(gx, gu_{n+1}) &= d(F(x,y,z), F(u_n,v_n,w_n)) \\
&\leq_P a_1 d(gx, gu_n) + a_2 d(F(x,y,z,gx)) + a_3 d(gy, gv_n) \\
&\quad + a_4 d(F(u_n,v_n,w_n,gu_n)) + a_5 d(F(x,y,z,gu_n)) \\
&\quad + a_6 d(F(u_n,v_n,w_n,gx)) + a_7 d(gz, gw_n) \\
&= a_1 d(gx, gu_n) + a_2 d(gx,gx) + a_3 d(gy, gv_n) + a_4 d(gu_{n+1}, gu_n) \\
&\quad + a_5 d(gx, gu_n) + a_6 d(gu_{n+1}, gx) + a_7 d(gz, gw_n). \\
&= a_1 d(gx_1, gu_n) + a_3 d(gy, gv_n) + a_4[d(gu_{n+1}, gx) \\
&\quad + d(gx, gx_n)] + a_6 d(gu_{n+1}, gx) + a_7 d(gz, gw_n). \\
&= (a_1 + a_4 + a_5) d(gx, gu_n) + a_3 d(gy, gv_n) + (a_4 + a_6) d(gu_{n+1}, gx) \\
&\quad + a_7 d(gz, gw_n).
\end{aligned}$$

which implies that

$$(1 - a_4 - a_6) d(gx, gu_{n+1}) \leq_P (a_1 + a_4 + a_5) d(gx, gu_n) + a_3 d(gy, gv_n) + a_7 d(gz, gw_n). \tag{3.14}$$

Similarly,

$$(1 - a_4 - a_6) d(gy, gv_{n+1}) \leq_P (a_1 + a_4 + a_5) d(gy, gv_n) + a_3 d(gz, gw_n) + a_7 d(gx, gu_n). \tag{3.15}$$

$$(1 - a_4 - a_6) d(gz, gw_{n+1}) \leq_P (a_1 + a_4 + a_5) d(gz, gw_n) + a_3 d(gx, gu_n) + a_7 d(gy, gv_n). \tag{3.16}$$

Adding up (3.14), (3.15) and (3.16) we obtain that

$$\begin{aligned}
&(1 - a_4 - a_6)[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \\
&\leq_P (a_1 + a_4 + a_5)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \\
&\qquad + a_3[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \qquad (3.17) \\
&\qquad + a_7[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \\
&= (a_1 + a_3 + a_4 + a_5 + a_7)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)]
\end{aligned}$$

Now stating from $d(gu_{n+1}, gx) = d(F(u_n, v_n, w_n), F(x, y, z))$ we get that

$$\begin{aligned}
d(gu_{n+1}, gx) &= d(F(u_n, v_n, w_n, F(x, y, z)) \\
&\leq_P a_1 d(gu_n, gx) + a_2 d(F(u_n, v_n, w_n, gu_n)) + a_3 d(gv_n, gy) \\
&\quad + a_4 d(F(x, y, z, gx)) + a_5 d(F(u_n, v_n, w_n, gx)) \\
&\quad + a_6 d(F(x, y, z, gu_n)) + a_7 d(gv_n, gz) \\
&= a_1 d(gu_n, gx) + a_2 d(gu_{n+1}, gu_n) + a_3 d(gv_n, gy) + a_4 d(gx, gx) \\
&\quad + a_5 d(gu_{n+1}, gx) + a_6 d(gx, gu_n)) + a_7 d(gv_n, gz) \\
&= a_1 d(gu_n, gx) + a_2 [d(gu_{n+1}, gx) + d(gx, gu_n)] + a_3 d(gv_n, gy) \\
&\quad + a_4 d(gx, gx) + a_5 d(gu_{n+1}, gx) + a_6 d(gx, gu_n)) + a_7 d(gv_n, gz)
\end{aligned}$$

$$(1 - a_2 - a_5) d(gu_{n+1}, gx) = (a_1 + a_2 + a_6) d(gx, gu_n) + a_3 d(gv_n, gy) + a_7 d(gv_n, gz) \tag{3.18}$$

Similarly,

$$(1 - a_2 - a_5) d(gv_{n+1}, gy) = (a_1 + a_2 + a_6) d(gy, gv_n) + a_3 d(gw_n, gz) + a_7 d(gu_n, gx) \tag{3.19}$$

$$(1 - a_2 - a_5) d(gw_{n+1}, gz) = (a_1 + a_2 + a_6) d(gz, gw_n) + a_3 d(gu_n, gx) + a_7 d(gw_n, gy) \tag{3.20}$$

Adding (3.18), (3.19) and (3.20) we obtain that

$$\begin{aligned}
&(1 - a_2 - a_5)[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \\
\leq_P\ & (a_1 + a_2 + a_6)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] + a_3 [d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \\
& + a_7 [d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \\
=\ & (a_1 + a_2 + a_3 + a_6 + a_7)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)]
\end{aligned} \tag{3.21}$$

Now adding (3.17)and(3.21) we get

$$\leq_P \left(\frac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6 + 2a_7}{2 - a_2 - a_4 - a_5 - a_6}\right)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \tag{3.22}$$

with $0 \leq \lambda < 1$. Where $\lambda = \frac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6 + 2a_7}{2 - a_2 - a_4 - a_5 - a_6}$.
Since $\sum_{i=1}^{7} a_i < 1$. using relation (3.22) n - times, we obtain

$$\leq_P \lambda[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)]$$
$$\leq_P \lambda^2[d(gx, gu_{n-1}) + d(gy, gv_{n-1}) + d(gz, gw_{n-1})]$$
$$\vdots$$
$$\leq_P \lambda^n[d(gx, gu_0) + d(gy, gv_0) + d(gz, gw_0)].$$

Then $[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \to 0_E$ as $n \to \infty$.

Thus $d(gx, gu_{n+1}) = d(gy, gv_{n+1}) = d(gz, gw_{n+1}) \to 0_E$ as $n \to \infty$. Since $0 \leq \lambda < 1$.

Hence $[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \ll c$ for each $c \in intP$ and large n. Since $0_E \leq_P d(gx, gu_{n+1}) \leq_P [d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})]$, it follows by (P_3) that $d(gu_{n+1}, gx) \ll c$, for n large enough and so $\{gu_n\} \to gx$ when $n \to \infty$. Similarly, $\{gv_{n+1}\} \to gy$ and $\{gw_{n+1}\} \to gz$. By the same procedure one can show that $\{gu^*_{n+1}\} \to gx^*, \{gv^*_{n+1}\} \to gy^*$ and $\{gw^*_{n+1}\} \to gz^*$. By the uniqueness of the limit, it follows that $gx = gx^*, gy = gy^*$ and $gz = gz^*$. as $n \to \infty$. Hence (gx, gy, gz) is the unique tripled point of coincidence of F and g.

Now we show that F and g have a unique common tripled fixed point. For this, let $gx = u$. Then we have $u = gx = F(x, y, z)$. By w - compatibility of F and g, we have

$$gu = g(gx) = g(F(x, y, z)) = F(gx, gy, gz) = F(u, v, w),$$

$$gv = g(gy) = g(F(y, z, x)) = F(gy, gz, gx) = F(v, w, u),$$

and

$$gw = g(gz) = g(F(z, x, y)) = F(gz, gy, gx) = F(w, u, v).$$

Hence the triple (u, v, w) is also triple coincidence point of F and g. Thus we have

$$gu = gx, gv = gy \quad \text{and} \quad gw = gz.$$

Therefore

$$u = gu = F(u, v, w), v = gv = F(v, w, u) \quad \text{and} \quad w = F(w, u, v).$$

Thus (u, v, w) is common triple fixed point of F and g.

To prove the uniqueness, let (u^*, v^*, w^*) be any common triple fixed point of F and g.

Then $u^* = gu = F(u^*, v^*, w^*), v^* = gv^* = F(v^*, w^*, u^*)$ and $w^* = F(w^*, u^*, v^*)$.

Since the (u^*, v^*, w^*) is a triple coincidence point of F and g.

We have

$$gu^* = gx, gv^* = gy \quad \text{and} \quad gw^* = gz.$$

Thus

$$u^* = gu^* = gx = u, v^* = gv^* = gy = v \quad \text{and} \quad w^* = gw^* = gz = w.$$

Hence the common triple fixed point is unique.

This completes the proof.

Corollary 3.3. Let $(X, d, \preceq)$ be an ordered cone metric space over a solid cone P. Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g - monotone property on X and there exists three elements $x_0, y_0, z_0 \in X$ with $gx_0 \preceq F(x_0, y_0, z_0), gy_0 \succeq F(y_0, z_0, y_0)$ and $gz_0 \preceq F(z_0, y_0, x_0)$. Suppose further that F, g satisfy that

$$d(F(x, y, z), F(u, v, w)) \leq_p \alpha d(gx, gu) + \beta d(gy, gv) + \gamma d(gz, gw) + \delta d(F(x, y, z), gu),$$

for all $(x, y, z), (u, v, w) \in X$ with $(gx \preceq gu, \quad gy \succeq gv \quad \text{and} \quad gz \preceq gw)$, where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + \gamma + \delta < 1$. Further suppose

(1) $F(X \times X \times X) \subseteq g(X)$;
(2) $g(X)$ is a complete subspaces of X.

Also, suppose that X has the following properties:

(i) if a non - decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$, then $x_n \preceq x$ for all $n \in N$,
(ii) if a non - increasing sequence $\{y_n\}$ in X is such that $y_n \to y$, then $y_n \succeq y$ for all $n \in N$,
(iii) if a non - decreasing sequence $\{z_n\}$ in X is such that $z_n \to z$, then $z_n \preceq z$ for all $n \in N$,

Then there exists $x, y, z \in X$ such that $g(x) = F(x, y, z), \quad g(y) = F(y, x, z)$ and $g(z) = F(z, y, x)$, that is, F and g have tripled coincidence point in X. Similarly corollary can be stated as a consequence of Previous theorem.

Putting $g = i_X$ (the identity map) in previous theorem we get the following corollary.

Corollary 3.4. Let $(X, d, \preceq)$ be complete ordered cone metric space over a solid cone P. Let $F : X \times X \times X \to X$ be a mappings having the mixed monotone property on X and there exists three elements $x_0, y_0, z_0 \in X$ with $x_0 \preceq F(x_0, y_0, z_0), y_0 \succeq F(y_0, z_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$. Suppose further that F, g satisfy

$$d(F(x,y,z), F(u,v,w)) \leq_p a_1 d(x,u) + a_2 d(F(x,y,z),x) + a_3 d(y,v)$$
$$+ a_4 d(F(u,v,w),u) + a_5 d(F(x,y,z),u) + a_6 d(F(u,v,w),x) + a_7 d(z,w), \tag{3.23}$$

for all $(x, y, z), (u, v, w) \in X$ with $(x \preceq u, \quad y \succeq v \quad \text{and} \quad z \preceq w)$, where $a_i \geq 0$, for $i = 1, 2, ..., 7$ and $\sum_{i=1}^{7} a_i < 1$. Also suppose that X has the following properties:

(i) if a non - decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$, then $x_n \preceq x$ for all $n \in N$,

(ii) if a non - increasing sequence $\{y_n\}$ in X is such that $y_n \to y$, then $y_n \succeq y$ for all $n \in N$,

(iii) if a non - decreasing sequence $\{z_n\}$ in X is such that $z_n \to z$, then $z_n \preceq z$ for all $n \in N$,

Then there exists $x, y, z \in X$ such that $x = F(x, y, z), \quad y = F(y, x, z) \quad \text{and} \quad z = F(z, y, x)$, that is, F and g have tripled coincidence point in X.

If we put $a_2 = a_4 = a_5 = a_6 = 0$ and $a_1 = j, a_3 = k \quad \text{and} \quad a_7 = l$ in contractive condition (3.1) then we get the result of Rao and Kishor([2]).

Corollary 3.5. Let $(X, \preceq, d)$ be a partially ordered cone metric space and let $T : X \times X \times X \to X$ and $f : X \to X$ be a mappings satisfying

(i) $d(T(x,y,z), T(u,v,w)) \leq jd(x,u) + kd(y,v) + ld(z,w)$, for all $(x,y,z), (u,v,w) \in X$ with $(fx \preceq fu, fy \succeq fv, fz \preceq fw)$ and $j, k, l \in [0, 1)$ with $j + k + l < 1$,

(ii) $T(X \times X \times X) \subseteq f(X)$ and $f(X)$ is complete subspaces of X,

(iii) T has the mixed f - monotone property,

(a) if a non - decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,

(b) if a non - increasing sequence $\{y_n\} \to y$, then $y_n \leq y$ for all n,

(c) if a non - decreasing sequence $\{z_n\} \to z$, then $z_n \leq z$ for all n.

Then there exists $x_0, y_0, z_0 \in X$ such that $fx_0 \succeq T(x_0, y_0, z_0), \quad fy_0 \preceq T(y_0, x_0, z_0) \quad \text{and}$ $fz_0 = T(z_0, y_0, x_0)$, than T and f have tripled coincidence point in X.

Example 3.6. Let $X = [0, 1]$ be taken with the standard order and with the cone metric given by $d(x, y) = (\mid x - y \mid, \alpha \mid x - y \mid)$ for fixed $\alpha > 0$.(Here $E = R^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$ is a solid cone.) Consider the mappings $F : X \times X \times X \to X$ and $g : X \to X$ given by

$$F(x,y,z) = \begin{cases} \frac{1}{3}(x^2 - y^2 - z^2), & \text{if } x > y > z \\ 0 & \text{if } otherwise; \end{cases} \quad \text{and} \quad gx = x^2,$$

and the contractive condition taken with $a_1 = a_3 = \frac{1}{8}, a_2 = a_4 = \frac{1}{4}$ and $a_5 = a_6 = a_7 = 0$. We will check that this condition is satisfied for all $x, y, z, u, v, w \in X$ with ($x \preceq u, y \succeq v$ and $z \preceq w$). The other conditions of Theorems are obviously satisfied. Consider the following possibilities.

Case1. $x > y > z$ and $u > v > w$. (and hence $u \geq x > y \geq v > w \geq z$). Then
$d(F(x,y,z), F(u,v,w)) = d(\frac{1}{3}(x^2 - y^2 - z^2), \frac{1}{3}(u^2 - v^2 - w^2)) = (L, \alpha L)$
where

$$L = \frac{1}{3}(u^2 - x^2 + v^2 - y^2 + z^2 - w^2),$$

and
$\frac{1}{8}d(gx, gu) + \frac{1}{4}d(F(x,y,z), gx) + \frac{1}{8}d(gy, gv) + \frac{1}{4}d(F(u,v,w), gu) = (D, \alpha D)$,
where $D = \frac{1}{8}(u^2 - x^2) + \frac{1}{12}(2x^2 + y^2 + z^2)) + \frac{1}{8}(y^2 - v^2) + \frac{1}{12}(2u^2 + v^2 + w^2))$ clearly $L \leq D$, Hence contraction condition (3.1) holds true.

Case2. $x > y > z$ and $u > w > v$ (and hence $u \geq x > y \geq w > v \geq z$). Then
$d(F(x,y,z), F(u,v,w)) = d(\frac{1}{3}(x^2 - y^2 - z^2), 0) = (L, \alpha L)$
where

$$L = \frac{1}{3}(x^2 - y^2 - z^2),$$

and
$\frac{1}{8}d(gx, gu) + \frac{1}{4}d(F(x,y,z), gx) + \frac{1}{8}d(gy, gv) + \frac{1}{4}d(0, gu) = (D, \alpha D)$,
where

$$D = \frac{1}{8}(u^2 - x^2) + \frac{1}{12}(2x^2 + y^2 + z^2)) + \frac{1}{8}(y^2 - v^2) + \frac{1}{4}u^2$$

clearly $L \leq D$, Hence contraction condition (3.1) holds true.

Case3. $x > y > z$ and any other combination between u, v, w other than $u > v > w$. Then
$d(F(x,y,z), F(u,v,w)) = d(\frac{1}{3}(x^2 - y^2 - z^2), 0) = (L, \alpha L)$
where

$$L = \frac{1}{3}(x^2 - y^2 - z^2),$$

and
$\frac{1}{8}d(gx, gu) + \frac{1}{4}d(F(x,y,z), gx) + \frac{1}{8}d(gy, gv) + \frac{1}{4}d(0, gu) = (D, \alpha D)$,
where

$$D = \frac{1}{8}(u^2 - x^2) + \frac{1}{12}(2x^2 + y^2 + z^2)) + \frac{1}{8}(y^2 - v^2) + \frac{1}{4}u^2$$

clearly $L \leq D$, Hence contraction condition (3.1) holds true.

Case4. $u > v > w$ and any other combination between x, y, z other than $x > y > z$. This case is treated analogously to the previous one.

Case5. Any other combination between x, y, z other than $x > y > z$ and also in between u, v, w other than $u > v > w$. Then

$d(F(x,y,z), F(u,v,w)) = d(0,0) = 0_E$ and the contractive condition is trivially satisfied.

Thus all the condition of Theorem 3.1 and 3.2 are satisfied. The mapping F and g have a unique common tripled fixed point $(0,0)$.

References

[1] H. K. Nashine, Z. Kadelburg and S. Radenovic, Coupled common fixed point theorems for w^*-compatible mappings in ordered cone metric spaces, Appl. Math. Comput. 218 (2012), 5422-5432.

[2] K. P. R. Rao and G. N. V. Kishore, A unique common triple fixed point theorem in partially ordered cone metric spaces, Bull. Math. Anal. Appl. 3 (2011), 213-222.

[3] L.-G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[4] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered set and some application to matix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.

[5] 14. Dajun Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal., Theory Methods Appl. 11 (1987), 623-632.

[6] J. J. Nieto and R. R. Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005), 223-239.

[7] J. J. Nieto and R. R. Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation, Acta Math. Sinica (English Ser.), 23 (2007), 2205-2212.

[8] T. G. Bhaskar, V. Lakshmikantham, Fixed point theory in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379-1393.

[9] E. Karapinar, Coupled fixed point theorems for nonlinear contractions in cone metric spaces, Comput. Math. Appl. 59 (2010), 3656-3668.

[10] F. Sabetghadam, H. P. Masiha, and A. H. Sanatpour, some coupled fixed point theorems in cone metric spaces,Fixed Point Theory Appl. Article ID 125426.

[11] W. Shatanawi, Partially ordered cone metric spaces and coupled fixed point results, Comput. Math. Appl. 59 (2010), 3656-3668.

[12] W. Shatanawi, Some common coupled fixed point results in cone metric spaces, Int. J. Math. Anal. 4 (2010), 2381-2388.

[13] H. S. Ding, L. Li, Coupled fixed point theorems in partially ordered mertric spaces, Filomate 25 (2011), 137-149.

[14] H. Aydi, B. Samet, C. Vetro, Coupled fixed point results in cone metric spaces for $\tilde{w}$-compatible mappings, Fixed Point Theory Appl. 2011, 2011:27.

[15] V. Bernide and M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 4889-4897.