Available online at http://scik.org
Adv. Fixed Point Theory, 9 (2019), No. 2, 165-177
https://doi.org/10.28919/afpt/3925
ISSN: 1927-6303

SOME COMMON COUPLED FIXED POINT THEOREMS FOR GENERALIZED CONTRACTION MAPPINGS IN C* ALGEBRAS-VALUED B-METRIC SPACES

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Abstract. In this paper, we establish the existence of common coupled fixed point and prove common coupled fixed point theorems for generalized contractive mappings in C*-algebras-valued b-metric spaces. Our results generalize some known results in C*-algebras-valued metric spaces.

Keywords: C*-algebras-valued b-metric spaces; generalized contraction; common coupled fixed point.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction

We all know, fixed point theory is one of the active research areas in mathematics. It focuses on the study of fixed points of mappings satisfy certain contractions in abstract spaces. In 1922, Banach proved the Banach theorem. The theorem is one of the main tools in fixed point theory, also, it is a very effective tools in solving existence problems in many branches of mathematical

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Received September 19, 2018

analysis and engineering. So fixed point theory has many applications in various branches of mathematics and branches of science.

The notion of b-metric space was introduced by *Bakhtin* in[1]. Since then, many actions generalized the b-metric spaces(see [2], [3], [4]). Recently, Ma and Jiang[5] introduced the concept of a C* algebras-valued b-metric spaces,and they obtained the basic fixed point theorems for self-map with contractive condition in C* algebras-valued b-metric spaces. In 2016, *Kamranetal* [6] also introduced the concept of this space, and generalized the Banach contraction principle on this space.

Gao and *Lakshmikantham*[7] introduced the concept of coupled fixed point. Since then, many pursuers investigated coupled fixed point theorems in ordered metric spaces[8-11]. Recently, *NidhiMalhotra* and *BineluBansal*[12] studied some coupled fixed point theorems for generalized contraction in b-metric spaces.

In this paper,we will establish the existence of common coupled fixed point and prove common coupled fixed point theorem for generalized contractive mappings in C*-algebras-valued b-metric spaces. Our results generalize some known results in b-metric spaces.

2. **Preliminaries**

In this section, we recall some basic definitions ,notations and results of C* algebras, which will be used in the sequel. The details of C* algebras can be found in [3].

Let $\mathcal{A}$ be an algebra. An involution on $\mathcal{A}$ is a conjugate linear map $a \longrightarrow \alpha^*$ such that $(a^*)^* = a$ and $(ab)^* = a^*b^*$ for all $a, b \in \mathcal{A}$. The pair $(\mathcal{A}, *)$ is called a $*$-algebra. A $*$-algebra $\mathcal{A}$ together with a complete substantiative norm such that $||a^*|| = ||a||$ is said to be a Banach $*$-algebra. Furthermore, if we have $||a^*a|| = ||a||^2$ in Banach $*$-algebra for all $a \in \mathcal{A}$, then $\mathcal{A}$ is a C*-algebra. An element a of a C*-algebra is positive if $a = a^*$ and $\sigma(a) \subseteq [0, +\infty)$, where $\sigma(a)$ is the spectrum of a, and $\sigma(a) = \{\lambda \in \mathcal{R} : \lambda I_{\mathcal{A}} - a$ is not invertible$\}$, where $I_{\mathcal{A}}$ is identity element of C*-algebra. Every positive element a of C*-algebra $\mathcal{A}$ has a unique positive square root, and denote by $\mathcal{A}_+$ be the set of positive elements, where $0_{\mathcal{A}} \geq a$, $0_{\mathcal{A}}$ is the zero element in $\mathcal{A}$.

we define a natural partial ordering $\preceq$ with respect to $\mathcal{A}$ by $x \preceq y$ if and only if $0_{\mathcal{A}} \succeq y - x$. Let $\mathcal{A}' = \{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{A}\}$,and $\mathcal{A}'_+ = \mathcal{A}_+ \cap \mathcal{A}'$.Now we give some basic concepts and results, which will be needed as follows.

Definition 2.1. *(see[13]) Let $\mathcal{A}$ always be a C^*-algebra. X be a nonempty set. Let $b \in \mathcal{A}'$ be such that $\|b\| \geqslant 1$. Suppose that the mapping $d_b : X \times X \longrightarrow \mathcal{A}_+$ is defined with the following properties hold for all $x, y, z \in \mathcal{A}$ is a real Banach space in which an operation of multiplication is defined,subject to the following properties ,for all $x, y, z \in \mathcal{A}, \alpha \in \mathcal{R}$:*

1. $d_b(x, y) \succeq 0_{\mathcal{A}}$;

2. $d_b(x, y) = 0$ if and only if $x = y$;

3. $d_b(x, y) = d_b(y, x)$;

4. $d_b(x, y) \preceq b[d_b(x, z) + d_b(z, x)]$.

then d_b is said to be a C^*-algebras-valued b-metric mapping on X , and the triplet $(X, \mathcal{A}, d_b)$ is called a C^*-algebras-valued b-metric space with coefficient b.

Remark 2.2. *From[3], we know that a C^*-algebras-valued b-metric space are not C^*-algebras-valued metric spaces.*

Definition 2.3. *(see[3]) Suppose that $(X, \mathcal{A}, d_b)$ be a C^*-algebras-valued b-metric space, let $x \in X$ and $\{x_n\}$ is a sequence in X.*

1.$\{x_n\}$ converges to x whenever for every $c > \theta$, there is a natural number N such that $\|d_b(x_n, x)\| < c$ for all $n \geq N$.we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x(n \longrightarrow \propto)$;

2.$\{x_n\}$ is a Cauchy sequence whenever for every $c > \theta$, there is a natural number N such that $\|d_b(x_n, x_m)\| < c$ for all $n, m \geq N$;

3. $(X, \mathcal{A}, d_b)$ is a complete cone metric space if every Cauchy sequence is convergent.

Definition 2.4. *(see[14]) An element $(x, y) \in X \times X$ is called a coupled fixed point of mappings $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$*

Definition 2.5. *(see[14]) An element $(x, y) \in X \times X$ is called*

(1)a coupled coincidence point of mappings $S, F : X \times X \to X$, if

$$S(x, y) = F(x, y), S(y, x) = F(y, x);$$

(2)a common coupled fixed point of mappings $S,F : X \times X \to X$, if

$$x = S(x,y) = F(x,y), y = S(y,x) = F(y,x).$$

Lemma 2.6. *(see[1,3]) Suppose that $\mathcal{A}$ is a uncial C^*-algebras with a unite $I_{\mathcal{A}}$.*

(1) if $a \in \mathcal{A}_+$, with $\|a\| < \frac{1}{2}$, then $I_{\mathcal{A}} - a$ is invertible and $\|a(I_{\mathcal{A}} - a)^{-1}\| < 1$;

(2) suppose that $a,b \in \mathcal{A}$,with $a,b \geq \theta$ andab $= ba$,then $ab \geq \theta$;

(3) let $a \in \mathcal{A}'$, if $b,c \in \mathcal{A}$ with $b \geq c \geq 0_{\mathcal{A}}$, and $(I_{\mathcal{A}} - a) \in \mathcal{A}'_+$ is an invertible operator , then

$$(I_{\mathcal{A}} - a)^{-1}b \geq (I_{\mathcal{A}} - a)^{-1}c.$$

Lemma 2.7. *(see[1]) the sum of two positive elements in a C^*-algebras is a positive element.*

Remark 2.8. *From Lemma 2.1, we know that a $b[d_b(x,z) + d_b(z,y)]$ is a positive element.*

3. **Main Results**

Theorem 3.1. *Let $(X,\mathcal{A},d_b)$ be a complete C^*-algebras-valued b-metric space.Assume that the mappings $S,T : X \to X$ satisfies the following contractive condition:*

$$d_b(S(x,y),T(u,v)) \leq \alpha\frac{d_b(x,u)+d_b(y,v)}{2} + \beta\frac{d_b(x,S(x,y))d_b(u,T(u,v))}{1+d_b(x,u)+d_b(y,v)} + \gamma\frac{d_b(u,S(x,y))d_b(x,T(u,v))}{1+d_b(x,u)+d_b(y,v)}$$

for all $x,y,u,v \in X$ and $\alpha,\beta,\gamma \geq 0$, with $\|\beta\| < \frac{1}{2}$, $\|b\|\|\alpha\| + \|\beta\| < 1, \|\frac{\alpha}{2}\| + \|\gamma\| < \frac{1}{2}$, then S and T have a unique common coupled fixed point in X.

Proof: Let x_0, y_0 be any two arbitrary points in X, set $x_{2k+1} = S(x_{2k}, y_{2k})$ and $y_{2k+1} = S(y_{2k}, x_{2k})$, $x_{2k+2} = T(x_{2k+1}, y_{2k+1})$ and $y_{2k+2} = T(y_{2k+1}, x_{2k+1})$, where $k = 0,1,2,\cdots$, then we have

$$\begin{aligned}
& d_b(x_{2k+1},x_{2k+2}) = d_b(S(x_{2k},y_{2k}),T(x_{2k+1},y_{2k+1})) \\
\leq\ & \alpha\frac{d_b(x_{2k},x_{2k+1})+d_b(y_{2k},y_{2k+1})}{2} + \beta\frac{d_b(x_{2k},S(x_{2k},y_{2k}))d_b(x_{2k+1},T(x_{2k+1},y_{2k+1}))}{1+d_b(x_{2k},x_{2k+1})+d_b(y_{2k},y_{2k+1})} \\
+\ & \gamma\frac{d_b(x_{2k+1},S(x_{2k},y_{2k}))d_b(x_{2k},T(x_{2k+1},y_{2k+1}))}{1+d_b(x_{2k},x_{2k+1})+d_b(y_{2k},y_{2k+1})} \\
=\ & \alpha\frac{d_b(x_{2k},x_{2k+1})+d_b(y_{2k},y_{2k+1})}{2} + \beta\frac{d_b(x_{2k},x_{2k+1})d_b(x_{2k+1},x_{2k+2})}{1+d_b(x_{2k},x_{2k+1})+d_b(y_{2k},y_{2k+1})}
\end{aligned}$$

$$
\begin{aligned}
(3.1) \qquad & + \gamma \frac{d_b(x_{2k+1}, x_{2k+1}) d_b(x_{2k}, x_{2k+2})}{1 + d_b(x_{2k}, x_{2k+1}) + d_b(y_{2k}, y_{2k+1})} \\
& = \alpha \frac{d_b(x_{2k}, x_{2k+1}) + d_b(y_{2k}, y_{2k+1})}{2} + \beta \frac{d_b(x_{2k}, x_{2k+1}) d_b(x_{2k+1}, x_{2k+2})}{1 + d_b(x_{2k}, x_{2k+1}) + d_b(y_{2k}, y_{2k+1})} \\
& \leq \frac{\alpha}{2} d_b(x_{2k}, x_{2k+1}) + \frac{\alpha}{2} d_b(y_{2k}, y_{2k+1}) + \beta d_b(x_{2k+1}, x_{2k+2}).
\end{aligned}
$$

From which it follows:

$$
(3.2) \qquad (I_{\mathcal{A}} - \beta) d_b(x_{2k+1}, x_{2k+2}) \leq \alpha \frac{d_b(x_{2k}, x_{2k+1})}{2} + \alpha \frac{d_b(y_{2k}, y_{2k+1})}{2}.
$$

Similarly

$$
(3.3) \qquad (I_{\mathcal{A}} - \beta) d_b(y_{2k+1}, y_{2k+2}) \leq \alpha \frac{d_b(y_{2k}, y_{2k+1})}{2} + \alpha \frac{d_b(x_{2k}, x_{2k+1})}{2}.
$$

From (3.1) and (3.2), we get

$$
(3.4) \quad (I_{\mathcal{A}} - \beta)[d_b(x_{2k+1}, x_{2k+2}) + d_b(y_{2k+1}, y_{2k+2})] \leq \alpha [d_b(x_{2k}, x_{2k+1}) + d_b(y_{2k}, y_{2k+1})].
$$

Since $\beta \in \mathcal{A}'_+$, moreover from the condition $\|\beta\| < \frac{1}{2}$, then by Lemma2.1, it follows that $I_{\mathcal{A}} - \beta$ is invertible.

By multiplying in both side of (3.3) by $(I_{\mathcal{A}} - \beta)^{-1}$, we arrive at

$$
(3.5) \quad d_b(x_{2k+1}, x_{2k+2}) + d_b(y_{2k+1}, y_{2k+2}) \leq (I_{\mathcal{A}} - \beta)^{-1} \alpha [d_b(x_{2k}, x_{2k+1}) + d_b(y_{2k}, y_{2k+1})].
$$

Furthermore we can obtain similarly:

$$
(3.6) \qquad (I_{\mathcal{A}} - \beta) d_b(x_{2k+2}, x_{2k+3}) \leq \alpha \frac{d_b(x_{2k+1}, x_{2k+2})}{2} + \alpha \frac{d_b(y_{2k+1}, y_{2k+2})}{2}.
$$

and

$$
(3.7) \qquad (I_{\mathcal{A}} - \beta) d_b(y_{2k+2}, y_{2k+3}) \leq \alpha \frac{d_b(y_{2k+1}, y_{2k+2})}{2} + \alpha \frac{d_b(x_{2k+1}, x_{2k+2})}{2}.
$$

Adding (3.1) and (3.2), we get

$$
(3.8) (I_{\mathcal{A}} - \beta)[d_b(x_{2k+2}, x_{2k+3}) + d_b(y_{2k+2}, y_{2k+3})] \leq \alpha [d_b(x_{2k+1}, x_{2k+2}) + d_b(y_{2k+1}, y_{2k+2})].
$$

By multiplying in both side of (3.7) by $I_{\mathcal{A}} - \beta$,we arrive at

$$
(3.9) d_b(x_{2k+2}, x_{2k+3}) + d_b(y_{2k+2}, y_{2k+3}) \leq (I_{\mathcal{A}} - \beta)^{-1} \alpha [d_b(x_{2k+1}, x_{2k+2}) + d_b(y_{2k+1}, y_{2k+2})].
$$

from (3.4) and (3.5), we have

$$d_b(x_n, x_{n+1}) + d_b(y_n, y_{n+1}) \leq \frac{\alpha}{I_{\mathcal{A}} - \beta}[d_b(x_{n-1}, x_n) + d_b(y_{n-1}, y_n)]. \tag{3.10}$$

Denote $\eta_n = d_b(x_n, x_{n+1}) + d_b(y_n, y_{n+1})$, $h = \frac{\alpha}{I_{\mathcal{A}}-\beta}$,*then by (3.9), we have*

$$\eta_n \leq h\eta_{n-1} \leq h^2\eta_{n-2} \leq \cdots \leq h^n\eta_0.$$

If $\eta_0 = 0_{\mathcal{A}}$,*then from definition 2.1, we easily know that* (x_0, y_0) *is a common coupled fixed point of the mappings S and T. Now,we set* $\eta_0 > 0_{\mathcal{A}}$, *and set* $n, m \in N$ *with* $m > n$, *by using Definition 2.1, it following that:*

$$\begin{aligned} d_b(x_n, x_m) &\leq b[d_b(x_n, x_{n+1}) + d_b(x_{n+1}, x_m)] \\ &\leq bd_b(x_n, x_{n+1}) + b^2[d_b(x_{n+1}, x_{n+2}) + d_b(x_{n+2}, x_m)) \\ &\leq bd_b(x_n, x_{n+1}) + b^2d_b(x_{n+1}, x_{n+2}) + \cdots + b^{m-n-1}d_b(x_{m-2}, x_{m-1}) + b^{m-n-1}d_b(x_{m-1}, x_m). \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} d_b(y_n, y_m) &\leq bd_b(y_n, y_{n+1}) + b^2d_b(y_{n+1}, y_{n+2}) + \cdots \\ &+ b^{m-n-1}d_b(y_{m-2}, y_{m-1}) + b^{m-n-1}d_b(y_{m-1}, y_m). \end{aligned} \tag{3.11}$$

Hence, adding (3.10) and (3.11), we have

$$\begin{aligned} d_b(x_n, x_m) + d_b(y_n, y_m) &\leq b\eta_n + b^2\eta_{n+1} + \cdots + b^{m-n-1}\eta_{m-2} + b^{m-n-1}\eta_{m-1} \\ &\leq bh^n\eta_0 + b^2h^{n+1}\eta_0 + \cdots + b^{m-n-1}h^{m-2}\eta_0 + b^{m-n-1}h^{m-1}\eta_0. \\ &= b\sum_{i=n}^{m-2} b^{i-n}h^i\eta_0 + b^{m-n-1}h^{m-1}\eta_0 \\ &= b\sum_{i=n}^{m-2} b^{i-n}h^{\frac{i}{2}}\eta_0^{\frac{1}{2}}\eta_0^{\frac{1}{2}}h^{\frac{i}{2}} + b^{m-n-1}h^{\frac{m-1}{2}}\eta_0^{\frac{1}{2}}\eta_0^{\frac{1}{2}}h^{\frac{m-1}{2}} \\ &\leq \|b\sum_{i=n}^{m-2} b^{i-n}h^{\frac{i}{2}}\eta_0^{\frac{1}{2}}\|^2 + \|b^{m-n-1}h^{\frac{m-1}{2}}\eta_0^{\frac{1}{2}}\|^2 \end{aligned}$$

$$
\begin{aligned}
(3.12) \qquad & \leq \|b\| \sum_{i=n}^{m-2} \|b^{i-n}\| \| h^{\frac{i}{2}}\|^2 \|\eta_0^{\frac{1}{2}}\|^2 I_{\mathcal{A}} + \|b^{n-m-1}\| \| h^{\frac{m-1}{2}}\|^2 \|\eta_0^{\frac{1}{2}}\|^2 I_{\mathcal{A}} \\
& \leq \|b\|^{1-n} \|\eta_0^{\frac{1}{2}}\|^2 \sum_{i=n}^{m-2} \|bh\|^i I_{\mathcal{A}} + \|b\|^{-n} \|\eta_0^{\frac{1}{2}}\|^2 \|bh\|^{m-1} I_{\mathcal{A}}.
\end{aligned}
$$

by the condition, $\|b\| > 1$, $\|b\|\|\alpha\| + \|\beta\| < 1$. *So* $d_b(x_n, x_m) + d_b(y_n, y_m) \to 0_{\mathcal{A}}$, *as* $m, n \to \infty$. *Hence,* $\{x_n\}$ *and* $\{y_n\}$ *are Cauchy sequences in* X. *Since* X *is a complete space, there exists* $x^*, y^* \in X$ *such that* $x_n \to x^*$ *and* $y_n \to y^*$ *as* $n \to \infty$. *Now we will prove that* $x^* = S(x^*, y^*)$ *and* $y^* = S(y^*, x^*)$.

$$
\begin{aligned}
d_b(x^*, S(x^*, y^*)) &\leq b[d_b(x^*, x_{2k+2}) + d_b(x_{2k+2}, S(x^*, y^*))] \\
&\leq bd_b(x^*, x_{2k+2}) + bd_b(T(x_{2k+1}, y_{2k+1}), S(x^*, y^*)) \\
&\leq bd_b(x^*, x_{2k+2}) + b\alpha \frac{d_b(x_{2k+1}, x) + d_b(y_{2k+1}, y)}{2} \\
&+ b\beta \frac{d_b(x^*, S(x^*, y^*)))d_b(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{1 + d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)} \\
&+ b\gamma \frac{d_b(x_{2k+1}, S(x^*, y^*))d_b(x^*, T(x_{2k+1}, y_{2k+1}))}{1 + d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)} \\
&\leq bd_b(x^*, x_{2k+2}) + b\alpha \frac{d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)}{2} \\
&+ b\beta \frac{d_b(x^*, S(x^*, y^*))d_b(x_{2k+1}, x_{2k+2})}{1 + d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)} \\
&+ b\gamma \frac{d_b(x_{2k+1}, S(x^*, y^*))d_b(x^*, x_{2k+2})}{1 + d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)}.
\end{aligned}
$$

by taking $k \to \infty$, *we have* $d_b(x^*, S(x^*, y^*)) \leq 0$, *so* $d_b(x^*, S(x^*, y^*)) = 0$ *That is,* $x^* = S(x^*, y^*)$. *Similarly, we can prove that* $y^* = S(y^*, x^*)$. *We can similarly prove that* $x^* = T(x^*, y^*)$ *and* $y^* = T(y^*, x^*)$. *Therefore, we have obtain that* (x^*, y^*) *is a common coupled fixed point of* S *and* T.

Now we show that S *and* T *have a unique common coupled fixed point. Let* $(x, y) \in X \to X$ *be another common coupled fixed point of* S *and* T,*then*

$$
\begin{aligned}
d_b(x^*, x) &= d_b(S(x^*, y^*), T(x, y)) \\
&\leq \alpha \frac{d_b(x^*, x) + d_b(y^*, y)}{2} + b\beta \frac{d_b(x^*, S(x^*, y^*))d_b(x, T(x, y))}{1 + d_b(x^*, x) + d_b(y^*, y)}
\end{aligned}
$$

$$
\begin{aligned}
(3.13) \qquad & + \quad \gamma \frac{d_b(x,S(x^*,y^*))d_b(x^*,T(x,y))}{1+d_b(x^*,x)+d_b(y^*,y)} \\
& = \quad \alpha \frac{d_b(x^*,x)+d_b(y^*,y)}{2} + b\beta \frac{d_b(x^*,x^*)d_b(x,x)}{1+d_b(x^*,x)+d_b(y^*,y)} \\
& + \quad \gamma \frac{d_b(x,x^*)d_b(x^*,x)}{1+d_b(x^*,x)+d_b(y^*,y)} \\
& \leq \quad \alpha \frac{d_b(x^*,x)}{2} + \alpha \frac{d_b(y^*,y)}{2} + \gamma d_b(x^*,x).
\end{aligned}
$$

So $(I_{\mathcal{A}} - \frac{\alpha}{2} - \gamma)d_b(x^*,x) \leq \alpha \frac{d_b(y^*,y)}{2}$

by the condition $\|\frac{\alpha}{2}\| + \|\gamma\| < \frac{1}{2}$ *and Lemma2.1, it follows that* $I_{\mathcal{A}} - \frac{\alpha}{2} - \gamma$ *is invertible, then and*

$$(3.14) \qquad d_b(x^*,x) \quad \leq \quad (I_{\mathcal{A}} - \frac{\alpha}{2} - \gamma)^{-1} \frac{\alpha}{2} d_b(y^*,y) = \frac{\alpha}{2I_{\mathcal{A}} - \alpha - 2\gamma} d_b(y^*,y).$$

Similarly we can easily prove that

$$
\begin{aligned}
(3.15) \qquad d_b(y^*,y) \quad & \leq \quad (I_{\mathcal{A}} - \frac{\alpha}{2} - \gamma)^{-1} \frac{\alpha}{2} d_b(x^*,x) \\
& = \quad \frac{\alpha}{2I_{\mathcal{A}} - \alpha - 2\gamma} d_b(x^*,x).
\end{aligned}
$$

Adding (3.12) and (3.13), we have

$d_b(x^*,x) + d_b(y^*,y) \leq \frac{\alpha}{2I_{\mathcal{A}} - \alpha - 2\gamma}[d_b(x^*,x) + d_b(y^*,y)]$

By the condition $\|\frac{\alpha}{2}\| + \|\gamma\| < \frac{1}{2}$, *we have* $\|\alpha\| + \|2\gamma\| < 1$. *So* $\|\alpha\| + \|\gamma\| < 1$, *then we get* $(2I_{\mathcal{A}} - 2\alpha - 2\gamma)[d_b(x^*,x) + d_b(y^*,y)] \leq 0$. *So* $d_b(x^*,x) + d_b(y^*,y) = 0$. *That is* $x = x^*$ *and* $y = y^*$.

Corollary 3.2. *Let* $(X, \mathcal{A}, d_b)$ *be a complete* C^**-algebras-valued b-metric space. Assume that the mapping* $F : X \to X$ *satisfies the following contractive condition:*

$$d_b(F(x,y),F(u,v)) \leq \alpha \frac{d_b(x,u)+d_b(y,v)}{2} + \beta \frac{d_b(x,F(x,y))d_b(u,F(u,v))}{1+d_b(x,u)+d_b(y,v)} + \gamma \frac{d_b(u,F(x,y))d_b(x,F(u,v))}{1+d_b(x,u)+d_b(y,v)}$$

for all $x,y,u,v \in X$ *and* $\alpha,\beta,\gamma \geq 0$, *with* $\|\beta\| < \frac{1}{2}, \|b\|\|\alpha\| + \|\beta\| < 1, \|\alpha\| + \|\gamma\| < 1$, *then* F *have a unique coupled fixed point in* X.

Proof: Take $S = T = F$ *In Theorem 3.1, we can obtain the conclusion.*

Theorem 3.3. *Let* $(X, \mathcal{A}, d_b)$ *be a complete* C^**-algebras-valued b-metric space.Assume that the mappings* $S,T : X \to X$ *satisfies the following contractive condition:*

$$d_b(S(x,y),F(u,v)) \leq \begin{cases} \alpha \frac{d_b(x,u)+d_b(y,v)}{2} + \beta \frac{d_b(x,S(x,y))d_b(u,F(u,v))}{b[d_b(x,F(u,v))+d_b(u,S(x,y))+d_b(x,u)+d_b(y,v)]}, & if D \neq 0 \\ 0, & if D = 0 \end{cases}$$

for all $x, y, u, v \in X$, *where* $D = D(x, y, u, v) = b[d_b(x, F(u, v)) + d_b(u, S(x, y)) + d_b(x, u) + d_b(y, v)]$,*and* $\alpha, \beta \geq 0$, *with* $\|\beta\| < \frac{1}{2}, \|b\alpha\|\|\beta\| < 1$, *then* S *and* F *have a unique common coupled fixed point in* X.

Proof: Let x_0, y_0 *be any two arbitrary points in* X, *set* $x_{2k+1} = S(x_{2k}, y_{2k})$ *and* $y_{2k+1} = S(y_{2k}, x_{2k})$, $x_{2k+2} = F(x_{2k+1}, y_{2k+1})$ *and* $y_{2k+2} = F(y_{2k+1}, x_{2k+1})$, *where* $k = 0, 1, 2, \cdots$, *we can assume that* $D_1 = D(x_{2k}, y_{2k}, x_{2k+1}, y_{2k+1}) \neq 0$, *and* $D_2 = D(y_{2k}, x_{2k}, y_{2k+1}, x_{2k+1}) \neq 0$ *then we have*

$$\begin{aligned} & d_b(x_{2k+1}, x_{2k+2}) = d_b(S(x_{2k}, y_{2k}), F(x_{2k+1}, y_{2k+1})) \\ \leq\ & \alpha \frac{d_b(x_{2k}, x_{2k+1}) + d_b(y_{2k}, y_{2k+1})}{2} + \beta \frac{d_b(x_{2k}, S(x_{2k}, y_{2k})) d_b(x_{2k+1}, F(x_{2k+1}, y_{2k+1}))}{b[d_b(x_{2k}, x_{2k+2}) + d_b(x_{2k}, x_{2k+1}) + d_b(y_{2k}, y_{2k+1})]} \\ \leq\ & \alpha \frac{d_b(x_{2k}, x_{2k+1}) + d_b(y_{2k}, y_{2k+1})}{2} + \beta d_b(x_{2k+1}, x_{2k+2}). \end{aligned}$$

So

$$(1 - \beta) d_b(x_{2k+1}, x_{2k+2}) \leq \frac{\alpha}{2} d_b(x_{2k}, x_{2k+1}) + \frac{\alpha}{2} d_b(y_{2k}, y_{2k+1}). \tag{3.16}$$

Similarly,we can easily prove that

$$(1 - \beta) d_b(y_{2k+1}, y_{2k+2}) \leq \frac{\alpha}{2} d_b(y_{2k}, y_{2k+1}) + \frac{\alpha}{2} d_b(x_{2k}, x_{2k+1}). \tag{3.17}$$

adding (3.14) and (3.15), we have

$$(1 - \beta)[d_b(x_{2k+1}, x_{2k+2}) + d_b(y_{2k+1}, y_{2k+2})] \leq \alpha[d_b(y_{2k}, y_{2k+1}) + d_b(x_{2k}, x_{2k+1})].$$

Since $\beta \in \mathcal{A}'_+$, *moreover from the condition* $\|\beta\| < \frac{1}{2}$, *then by Lemma2.1, it follows that* $I_{\mathcal{A}} - \beta$ *is invertible. Now, if*

$D_3 = D(x_{2k+1}, y_{2k+1}, x_{2k+2}, y_{2k+2}) \neq 0$, *and* $D_2 = D(y_{2k+1}, x_{2k+1}, y_{2k+2}, x_{2k+2}) \neq 0$. *We get*

$$\begin{aligned} d_b(x_{2k+2}, x_{2k+3}) &= d_b(F(x_{2k+1}, y_{2k+1}), S(x_{2k+2}, y_{2k+2})) \\ &\leq \alpha \frac{d_b(x_{2k+2}, x_{2k+1}) + d_b(y_{2k+2}, y_{2k+1})}{2} \\ &+ \beta \frac{d_b(x_{2k+2}, S(x_{2k+2}, y_{2k+2})) d_b(x_{2k+1}, F(x_{2k+1}, y_{2k+1}))}{b[d_b(x_{2k+2}, x_{2k+3}) + d_b(x_{2k+1}, x_{2k+2}) + d_b(y_{2k+2}, y_{2k+1})]} \\ &\leq \alpha \frac{d_b(x_{2k+2}, x_{2k+1}) + d_b(y_{2k+2}, y_{2k+1})}{2} + \beta d_b(x_{2k+2}, x_{2k+3}). \end{aligned}$$

$$so (1 - \beta) d_b(x_{2k+2}, x_{2k+3}) \leq \frac{\alpha}{2} d_b(x_{2k+1}, x_{2k+2}) + \frac{\alpha}{2} d_b(y_{2k+1}, y_{2k+2}). \tag{3.18}$$

Similarly, we also can prove that

$$(1-\beta)d_b(y_{2k+2},y_{2k+3}) \le \frac{\alpha}{2}d_b(y_{2k+1},y_{2k+2}) + \frac{\alpha}{2}d_b(x_{2k+1},x_{2k+2}). \tag{3.19}$$

adding (3.16) and (3.17), we have

$$(1-\beta)[d_b(x_{2k+2},x_{2k+3}) + d_b(y_{2k+2},y_{2k+3})] \le \alpha[d_b(x_{2k+1},x_{2k+2}) + d_b(y_{2k+1},y_{2k+2}].$$

Therefore, we get

$$d_b(x_n,x_{n+1}) + d_b(y_n,y_{n+1}) \le (1-\beta)^{-1}\alpha[d_b(x_{n-1},x_n) + d_b(y_{n-1},y_n].$$

Notion $\delta = (1-\beta)^{-1}\alpha$, *then* $\|\delta\|$*and* $h_n = d_b(x_n,x_{n+1}) + d_b(y_n,y_{n+1})$, *then we have*

$$h_n \le \delta h_{n-1} \le \delta^2 h_{n-2} \le \cdots \le \delta^n h_0.$$

Set $m,n \in N$ *and* $m > n$

$$\begin{aligned} d_b(x_n,x_m) + d_b(y_n,y_m) &\le b[d_b(x_n,x_{n+1}) + d_b(y_n,y_{n+1})] + b^2[d_b(x_{n+1},x_{n+2}) + d_b(y_{n+1},y_{n+2})] \\ &+ \cdots + b^{m-n}[d_b(x_{m-1},x_m) + d_b(y_{m-1},y_m)] \\ &\le b\delta^n h_0 + b^2\delta^{n+1}h_0 + \cdots + b^{m-n}\delta^{m-1}h_0 \\ &\le b\delta^n h_0(1 + b\delta + (b\delta)^2 + \cdots + (b\delta)^{m-n-1}) \\ &= \frac{hb\delta^n(1-(b\delta)^{m-n})}{1-b\delta}. \end{aligned}$$

By the condition, $\|b\alpha\|\|\beta\| < 1$, *so we get*

$$d_b(x_n,x_m) + d_b(y_n,y_m) \to 0, (m,n \to \infty).$$

this shows that $\{x_n\}$ *and* $\{y_n\}$ *are Cauchy sequence in* X. *Since* $(X,\mathcal{A},d_b)$ *be a complete* C^*-*algebras-valued b-metric space, there exists* $x^*,y^* \in X$ *such that* $x_n \to x^*$ *and* $y_n \to y^*$ *as* $n \to \infty$.

Now we will prove that $x^* = S(x^*, y^*)$ *and* $y^* = S(y^*, x^*)$.

$$\begin{aligned}
d_b(x^*, S(x^*, y^*)) &\leq b[d_b(x^*, x_{2k+2}) + d_b(x_{2k+2}, S(x^*, y^*))] \\
&\leq bd_b(x^*, x_{2k+2}) + bd_b(F(x_{2k+1}, y_{2k+1}), S(x^*, y^*)) \\
&\leq bd_b(x^*, x_{2k+2}) + b\alpha \frac{d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)}{2} \\
&+ b\beta \frac{d_b(x^*, S(x^*, y^*)))d_b(x_{2k+1}, F(x_{2k+1}, y_{2k+1}))}{b[d_b(x_{2k+1}, S(x^*, y^*)) + d_b(x^*, F(x_{2k+1}, y_{2k+1})) + d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)]} \\
&= bd_b(x^*, x_{2k+2}) + \alpha \frac{d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)}{2} \\
&+ b\beta \frac{d_b(x^*, S(x^*, y^*))d_b(x_{2k+1}, x_{2k+2})}{d_b(x_{2k+1}, S(x^*, y^*)) + d_b(x^*, x_{2k+1}) + d_b(x_{2k+1}, x^*) + d_b(y_{2k+1}, y^*)}.
\end{aligned}$$

by taking $k \to \infty$, *we have* $d_b(x^*, S(x^*, y^*)) \leq 0$, *so* $d_b(x^*, S(x^*, y^*)) = 0$ *That is,* $x^* = S(x^*, y^*)$. *Similarly, we can prove that* $y^* = S(y^*, x^*)$. *We can similarly prove that* $x^* = F(x^*, y^*)$ *and* $y^* = F(y^*, x^*)$. *Therefore, we have obtain that* (x^*, y^*) *is a common coupled fixed point of* S *and* F.

Now we show that S *and* F *have a unique common coupled fixed point. Let* $(x, y) \in X \to X$ *be another common coupled fixed point of* S *and* F*,then*

$$\begin{aligned}
d_b(x^*, x) &= d_b(S(x^*, y^*), F(x, y)) \\
&\leq \alpha \frac{d_b(x^*, x) + d_b(y^*, y)}{2} + b\beta \frac{d_b(x^*, S(x^*, y^*))d_b(x, F(x, y))}{b[d_b(x^*, F(x, y)) + d_b(x, S(x^*, y^*)) + d_b(x^*, x) + d_b(y^*, y)]} \\
&= \alpha \frac{d_b(x^*, x) + d_b(y^*, y)}{2} + b\beta \frac{d_b(x^*, x^*)d_b(x, x)}{b[3d_b(x^*, x) + d_b(y^*, y)]} \\
&= \alpha \frac{d_b(x^*, x) + d_b(y^*, y)}{2}.
\end{aligned}$$

So $(I_{\mathcal{A}} - \frac{\alpha}{2})d_b(x^*, x) \leq \alpha d_b(y^*, y)$

We consider the condition $\|\beta\| < \frac{1}{2}, \|b\alpha\|\|\beta\| < 1$, *we have* $\|\alpha\| < \frac{1}{2}$, *by Lemma2.2, it follows that* $I_{\mathcal{A}} - \frac{\alpha}{2}$ *is invertible, then*

$$d_b(x^*, x) \leq (I_{\mathcal{A}} - \frac{\alpha}{2})^{-1} \alpha d_b(y^*, y) = \frac{\alpha}{2I_{\mathcal{A}} - \alpha} d_b(y^*, y). \tag{3.20}$$

Similarly we can easily prove that

$$d_b(y^*,y) \leq (2I_{\mathcal{A}}-\alpha)^{-1}\alpha d_b(x^*,x) = \frac{\alpha}{2I_{\mathcal{A}}-\alpha}d_b(x^*,x). \tag{3.21}$$

Adding (3.18) and (3.19), we have $d_b(x^*,x)+d_b(y^*,y) \leq \frac{\alpha}{2I_{\mathcal{A}}-\alpha}[d_b(x^*,x)+d_b(y^*,y)]$ *By the condition* $\|\alpha\|<1$*, we have* $(2I_{\mathcal{A}}-2\alpha)[d_b(x^*,x)+d_b(y^*,y)] \leq 0$*. So* $d_b(x^*,x)+d_b(y^*,y)=0$*. That is* $x=x^*$ *and* $y=y^*$*. The proof is over.*

Corollary 3.4. *Let* $(X,\mathcal{A},d_b)$ *be a complete* C^**-algebras-valued b-metric space.Assume that the mapping* $T: X \to X$ *satisfies the following contractive condition:*

$$d_b(T(x,y),T(u,v)) \leq \begin{cases} \alpha\frac{d_b(x,u)+d_b(y,v)}{2}+\beta\frac{d_b(x,T(x,y))d_b(u,T(u,v))}{b[d_b(x,T(u,v))+d_b(u,T(x,y))+d_b(x,u)+d_b(y,v)]}, & if D \neq 0 \\ 0, & if D = 0 \end{cases}$$

for all $x,y,u,v \in X$*, where* $D=D(x,y,u,v)=b[d_b(x,T(u,v))+d_b(u,T(x,y))+d_b(x,u)+d_b(y,v)]$*, and* $\alpha,\beta \geq 0$*, with* $\|b\|\|\alpha+\beta\|<1$*, then* S *and* F *have a unique common coupled fixed point in X.*

Proof: Take $S=T=F$ *In Theorem 3.1, we can obtain the conclusion.*

Competing Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgment

This research was supported by the Natural Science Foundation of Shandong Province of China (ZR2017LA001) and Youth Foundation of Linyi University (LYDX2016BS023).

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