Available online at http://scik.org
Adv. Fixed Point Theory, 9 (2019), No. 3, 260-272
https://doi.org/10.28919/afpt/4091
ISSN: 1927-6303

FIXED POINT THEOREM BASED SOLVABILITY OF FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATION WITH FOUR-POINT BOUNDARY VALUE CONDITIONS

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Abstract: In this paper, an existence criterion of nontrivial solution to the four-point boundary value problem for fourth order ordinary differential equation has been established. The analysis of this paper is based on the well-known Krasnosclskii-Zabreiko fixed point theorem. An illustrative example is included to support the analytic proof of established existence criteria.

Keywords: four-point boundary value problem; Krasnosclskii-Zabreiko fixed point theorem; existence of nontrivial Solution.

2010 AMS Subject Classification: 34B09, 34B10, 34B15.

1. INTRODUCTION

Boundary value problems (BVPs for short) for fourth order ordinary differential equations (ODEs for short) are used to describe a huge number of physical, biological and chemical phenomena, see for instance [1-4] and references therein. If we put $f(t,u(t)) = p(t)g(u(t))$ in our considered

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Received April 11, 2019

boundary value problem, then it will be refer as the beam equation and further physical interpretation of that equation can be found in the work of Zill and Cullen [4, pp. 237-243]. It is well established that the fixed-point technique is the most important technique for checking the existence of solutions of ordinary differential equations. In the last few decades, two-point, three-point and four-point boundary value problems for second order, three order, fourth order as well as higher order has extensively been studied by using various techniques, see for instance [5-18] and references therein. Inspiring by the above mentioned works, we have interested to check the existence of solutions of four-point boundary value problem (BVP for short) for fourth-order nonlinear ordinary differential equations (FONLODEs for short) by applying Krasnosclskii-Zabreiko fixed point theorem and from this context, here we only described the most recent analogous literature about the existence of solutions of four-point BVP for FONLODEs.

In 2006, Chen *et al.* [5] checked the existence of solutions of following four-point BVP for FONLODEs by applying the upper and lower solution method and Schauder fixed point theorem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \qquad t \in (0,1) \\ u(0) = u(1) = 0, \\ au''(\xi_1) - bu'''(\xi_1) = 0,\ cu''(\xi_2) + du'''(\xi_2) = 0, \end{cases} \tag{1.1}$$

where, a, b, c, d are nonnegative constants satisfying, $ad + bc + ac(\xi_2 - \xi_1) > 0$, $0 \le \xi_1 < \xi_2 \le 1$ and $f \in C([0,1] \times \mathbb{R})$. They established their main result on basis the following lemma:

Lemma 1.1 (See [5], Lemma 2.2). *Suppose* a, b, c, d, ξ_1, ξ_2 *are nonnegative constants satisfying* $0 \le \xi_1 < \xi_2 \le 1$, $b - a\xi_1 \ge 0$, $d - c + c\xi_2 \ge 0$ *and*

$$\delta = ad + bc + ac(\xi_2 - \xi_1) \ne 0.$$

If $u(t) \in C^4[0,1]$ *satisfies*

$$u^{(4)}(t) \ge 0,\ \ t \in (0,1),$$

$$u(0) \ge 0,\ \ u(1) \ge 0,$$

$$au''(\xi_1) - bu'''(\xi_1) \le 0,\ \ cu''(\xi_2) + du'''(\xi_2) \le 0,$$

then $u(t) \ge 0$ *and* $u''(t) \le 0$ *for* $t \in [0,1]$.

Unfortunately, this lemma is incorrect. Now we provide a counter example to demonstrate it.

Counter example to [5, Lemma 2.2]. Let $u(t) = \frac{1}{6}t^4 - \frac{1}{3}t^3 + \frac{15}{196}t^2 + \frac{7}{64}$, $\xi_1 = \frac{5}{8}, \xi_2 = \frac{8}{15}$ and

a, b, c, d are four positive constants such that $a = b$, and $c = d$. Then we have

$$u^{(4)}(t) \geq 0, \quad t \in (0,1)$$

$$u(0) = \frac{7}{64} \geq 0, \quad u(1) = \frac{181}{9408} \geq 0,$$

$$au''(\xi_1) - bu'''(\xi_1) = -a\frac{1279}{1568} \leq 0 \text{ and } cu''(\xi_2) + du'''(\xi_2) = -c\frac{4661}{22050} \leq 0.$$

But

$$u''\left(\frac{1}{16}\right) = \frac{225}{6272} > 0,$$

which means that Lemma 2.2 of [5] is not correct.

Therefore, the results of Chen *et al.* [5] should be reassessed.

From this ground here we considered the fourth order four-point BVP defined by (1.1) and check the existence of solutions of that BVP by applying Krasnosclskii-Zabreiko fixed point theorem [19] instead of upper and lower solution method. The rest of this paper is organized as follows:
The Section 2, is used to provide some necessary definitions, lemmas and Krasnosclskii-Zabreiko fixed point theorem [19] associated with BVP (1.1) and (1.2). In Section 3, the main results have been stated and proved. Finally, we give an example to illustrate our main results.

2. PRELIMINARIES

In this section, we establish some lemmas and state Krasnosclskii-Zabreiko fixed point theorem which are used as tools to proof of our main results.

Lemma 2.1 Assume a, b, c, d, ξ_1, ξ_2 *are nonnegative constants satisfying* $0 \leq \xi_1 < \xi_2 \leq 1$, *and* $\delta = ad + bc + ac(\xi_2 - \xi_1) \neq 0$. *If* $h(t) \in C[\xi_1, \xi_2]$, *then the BVP*

$$\begin{cases} u''(t) = h(t), \quad t \in [\xi_1, \xi_2], \\ au(\xi_1) - bu'(\xi_1) = 0, \; cu(\xi_2) + du'(\xi_2) = 0, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = -\int_{\xi_1}^{\xi_2} G(t,s)h(s)ds,$$

where, $G(t,s) = \frac{1}{\delta}\begin{cases} (a(t-\xi_1)+b)\big(d + c(\xi_2 - s)\big), & \xi_1 \leq t < s \leq \xi_2, \\ (a(s-\xi_1)+b)\big(d + c(\xi_2 - t)\big), & \xi_1 \leq s \leq t \leq \xi_2 \end{cases}$

is the Green's function of the BVP

$$\begin{cases} u''(t) = 0, \quad t \in [0,1], \\ au(\xi_1) - bu'(\xi_1) = 0, \; cu(\xi_2) + du'(\xi_2) = 0, \end{cases} \tag{2.2}$$

Proof. Here first we solve the BVP (2.2) by using Green's function.

The general solution of (2.2) is

$$u(t) = At + B \tag{2.3}$$

Using the boundary conditions of (2.2), we obtain $A = B = 0$. Hence (2.3) yields only trivial solution $u(t) = 0$. Therefore, the unique Green's function exists for BVP (2.2) and is given by

$$G(t,s) = \begin{cases} a_1 t + a_2, & \xi_1 \leq t < s \leq \xi_2 \\ b_1 t + b_2, & \xi_1 \leq s \leq t \leq \xi_2 \end{cases} \tag{2.4}$$

Now, by the properties of Green's function, we have

$$(b_1 - a_1)s + (b_2 - a_2) = 0 \tag{2.5}$$

$$b_1 - a_1 = -1 \Rightarrow b_1 = a_1 - 1 \tag{2.6}$$

$$aG(\xi_1, s) - bG'(\xi_1, s) = 0 \Rightarrow (a\xi_1 - b)a_1 + aa_2 = 0 \tag{2.7}$$

$$cG(\xi_2, s) + dG'(\xi_2, s) = 0 \Rightarrow (c\xi_2 + d)b_1 + cb_2 = 0 \tag{2.8}$$

Solving (2.5), (2.6), (2.7) and (2.8), we obtain

$a_1 = \frac{ac(\xi_2 - s) + ad}{\delta}, a_2 = -\frac{(a\xi_1 - b)(c(\xi_2 - s) + d)}{\delta}, b_1 = \frac{ac(\xi_1 - s) - bc}{\delta}$ and

$b_2 = -\frac{(c\xi_2 + d)(a(\xi_1 - s) - b)}{\delta}$,

where, $\delta = ad + bc + ac(\xi_2 - \xi_1)$.

Putting the values of $a_1, a_2, b_1,$ and b_2 in (2.4), we obtain the unique Green's function

$$G(t,s) = \frac{1}{\delta}\begin{cases} (a(t - \xi_1) + b)\big(d + c(\xi_2 - s)\big), & \xi_1 \leq t < s \leq \xi_2, \\ (a(s - \xi_1) + b)\big(d + c(\xi_2 - t)\big), & \xi_1 \leq s \leq t \leq \xi_2. \end{cases}$$

Therefore, the unique solution of BVP (2.2) is

$$u(t) = -\int_{\xi_1}^{\xi_2} G(t,s)ds,$$

where, $G(t,s) = \frac{1}{\delta}\begin{cases} (a(t - \xi_1) + b)\big(d + c(\xi_2 - s)\big), & \xi_1 \leq t < s \leq \xi_2, \\ (a(s - \xi_1) + b)\big(d + c(\xi_2 - t)\big), & \xi_1 \leq s \leq t \leq \xi_2, \end{cases}$ and this solution ensure that the BVP (2.1) has a unique solution and which is

$$u(t) = -\int_{\xi_1}^{\xi_2} G(t,s)h(s)ds.$$

This completes the lemma. ■

Remark 2.2 Considering

$$R(t) = \frac{1}{\delta}\big((a(t-\xi_1)+b)x_3 + (c(\xi_2 - t)+d)x_2\big), \tag{2.9}$$

$$G_1(t,s) = \begin{cases} t(1-s), & 0 \le t < s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \tag{2.10}$$

$$G_2(t,s) = \frac{1}{\delta}\begin{cases} (a(t-\xi_1)+b)\big(d + c(\xi_2 - s)\big), & \xi_1 \le t < s \le \xi_2, \\ (a(s-\xi_1)+b)\big(d + c(\xi_2 - t)\big), & \xi_1 \le s \le t \le \xi_2, \end{cases} \tag{2.11}$$

in [5, Lemma 2.2], Chen *et al.* claimed that

$$u(t) = tx_1 + (1-t)x_0 - \int_0^1 G_1(t,\xi)R(\xi)d\xi + \int_0^1 G_1(t,\xi)\int_{\xi_1}^{\xi_2} G_2(\xi,s)h(s)ds\,d\xi, \tag{2.12}$$

is the solution of the following BVP

$$\begin{cases} u^{(4)}(t) = h(t), & t \in (0,1) \\ u(0) = x_0, & u(1) = x_1, \\ au''(\xi_1) - bu'''(\xi_1) = x_2, & cu''(\xi_2) + du'''(\xi_2) = x_3, \end{cases}$$

where, $x_0 \ge 0, x_1 \ge 0, x_2 \le 0, x_3 \le 0, h(t) \in C[0,1]$ and $h(t) \ge 0$.

But, the solution defined by (2.12) is incorrect. Definitely, by our Lemma 2.1, (2.12) should be replaced as follows:

$$u(t) = tx_1 + (1-t)x_0 - \int_0^1 G_1(t,\xi)R(\xi)d\xi - \int_0^1 G_1(t,\theta)v(\theta)d\theta,$$

where,

$$v(\theta) = \int_{\xi_1}^{\theta}(\theta - s)h(s)ds + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}(a(\xi_1-\theta)-b)(c(\xi_2-s)+d)h(s)ds.$$

Remark 2.3. In [5, Theorem 3.1], the operator $A: C[0,1] \to C[0,1]$ is defined as

$$Au(t) = \int_0^1 G_1(t,\theta)\int_{\xi_1}^{\xi_2} G_2(\theta,s)f(s,u(s))ds\,d\theta,$$

where, $G_1(t,\theta)$ and $G_2(\theta,s)$ are as in Remark 2.2. But, according to our Lemma 2.1 and Remark 2.2, this definition is not accurate. So, according to our Lemma 2.1 and Remark 2.2, the operator $A: C[0,1] \to C[0,1]$ should be defined as follows:

$$Au(t) = \int_0^1 G_1(t,\theta)\int_{\xi_1}^{\theta}(s-\theta)f(s,u(s))dsd\theta,$$

$$+\frac{1}{\delta}\int_0^1 G_1(t,\theta)\int_{\xi_1}^{\xi_2}\big(b - a(\xi_1-\theta)\big)(c(\xi_2-s)+d)f(s,u(s))dsd\theta. \tag{2.13}$$

Now, we state the well-known Krasnosclskii-Zabreiko fixed point theorem [13], which will help to establish our main result.

Theorem 2.4 [18]**.** *Let B be a Banach space, and $T: B \to B$ be completely continuous operator.*

Assume that $L: B \to B$ *is a bounded linear operator such that 1 is not an eigenvalue of* L *and*

$$\lim_{\|x\|\to\infty} \frac{\|Tx - Lx\|}{\|x\|} = 0.$$

Then T *has a fixed point in* B.

3. Main results

In this section, we present and prove our main result and finally, justify it by a suitable example of fourth order nonlinear ordinary differential equation with four-point boundary conditions.

To establish our main result, we need the following assumptions:

(A_1) *Let* a, b, c, d, ξ_1, ξ_2 *are nonnegative constants satisfying* $0 \le \xi_1 < \xi_2 \le 1,\ b - a\xi_1 \ge 0,$ $d - c + c\xi_2 \ge 0$ *and* $\delta = ad + bc + ac(\xi_2 - \xi_1) \neq 0$.

(A_2) *Let* $f\big(t, u(t)\big) = p(t)g(u(t))$, *where* $g: \mathbb{R} \to \mathbb{R}$ *is continuous with* $\lim_{u\to\infty} \frac{g(u)}{u} = \kappa$, *and* $p \in C[0,1]$. *Furthermore, there exists some* $t_0 \in (0,1]$ *such that* $p(t_0)g(0) \neq 0$, *and there exists a continuous nonnegative function* $\omega: (0,1] \to \mathbb{R}^+$ *such that* $|p(s)| \le \omega(s)$ *for each* $s \in (0,1]$.

Theorem 3.1. *Suppose* $B = C^2[0,1]$ *and* $\|u\|_0 = max\{\|u\|,\ \|u'\|\}$, *where* $\|u\| = max_{0\le t\le 1}|u(t)|$ *and assume* (A_1) *and* (A_1). *Then the BVP (1.1) has at least one nontrivial solution* $u \in B$ *if* $|\kappa| < min\left\{\frac{1}{M_1}, \frac{1}{M_2}\right\}$, *where*

$$M_1 = \frac{1}{12}\Big[\int_0^{\xi_1} r^3(2-r)\omega(r)dr + \int_{\xi_1}^1 (1-r)^3(1+r)\omega(r)dr$$
$$+ \frac{2(b - a\xi_1) + a}{\delta}\int_{\xi_1}^{\xi_2}(c(\xi_2 - r) + d)\omega(r)dr\Big],$$

and

$$M_2 = \int_{\xi_1}^1 (1-s)\omega(s)ds + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}\big(b + a(1-\xi_1)\big)(c(\xi_2 - s) + d)\omega(s)ds.$$

Proof. We prove this theorem by using Krasnosclskii-Zabreiko fixed point theorem (Theorem 2.4). First we define an operator $T: B \to B$ according to (2.13) by

$$Tu(t) := \int_0^1 G_1(t,s)\int_{\xi_1}^{s}(r - s)p(r)g(u(r))drds$$

$$+\frac{1}{\delta}\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2}\big(b-a(\xi_1-s)\big)(c(\xi_2-r)+d)p(r)g(u(r))drds \tag{3.1}$$

where $G_1(t,s)$is as in (2.10). Then by our Lemma 2.1 and Remark 2.3, it is clear that the fixed points of the operator $T: B \to B$ are the solutions to the BVP (1.1). From the definition of the operator T it is also clear that this operator is a completely continuous operator (see for proof [20]).

Now, we consider an analogous BVP of BVP (1.1)

$$\begin{cases} u^{(4)}(t) = \kappa p(t)u(t), \qquad t \in (0,1) \\ u(0) = u(1) = 0, \\ au''(\xi_1) - bu'''(\xi_1) = 0,\ cu''(\xi_2) + du''(\xi_2) = 0, \end{cases} \tag{3.2}$$

and define an operator $L: B \to B$ by

$$Lu(t) := \int_0^1 G_1(t,s)\int_{\xi_1}^{s}(r-s)\kappa p(r)u(r)drds$$

$$+\frac{1}{\delta}\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2}\big(b-a(\xi_1-s)\big)(c(\xi_2-r)+d)\kappa p(r)u(r)drds \tag{3.3}$$

Clearly, L is a bounded linear operator and the fixed point of L is a solution of the BVP (3.2) and conversely.

Now, we prove that 1 is not an eigenvalue of L. In fact, if $\kappa = 0$, then the BVP (3.2) has no nontrivial solution. So, If we let $\kappa \neq 0$ and suppose the BVP (3.2) has a nontrivial solution $u \in B$ and $\|u\|_0 > 0$, then we have

$$|Lu(t)| \le \int_0^1 G_1(t,s)\int_{\xi_1}^{s}|(r-s)\kappa p(r)u(r)|drds$$

$$+\frac{1}{\delta}\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2}\Big|\big(b-a(\xi_1-s)\big)(c(\xi_2-r)+d)\kappa p(r)u(r)\Big|drds$$

$$\le \int_0^1 s(1-s)\int_{\xi_1}^{s}(s-r)|\kappa||p(r)||u(r)|drds$$

$$+\frac{1}{\delta}\int_0^1 s(1-s)\int_{\xi_1}^{\xi_2}\big(b-a(\xi_1-s)\big)(c(\xi_2-r)+d)|\kappa||p(r)||u(r)|drds$$

$$\le \Big[\int_0^1 s(1-s)\int_{\xi_1}^{s}(s-r)|\kappa||p(r)|drds$$

$$+\frac{1}{\delta}\int_0^1 s(1-s)\int_{\xi_1}^{\xi_2}\big(b-a(\xi_1-s)\big)(c(\xi_2-r)+d)|\kappa||p(r)|drds\Big]\|u\|_0$$

$$=\frac{1}{12}\Big[\int_0^{\xi_1} r^3(2-r)|\kappa||p(r)|dr + \int_{\xi_1}^{1}(1-r)^3(1+r)|\kappa||p(r)|dr$$

$$+\frac{2(b-a\xi_1)+a}{\delta}\int_{\xi_1}^{\xi_2}(c(\xi_2-r)+d)|\kappa||p(r)|dr\Big]\|u\|_0$$

$$\leq |\kappa| \frac{1}{12}\left[\int_0^{\xi_1} r^3(2-r)\omega(r)dr + \int_{\xi_1}^1 (1-r)^3(1+r)\omega(r)dr\right.$$
$$\left. + \frac{2(b-a\xi_1)+a}{\delta}\int_{\xi_1}^{\xi_2}(c(\xi_2-r)+d)\omega(r)dr\right]\|u\|_0$$
$$\leq |\kappa| M_1\|u\|_0 < \frac{1}{M_1}M_1\|u\|_0 = \|u\|_0.$$

That is,

$$|Lu(t)| < \|u\|_0. \tag{3.4}$$

On the other hand, we have

$$\left|(Lu)'(t)\right| \leq \left|\int_{\xi_1}^{t}(s-t)\kappa p(s)u(s)ds\right.$$
$$\left. + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}\big(b-a(\xi_1-t)\big)(c(\xi_2-s)+d)\kappa p(s)u(s)ds\right|$$
$$\leq \left[\int_{\xi_1}^{1}(1-s)|\kappa||p(s)|ds\right.$$
$$\left. + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}\big(b+a(1-\xi_1)\big)(c(\xi_2-s)+d)|\kappa||p(s)|ds\right]\|u\|_0$$
$$\leq |\kappa|\left[\int_{\xi_1}^{1}(1-s)\omega(s)dr\right.$$
$$\left. + \frac{1}{\delta}\int_{\xi_1}^{\xi_2}\big(b+a(1-\xi_1)\big)(c(\xi_2-s)+d)\omega(s)ds\right]\|u\|_0$$
$$= |\kappa|M_2\|u\|_0 < \frac{1}{M_2}M_2\|u\|_0 = \|u\|_0.$$

That is,

$$|(Lu)'(t)| < \|u\|_0. \tag{3.5}$$

Therefore, according to the definition of $\|u\|_0$ and the inequalities (3.4) and (3.5), we have $\|Lu\|_0 < \|u\|_0$. That is, u is not a fixed point of L. This is a contradiction and this contradiction indicates that the BVP (3.2) has no nontrivial solution. Hence, 1 is not an eigenvalue of L.

Now, we prove that

$$\lim_{\|u\|_0\to\infty}\frac{\|Tu-Lu\|_0}{\|u\|_0} = 0.$$

Since, $\lim_{u\to\infty}\frac{g(u)}{u} = \kappa$, hence for any $\varepsilon > 0$, there must have an $D > 0$ such that

$$|g(u)-\kappa u| < \varepsilon|u| \text{ for all } |u| > D.$$

Put $D^* = max_{|u|\leq D}|g(u)|$ and choose $N > 0$ such that $(D^* + |\kappa|D) < \varepsilon N$.

Now, if we denote

$$F_1 = \{t \in [0,1]: |u(t)| \le D\} \text{ and } F_2 = \{t \in [0,1]: |u(t)| > D\},$$

then, for any $u \in C^2[0,1]$ with $\|u\|_0 > N$ and $t \in F_1$, we have

$$|g(u(t)) - \kappa u(t)| \le |g(u(t))| + |\kappa||u(t)| \le D^* + |\kappa| D < \varepsilon N < \varepsilon \|u\|_0.$$

Similarly, for any $u \in C^2[0,1]$ with $\|u\|_0 > N$ and $t \in F_2$, we have

$$|g(u(t)) - \kappa u(t)| < \varepsilon \|u\|_0.$$

From (3.1) and (3.3), we get

$$\begin{aligned}
&|Tu(t) - Lu(t)| \\
&= \Big| \int_0^1 G_1(t,s) \int_{\xi_1}^{s} (r-s) p(r) \big[g(u(r)) - \kappa u(r)\big] dr ds \\
&\qquad + \frac{1}{\delta} \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} \big(b - a(\xi_1 - s)\big)\big(c(\xi_2 - r) + d\big) p(r) \big[g(u(r)) - \kappa u(r)\big] dr ds \Big| \\
&\le \int_0^1 G_1(s,s) \int_{\xi_1}^{s} (s-r)|p(r)| \big|g(u(r)) - \kappa u(r)\big| dr ds \\
&\qquad + \frac{1}{\delta} \int_0^1 G_1(s,s) \int_{\xi_1}^{\xi_2} \big(b - a(\xi_1 - s)\big)\big(c(\xi_2 - r) + d\big) |p(r)| \big|g(u(r)) - \kappa u(r)\big| dr ds \\
&\le \Big[\int_0^1 G_1(s,s) \int_{\xi_1}^{s} (s-r)|p(r)| dr ds \\
&\qquad + \frac{1}{\delta} \int_0^1 G_1(s,s) \int_{\xi_1}^{\xi_2} \big(b - a(\xi_1 - s)\big)\big(c(\xi_2 - r) + d\big) |p(r)| dr ds \Big] \varepsilon \|u\|_0 \\
&= \Big[\int_0^1 s(1-s) \int_{\xi_1}^{s} (s-r)|p(r)| dr ds \\
&\qquad + \frac{1}{\delta} \int_0^1 s(1-s) \int_{\xi_1}^{\xi_2} \big(b - a(\xi_1 - s)\big)\big(c(\xi_2 - r) + d\big) |p(r)| dr ds \Big] \varepsilon \|u\|_0 \\
&\le \frac{1}{12} \Big[\int_0^{\xi_1} r^3 (2-r) \omega(r) dr + \int_{\xi_1}^{1} (1-r)^3 (1+r) \omega(r) dr \\
&\qquad + \frac{2(b - a\xi_1) + a}{\delta} \int_{\xi_1}^{\xi_2} \big(c(\xi_2 - r) + d\big) \omega(r) dr \Big] \varepsilon \|u\|_0 \\
&= \varepsilon M_1 \|u\|_0 .
\end{aligned}$$

That is,

$$|Tu(t) - Lu(t)| \le \varepsilon M_1 \|u\|_0 \quad . \tag{3.6}$$

On the other hand, we have

$$
\begin{aligned}
&|(Tu-Lu)'(t)| \\
&\quad = \left| \int_{\xi_1}^{t} (s-t)p(s)\big[g\big(u(s)\big)-\kappa u(s)\big]ds \right. \\
&\qquad \left. + \tfrac{1}{\delta}\int_{\xi_1}^{\xi_2}\big(b-a(\xi_1-t)\big)(c(\xi_2-s)+d)p(s)\big[g\big(u(s)\big)-\kappa u(s)\big]ds \right| \\
&\quad \le \int_{\xi_1}^{t} (s-t)|p(s)|\big|g\big(u(s)\big)-\kappa u(s)\big|ds \\
&\qquad + \tfrac{1}{\delta}\int_{\xi_1}^{\xi_2}\big(b-a(\xi_1-t)\big)(c(\xi_2-s)+d)|p(s)|\big|g\big(u(s)\big)-\kappa u(s)\big|ds \\
&\quad \le \left[\int_{\xi_1}^{1} (1-s)\omega(s)ds \right. \\
&\quad \left. + \tfrac{1}{\delta}\int_{\xi_1}^{\xi_2}\big(b+a(1-\xi_1)\big)(c(\xi_2-s)+d)\omega(s)ds \right] \varepsilon\|u\|_0 \\
&\quad = \varepsilon M_2\|u\|_0.
\end{aligned}
$$

That is,

$$
|(Tu-Lu)'(t)| \le \varepsilon M_2\|u\|_0. \tag{3.7}
$$

Combining the inequalities (3.6) and (3.7), we get

$$
\lim_{\|u\|_0\to\infty} \frac{\|Tu-Lu\|_0}{\|u\|_0} = 0.
$$

Hence, Krasnosclskii-Zabreiko fixed point theorem (Theorem 2.4) is satisfied and which assured that the BVP (1.1) has a solution $u \in B$.

Now, if we take $u=0$, then $(0)^{(4)} = p(t_0)g(0) = 0$, for some $t_0 \in (0,1]$, which leads a contradiction in our assumption (A_2). Therefore, $u \in B$ is a nontrivial solution of the BVP (1.1). This completes the proof. ■

Now we give an example to justify the Theorem 3.1.

Example 3.2. Consider a nonlinear fourth order four-point BVP as follows:

$$
\begin{cases}
u^{(4)}(t) = t\sin 2\pi t \cos u(t), \quad t \in (0,1) \\
u(0) = u(1) = 0, \\
u''\left(\frac{1}{3}\right) - u'''\left(\frac{1}{3}\right) = 0, \qquad u''\left(\frac{2}{3}\right) + u'''\left(\frac{2}{3}\right) = 0.
\end{cases} \tag{3.8}
$$

For proving that the BVP (3.8) has at least one nontrivial solution, we apply our Theorem 3.1 with

$$p(t) = t.sin2\pi t, g(u) = cosu, a = b = c = d = 1, \xi_1 = \frac{1}{3}, \text{and } \xi_2 = \frac{2}{3}.$$

Clearly, assumption (A_1) is satisfied with $\delta = \frac{7}{3} \neq 0$.

Now, $p(t_0)g(0) = t_0\, sin2\pi t_0. \cos 0 \neq 0$ for some $t_0 \in (0, 1]$, and $|p(s)| = |s.sin2\pi s| < |s.1| < s = \omega(s)$ for each $s \in (0, 1]$, we have

$$M_1 = \frac{1}{12}\left[\int_0^{\frac{1}{3}} r^3(2 - r)r\, dr + \int_{\frac{1}{3}}^{1}(1 - r)^3(1 + r)rdr + \int_{\frac{1}{3}}^{\frac{2}{3}}\left(\left(\frac{2}{3} - r\right) + 1\right)rdr\right]$$

$$= \frac{319}{9720} = 0.03281893 < 1, \tag{3.9}$$

and

$$M_2 = \int_{\frac{1}{3}}^{1}(1 - s)sds + \frac{3}{7}\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{5}{3}\left(\frac{5}{3} - s\right)sds = \frac{295}{1134} = 0.260141093 < 1. \tag{3.10}$$

So, from (3.9) and (3.10), we obtain

$$M_1 < M_2 < 1, \text{ that is } \frac{1}{M_1} > \frac{1}{M_2} > 1.$$

Notice that

$$\kappa = \lim_{u\to\infty}\frac{g(u)}{u} = \lim_{u\to\infty}\frac{cosu}{u} = 0,$$

which prove that,

$$|\kappa| < 1 < min\left\{\frac{1}{M_1}, \frac{1}{M_2}\right\}.$$

Hence, assumption (A_2) is satisfied. Therefore, Theorem 3.1 assurances that the BVP (3.8) has at least one nontrivial solution $u \in C^2[0, 1]$.

4. CONCLUSION

In this work, we develop a new approach to check the existence of nontrivial solution to four-point boundary problem for fourth order nonlinear ordinary differential equation given by (1.1) using Krasnosclskii-Zabreiko fixed point theorem. As the considered fourth order four-point boundary value problem of this paper represents a beam equation, so we can conclude that the Theorem 3.1 will play a vital role to check the existence of nontrivial solution of this type of beam equations. A justifying example also discussed here.

Conflict of Interests

The authors declare that there is no conflict of interests.

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