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# INEQUALITIES FOR FIXED POINTS OF THE SUBCLASS $P(j, \lambda, \alpha, n)$ OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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**Abstract.** We consider the subclass $P(j, \lambda, \alpha, n)$ of starlike functions with negative coefficients by using the differential $D^n$ operator and functions of the form $f(z) = z - \sum\limits_{k=j+1}^{\infty} a_k z^k$ which are analytic in the open unit disk. We examine the subclass $P(j, \lambda, \alpha, n, z_0)$ for which $f(z_0) = z_0$ or $f^{'}(z_0) = 1$, $z_0$ real. We determine coefficient inequalities for functions belonging to the class $P(j, \lambda, \alpha, n, z_0)$. As special cases, the results of our paper reduce to Silverman [1].

**Keywords**: Univalent function, starlike, convex, fixed point.

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## 1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the family of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the open unit disk $\mathcal{U} := \{z : z \in \mathbb{C} \ and \ |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha \, (0 \leq \alpha < 1)$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathcal{U}). \tag{2}$$

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We denote by $\mathcal{S}^*(\alpha)$, the class of all such functions. On the other hand, a function $f \in \mathcal{A}$ is said to be convex of order $\alpha\,(0 \leq \alpha < 1)$ if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathcal{U})\,. \tag{3}$$

Let $\mathcal{C}\,(\alpha)$ denote the class of all those functions which are convex of order $\alpha$ in $U$.

Note that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ are, respectively, the classes of starlike and convex functions in $\mathcal{U}$.

Let $\mathcal{A}\,(j)$ denote the class of functions of the form:

$$f\,(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad (\,j \in \mathbb{N} := \{1,2,3,...\}) \tag{4}$$

which are analytic in the open unit disk $\mathcal{U}$.

Sălăgean [3] has introduced the following operator called the Sălăgean operator for a function $f\,(z)$ in $\mathcal{A}\,(j)$

$$D^0 f\,(z) = f\,(z) \tag{5}$$

$$D^1 f\,(z) = Df\,(z) = zf^{'}\,(z) = z + \sum_{k=j+1}^{\infty} k a_k z^k \tag{6}$$

$$D^2 f\,(z) = D\,(Df\,(z)) = zf'\,(z) + z^2 f''\,(z) = z + \sum_{k=j+1}^{\infty} k^2 a_k z^k \tag{7}$$

$$\vdots \tag{8}$$

$$D^n f\,(z) = D\left(D^{n-1} f\,(z)\right) \;=\; z + \sum_{k=j+1}^{\infty} k^n a_k z^k \qquad (\,n \in \mathbb{N})\,. \tag{9}$$

With the help of the differential operator $D^n$, we say that a function $f\,(z)$ belonging to $\mathcal{A}\,(j)$ is in the class $Q\,(j, \lambda, \alpha, n)$ if and only if

$$\Re\left\{\frac{(1-\lambda)\,z\,(D^n f\,(z))^{'} + \lambda z\left(D^{n+1} f\,(z)\right)^{'}}{(1-\lambda)\,D^n f\,(z) + \lambda D^{n+1} f\,(z)}\right\} > \alpha \tag{10}$$

for some $\alpha$ $(0 \leq \alpha < 1)$ and $\lambda$ $(0 \leq \lambda \leq 1)$, and for all $z \in \mathcal{U}$.

In [2], M.K.Aouf and H.M.Srivastava, $T\,(j)$ denoted the subclass of $\mathcal{A}\,(j)$ consisting of functions of the form:

$$f\,(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \qquad (\,a_k \geq 0;\; j \in \mathbb{N})\,. \tag{11}$$

Further, M.K.Aouf and H.M.Srivastava defined the class $P\left(j,\lambda,\alpha,n\right)$ by

$$P\left(j,\lambda,\alpha,n\right)=Q\left(j,\lambda,\alpha,n\right)\cap T\left(j\right). \tag{12}$$

In [2], for a function $f\left(z\right)$ in $P\left(j,\lambda,\alpha,n\right)$ ,M.K.Aouf and H.M.Srivastava defined

$$D^{0}f\left(z\right)=f\left(z\right) \tag{13}$$

$$D^{1}f\left(z\right)=Df\left(z\right)=zf^{'}\left(z\right)=z-\sum_{k=j+1}^{\infty}ka_{k}z^{k} \tag{14}$$

$$D^{2}f\left(z\right)=D\left(Df\left(z\right)\right)=zf'\left(z\right)+z^{2}f''\left(z\right)=z-\sum_{k=j+1}^{\infty}k^{2}a_{k}z^{k} \tag{15}$$

$$\vdots \tag{16}$$

$$D^{n}f\left(z\right)=D\left(D^{n-1}f\left(z\right)\right)\;=\;z-\sum_{k=j+1}^{\infty}k^{n}a_{k}z^{k}\qquad\left(\,n\in\mathbb{N}_{0}=\mathbb{N}\cup\{0\}\right). \tag{17}$$

In [2], M.K.Aouf and H.M.Srivastava obtained coefficients inequalities, distortion theorems, closure thorems, and some properties involving the modified Hadamard products of several functions belonging to the class $P\left(j,\lambda,\alpha,n\right)$. They also determined the radii of close-to-convexity for, and considered integral operators associated with, functions belonging to the class $P\left(j,\lambda,\alpha,n\right)$. Finally, they extended some of the aforementioned distortion theorems to hold true for certain operators of fractional calculus ( that is, fractional integral and fractional derivative).

In order that prove our theorem, the following lemma is needed.

**Lemma 1.1.** [2] Let the function $f\left(z\right)$ be defined by (11). Then $f\left(z\right)\in P\left(j,\lambda,\alpha,n\right)$ if only if

$$\sum_{k=j+1}^{\infty}k^{n}\left(k-\alpha\right)\left\{1+\left(k-1\right)\lambda\right\}a_{k}\leq 1-\alpha \tag{18}$$

$(a_{k}\geq 0;\; j\in\mathbb{N};\; n\in\mathbb{N}_{0};\; 0\leq\alpha<1;\; z\in\mathcal{U};\; 0\leq\lambda<1\;)$.

In [4], Schild examined the class of polynomials of the form $f(z)=z-\sum\limits_{n=2}^{N}a_{n}z^{n}$, where $a_{n}\geq 0$ and $f\left(z\right)$ is univalent in the disk $|z|<1$. In [5], Piłat studied the class of univalent polynomials of the from $f(z)=a_{1}z-\sum\limits_{n=2}^{N}a_{n}z^{n}$, where $a_{n}\geq 0$ and $f\left(z\right)=z_{0}>0$. In [1], Silverman dealt with functions of the form

$$f(z)=a_{1}z-\sum_{n=2}^{\infty}a_{n}z^{n} \tag{19}$$

where either

$$a_n \geq 0,\ f(z_0) = z_0\ \ (-1 < z_0 < 1;\ z_0 \neq 0) \tag{20}$$

or

$$a_n \geq 0,\ f^{'}(z_0) = 1\ \ (-1 < z_0 < 1)\,. \tag{21}$$

Given $\alpha$ and $z_0$ fixed, $\mathcal{S}_0^*(\alpha, z_0)$ examined the subclass of functions starlike of order $\alpha$ that satisfy (20), and $\mathcal{S}_1^*(\alpha, z_0)$ examined the subclass of functions starlike of order $\alpha$ that satisfy (21). Also denoted by $K_0(\alpha, z_0)$ and $K_1(\alpha, z_0)$ the subclasses of functions convex of order $\alpha$ that satisfy, respectively, (20) and (21).

In [1], Silverman determined necessary and sufficient conditions for functions to be in these classes. Silverman found the extreme points for each of these classes. Besides, Silverman gove a necessary and sufficient condition for a subset $B$ of the real interval $(0,1)$ to have the property that $\bigcup_{z_\gamma \in B} \mathcal{S}_0^*(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} K_0(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} \mathcal{S}_1^*(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} K_1(\alpha, z_\gamma)$ each forms a convex family. The extreme points of each of these classes was then determined. Many of the results in [1] reduced to those in [6] in the special case $z_0 = 0$.

In this paper inspired and motivated by this facts, we consider the subclass $P(j, \lambda, \alpha, n)$ of starlike functions with negative coefficients by using the differential $D^n$ operator and functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k\ \ (\ a_k \geq 0;\ j \in \mathbb{N}) \tag{22}$$

which are analytic in the open unit disk. We examine the subclass $P(j, \lambda, \alpha, n, z_0)$ for which $f(z_0) = z_0$ or $f^{'}(z_0) = 1$, $z_0$ real. The purpose of this paper is to determine coefficient inequalities for functions belonging to the class $P(j, \lambda, \alpha, n, z_0)$. As special cases, the results of this paper reduce to Silverman [1].

## 2. Main results

**Theorem 2.1.** Let the function $f(z)$ be defined by (11). Then $f(z)$ is $P(j, \lambda, \alpha, n, z_0)$ if only if

$$\sum_{k=j+1}^{\infty} \left[ k^n \left( \frac{k-\alpha}{1-\alpha} \right) \{1 + (k-1)\lambda\} - z_0^{k-1} \right] a_k \leq 1. \tag{23}$$

$(a_k \geq 0;\ j \in \mathbb{N};\ n \in \mathbb{N}_0;\ 0 \leq \alpha < 1;\ z \in \mathcal{U};\ 0 \leq \lambda < 1; z_0 \in \mathbb{R} \text{ fixed point})\,.$

The result is sharp.

**Proof.** Assume that $f(z)$ is $P(j,\lambda,\alpha,n,z_0)$. We find that

$$\frac{f(z_0)}{z_0}=1\Rightarrow\frac{z_0-\sum\limits_{k=j+1}^{\infty}a_kz_0^k}{z_0}=1\Rightarrow 1-\sum_{k=j+1}^{\infty}a_kz_0^{k-1}=1 \tag{24}$$

$$\sum_{k=j+1}^{\infty}a_kz_0^{k-1}=0. \tag{25}$$

Then, by Lemma 1, we have

$$\sum_{k=j+1}^{\infty}k^n(k-\alpha)\{1+(k-1)\lambda\}a_k\leq 1-\alpha \tag{26}$$

$$\sum_{k=j+1}^{\infty}\left[k^n\left(\frac{k-\alpha}{1-\alpha}\right)\{1+(k-1)\lambda\}-z_0^{k-1}\right]a_k\leq 1. \tag{27}$$

Conversely assume that the inequlatiy (25) holds true. Then we find that

$$\sum_{k=j+1}^{\infty}\left[k^n\left(\frac{k-\alpha}{1-\alpha}\right)\{1+(k-1)\lambda\}-z_0^{k-1}\right]a_k\leq 1 \tag{28}$$

$$\frac{\sum\limits_{k=j+1}^{\infty}k^n(k-\alpha)\{1+(k-1)\lambda\}a_k}{1-\alpha}-\sum_{k=j+1}^{\infty}a_kz_0^{k-1}\leq 1 \tag{29}$$

$$\sum_{k=j+1}^{\infty}a_kz_0^{k-1}\geq 0\ . \tag{30}$$

Now, we have two case that both $\sum\limits_{k=j+1}^{\infty}a_kz_0^{k-1}=0$ and $\sum\limits_{k=j+1}^{\infty}a_kz_0^{k-1}>0$. Then, we have

$$\sum_{k=j+1}^{\infty}a_kz_0^{k-1}=0\ \Rightarrow\ 1-\sum_{k=j+1}^{\infty}a_kz_0^{k-1}=1\Rightarrow z_0-\sum_{k=j+1}^{\infty}a_kz_0^{k}=z_0\Rightarrow f(z_0)=z_0 \tag{31}$$

and

$$\sum_{k=j+1}^{\infty}a_kz_0^{k-1}>0\ \Rightarrow\ 1-\sum_{k=j+1}^{\infty}a_kz_0^{k-1}>1\Rightarrow z_0-\sum_{k=j+1}^{\infty}a_kz_0^{k}>z_0\Rightarrow f(z_0)\neq z_0. \tag{32}$$

Consequently, $f(z)\in P(j,\lambda,\alpha,n,z_0)$.

**Corollary 2.2.** If we set $j=1$, $\lambda=0$ and $n=0$ in Teorem 1, we immediately obtain. Then we find that

$$\sum_{k=j+1}^{\infty}\left[\frac{k-\alpha}{1-\alpha}-z_0^{k-1}\right]a_k\leq 1 \tag{33}$$

and $P(1,\alpha,z_0)=\mathcal{S}_0^*(\alpha,z_0)$. In [1] examined $\mathcal{S}_0^*(\alpha,z_0)$.

**Corollary 2.3.** A special case of Theorem 1 when $j = 1$ ,$\lambda = 1$ and $n = 0$ yields. Then we find that

$$\sum_{k=j+1}^{\infty} \left[ \frac{k\,(k-\alpha)}{1-\alpha} - z_0^{k-1} \right] a_k \leq 1 \tag{34}$$

and $P\,(1, 1, \alpha, z_0) = K_0\,(\alpha, z_0)$. In [1] examined $K_0\,(\alpha, z_0)$.

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