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COMMON FIXED POINT THEOREMS FOR GENERALIZED $\mathscr{F}$-GERAGHTY TYPE CONTRACTION IN PARTIAL b-METRIC-LIKE SPACES

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Abstract. In this paper, we introduce the notion of a pair of new generalized $\mathscr{F}$-Geraghty type contraction mappings and establish some new common fixed point theorems for such contraction in complete partial b-metric-like spaces. Examples are included to illustrate that our results are proper generalizations of previous results. We also discuss an application to the existence of solution for a nonlinear integral equation.

Keywords: fixed point; b-metric-like spaces; partial b-metric-like spaces; $\mathscr{F}$-Geraghty type contraction mapping.

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1. Introduction and Preliminaries

In 1973, Geraghty [7] extended and generalized the Banach contraction principle [1], and established the existence and uniqueness some new fixed points in the setting of complete metric spaces. Later, Geraghty contraction was improved and generalized in different spaces see [2, 4,

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10, 11, 12, 13, 14, 15]. On the other hand, the b-metric concept was launched by Bakhtin [3] as a generalization of a metric, and many articles have been dedicated to generalize the Geraghty contraction in both spaces. In 2017, Ameer et al [23]. introduce the notion of generalized $\alpha_* - \psi$-Geraghty contraction for multivalued mappings and establish common fixed point theorems for such mappings in an α-complete b-metric spaces, which were recently improved by Aleksić et al [24]. In 1994, Matthews [16], introduced the concepts of partial metric spaces wherein the distance of a point from itself may not be zero and obtained related fixed point theorems. Shukla [17] generalized the concept of partial b-metric space by combining the b-metric and partial metric spaces. after that Many authors obtained interesting generalized results of the Geraghty contraction in both spaces see [19, 20, 21, 22]. Alghamdi et al. [6] generalized the notion of a b-metric space by introduction of the concept of a b-metric-like space and proved some related fixed point results. Recently, Rao et al. [18] introduced the concept of partial b-metric-like by combining the b-metric-like and partial metric spaces and established new coupled coincidence point theorems.

Definition 1.1 [3]. A b-metric on a non empty set X is a function $d : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$ and $s \geq 1$, the following three conditions are satisfied:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$. As usual, the pair (X,d) is called a b-metric space.

Definition 1.2 [6]. A b-metric-like on a non empty set X is a function $d : X \times X \to [0,\infty)$ such that for all $x,y,z \in X$ and a constant $s \geq 1$, the following three conditions are satisfied:

(i) $d(x,y) = 0$ implies $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$. Then the pair (X,d) is called a b-metric-like space.

Definition 1.3 [16]**.** Let X be a non-empty set and $p : X \times X \to [0,\infty)$ be a function. Then p is called a partial metric on X, if for all $x,y,z \in X$;

(1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y) = 0$;

(2) $p(x,x) \leq p(x,y)$;

(3) $p(x,y) = p(y,x)$;

(4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$. Then the pair (X,p) is said to be a partial metric space.

Definition 1.4 [17]. Let X be a non-empty set and $p : X \times X \to [0,\infty)$ be a function, called a partial b-metric if there exists a real number $s \geq 1$ such that the following conditions hold for every $x,y,z \in X$,

(1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y) = 0$;

(2) $p(x,x) \leq p(x,y)$;

(3) $p(x,y) = p(y,x)$;

(4) $p(x,y) \leq s[p(x,z) + p(z,y)] - p(z,z)$. Then the pair (X,p) is said to be a partial b-metric space.

Definition 1.5 [18]**.** A partial b-metric-like on a non empty set X is a function $p : X \times X \to [0,\infty)$, wherein for all $x,y,z \in X$ and a constant $s \geq 1$, the following conditions are satisfied:

(1) $p(x,y) = 0$ implies $x = y$,

(2) $p(x,x) \leq p(x,y)$, $p(y,y) \leq p(x,y)$,

(3) $p(x,y) = p(y,x)$,

(4) $p(x,y) \leq s[p(x,z) + p(z,y) - p(z,z)]$. The pair (X,p) is called a partial b-metric-like space.

Definition 1.6 [18]. Let (X,p) be a partial b-metric-like space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Remark 1.7 [18]. In a partial b-metric-like space, the limit for a convergent sequence is not unique in general.

Definition 1.8 [18]. Let (X,p) be a partial b-metric-like space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

Definition 1.9 [18]. Let (X,p) be a partial b-metric-like space. We say that (X,p) is complete if and only if each Cauchy sequence in X converges to $x \in X$ so that

$$\lim_{n\to\infty} p(x_n,x) = p(x,x) = \lim_{m,n\to\infty} p(x_m,x_n).$$

Proposition 1.10 [18]. Let (X,p) be a partial b-metric-like space with constant $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n,x) = 0$. Then:

(1) x is unique.

(2) $\frac{1}{s}p(x,y) \leq \lim_{n\to\infty} p(x_n,y) \leq sp(x,y)$ for all $y \in X$.

(3) $p(x_n,x_0) \leq sp(x_0,x_1) + s^2p(x_1,x_2) + \ldots + s^{n-1}p(x_{n-2},x_{n-1}) + s^np(x_{n-1},x_n)$, whenever

$\{x_s\}_{s=0}^{n} \in X$.

Definition 1.11 [8]. Let $\alpha : X \times X \to [0,\infty)$ be a functional. A mapping $T : X \to X$ is said to be α-admissible, if for all $x, y \in X$, $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$.

Definition 1.12 [5]. Let $S,T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that the pair (S,T) is α-admissible if $x,y \in X$ such that $\alpha(x,y) \geq 1$, then we have $\alpha(Sx,Ty) \geq 1$ and $\alpha(Tx,Sy) \geq 1$.

Definition 1.13 [4]**.** Let X be a non-empty set, $T : X \to X$ and $\alpha, \beta : X \times X \longrightarrow \mathbb{R}^{+}$. We say that T is an (α,β)-admissible mapping, if

$$\alpha(x,y) \geq 1 \text{ and } \beta(x,y) \geq 1$$

implies

$$\alpha(Tx,Ty) \geq 1, \text{ and } \beta(Tx,Ty) \geq 1, \text{ for all } x,y \in X.$$

Theorem 1.14 [13]. Let (X,d) be a complete metric-like space and $T : X \to X$ be a mapping such that for all $x,y \in X$

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y),$$

where $\beta \in \xi$ and ξ is the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition $\beta(t_n) \to 1$ implies $t_n \to 0$ as $n \to \infty$. Then T has a unique fixed point $x^{*} \in X$ with $d(x^{*},x^{*}) = 0$.
In 2017 Aydi et al. [10] considered a new Geraghty type contraction in the complete metric-like space given as following

Theorem 1.15 [10]. Let (X,d) be a complete metric-like space and $T : X \to X$ be a given mapping. Suppose there exists$\beta \in \xi$ such that for all $x,y \in X$

$$d(Tx,Ty) \leq \beta(F(x,y))F(x,y),$$

where

$$F(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty)|.$$

Then T has a unique fixed point $u \in X$ with $d(u,u) = 0$.

2. GENERALIZED $\mathscr{F}$-GERAGHTY TYPE CONTRACTION MAPPINGS

In this section we give a real generalization of the results obtained in [10] we prove some new common fixed point theorems for new generalized $\mathscr{F}$-Geraghty type contraction in partial b-metric-like space using conditions of (α,φ)-admissible mappings, which generalize and extend Theorem 2.1 of Aydi et al. [10].

Definition 2.1. Let X be a non-empty set, $S,T : X \to X$ and $\alpha,\beta : X\times X \longrightarrow [0,\infty)$. The two mappings (S,T) is called a pair of (α,β)-admissible mappings, if

$$\alpha(x,y)\geq 1 \text{ and } \beta(x,y)\geq 1$$

implies

$$\alpha(Sx,Ty)\geq 1,\ \alpha(Tx,Sy)\geq 1 \text{ and } \beta(Sx,Ty)\geq 1,\ \beta(Tx,Sy)\geq 1$$

for all $x,y\in X$.

Definition 2.2. Let (X,p) be a partial b-metric-like space, $S,T : X\to X$ be two mappings. Suppose there exist functions $\beta\in\xi$ and $\alpha,\varphi : X\times X\to[0,\infty)$. Then (S,T) is said to be a pair of new generalized $\mathscr{F}$-Geraghty type contraction mappings, if for all $x,y\in X$, we have

$$\alpha(x,Sx)\,\varphi(y,Ty)\,s^3p(Sx,Ty)\leq\beta(\mathscr{F}(x,y))\,\mathscr{F}(x,y), \tag{1}$$

where

$$\mathscr{F}(x,y)=p(x,y)+|p(x,Sx)-p(y,Ty)|.$$

Theorem 2.3. Let (X,p) be a complete partial b-metric-like space and $S,T : X\to X$ be a pair of mappings. Suppose there exist functions $\beta\in\xi$ and $\alpha,\varphi : X\times X\to[0,\infty)$ such that the following conditions hold:

(i) (S,T) is a pair of new generalized $\mathscr{F}$-Geraghty type contraction mappings;

(ii) (S,T) is a pair of (α,φ)-admissible mappings;

(iii) there exists $x_0\in X$ such that $\alpha(x_0,Sx_0)\geq 1$ and $\varphi(x_0,Tx_0)\geq 1$;

(iv) for every sequence $\{x_n\}$ in X such that $\alpha(x_n,x_{n+1})\geq 1$ and $\varphi(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to x$, we have $\alpha(x,Sx)\geq 1$ and $\varphi(x,Tx)\geq 1$.

Then the pair (S,T) has a unique common fixed point $u\in X$.

Proof. Let $x_0\in A_0$. Since $Tx_0\in T(A_0)\subseteq B_0$, there exists x_1 in A_0 such that $\rho(x_1-Tx_0)=d_\rho(A,B)$. Moreover, $Tx_1\in T(A_0)\subseteq B_0$ implies the existence of an $x_2\in A_0$ such that $\rho(x_2-Tx_1)=d_\rho(A,B)$. Continuing in this way, we obtain a sequence $\{x_n\}$ in A_0 such that

$$\rho(x_{n+1}-Tx_n)=d_\rho(A,B)\text{, for all } n\in\mathbb{N}. \tag{1}$$

Since pair (A,B) has the $P-$property, from (10) we have,

$$\rho(x_n-x_{n+1})=\rho(Tx_{n-1}-Tx_n)\text{, for all } n\in\mathbb{N}. \tag{2}$$

We now prove that the sequence $\{x_n\}$ is ρ-convergent in A_0.If there exists $n_0\in\mathbb{N}$ such that $\rho(Tx_{n_0-1}-Tx_{n_0})=0$, then $\rho(x_{n_0}-x_{n_0+1})=0\Leftrightarrow x_{n_0}-x_{n_0+1}=0\Leftrightarrow x_{n_0}=x_{n_0+1}$ by (11). Thus

$$Tx_{n_0}=Tx_{n_0+1}\Leftrightarrow Tx_{n_0}-Tx_{n_0+1}=0\Leftrightarrow\rho(Tx_{n_0}-Tx_{n_0+1})=0. \tag{3}$$

From (11) and (12), we obtain

$$\rho(x_{n_0+2}-x_{n_0+1})=\rho(Tx_{n_0+1}-Tx_{n_0})=0\Rightarrow x_{n_0+2}=x_{n_0+1}.$$

Thus $x_n=x_{n_0}$ for all $n\geq n_0$ and hence $\{x_n\}$ is ρ-convergent in A_0.

Next let $\rho(Tx_{n-1}-Tx_n)\neq 0$ for all $n\in\mathbb{N}$. Then, for any positive integer n, using (11), we have

$$\tau+F(\rho(c(Tx_n-Tx_{n-1})))\leq F(\rho(l(x_n-x_{n-1}))).$$

because T is an F_ρ-contraction and this implies that

$$\begin{aligned}
F(\rho(c\ (x_{n+1}-\ x_n))) &\leq F(\rho(l(x_n-x_{n-1})))-\tau\\
F(\rho(c\ (x_{n+1}-\ x_n))) &\leq F(\rho(c(x_n-x_{n-1})))-\tau\\
F(\rho(c(x_{n+1}-x_n))) &\leq F(\rho(l(x_{n-1}-x_{n-2})))-2\tau\\
&\leq F(\rho(c(x_{n-2}-x_{n-3})))-3\tau\leq\ldots\leq F(\rho(c(x_1-x_0)))-n\tau. \qquad (4)
\end{aligned}$$

Denote $\beta_n := (\rho(c(x_{n+1}-x_n)))$. Then by (13), $\lim_{n\to\infty} F(\beta_n) = -\infty$. Appealing to (C_2), we get

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \rho(x_{n+1}-x_n) = 0. \tag{5}$$

A use of (C_3) guarantees the existence of a $k \in (0,1)$ such that

$$\lim_{n\to\infty} \beta_n^k F(\beta_n) = 0, \tag{6}$$

and so by (13), for all $n \in \mathbb{N}$, we have

$$\beta_n^k (F(\beta_n) - F(\beta_0)) \leq -\beta_n^k n\tau \leq 0.$$

Reading (14) and (15) together, we get

$$\lim_{n\to\infty} n\beta_n^k = 0.$$

Hence there exists $n_1 \in \mathbb{N}$ such that $n\beta_n^k \leq 1$ for all $n \geq n_1$. That is, for all $n \geq n_1$,

$$\beta_n \leq \frac{1}{n^{\frac{1}{k}}}, \tag{7}$$

or

$$\rho(x_n - x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}. \tag{8}$$

Similarly, there exists $n_2 \in \mathbb{N}$ such that

$$\begin{aligned} \rho(x_n - x_{n+2}) &\leq \omega(2)[\rho(x_n - x_{n+1}) + \rho(x_{n+1} - x_{n+2})] \\ &\leq \omega(2)\left(\frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}}\right) \\ &\leq \frac{\omega(2)}{n^{\frac{1}{k}}}. \end{aligned}$$

This implies that

$$\rho(x_n - x_{n+2}) \leq \frac{\omega(2)}{n^{\frac{1}{k}}}. \tag{9}$$

Now we have the following two cases.

CASE 1: If $m>2$ is odd, then $m=2L+1$, $L\geq 1$, using (17) for all $n\geq h$, $h=\max(n_0,n_1)$

$$\begin{aligned}\rho(x_n-x_{n+m}) &\leq \omega(2L+1)\left[\rho(x_n-x_{n+1})+\rho(x_{n+1}-x_{n+2})+...+\rho(x_{n+2L}-x_{n+2L+1})\right]\\ &\leq \omega(2L+1)\left[\frac{1}{n^{\frac{1}{k}}}+\frac{1}{(n+1)^{\frac{1}{k}}}+...+\frac{1}{(n+2L)^{\frac{1}{k}}}\right]\\ &\leq \omega(2L+1)\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

CASE 2: If $m>2$ is even, then $m=2L$,$L\geq 2$, using (17) and (18) for all $n\geq h$, $h=\max(n_0,n_1)$

$$\begin{aligned}\rho(x_n-x_{n+m}) &\leq \omega(2L)\left[\rho(x_n-x_{n+2})+\rho(x_{n+2}-x_{n+3})+...+\rho(x_{n+2L-1}-x_{n+2L})\right]\\ &\leq \omega(2L)\left[\frac{1}{n^{\frac{1}{k}}}+\frac{1}{(n+2)^{\frac{1}{k}}}+...+\frac{1}{(n+2L-1)^{\frac{1}{k}}}\right]\\ &\leq \omega(2L)\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

Combining these two cases, we have

$$\rho(x_n-x_{n+m})\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}\text{ for all }n\geq h,m\in\mathbb{N}.$$

Since the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ is convergent $\left(\text{as }\frac{1}{k}>1\right)$, we deduce that $\{x_n\}$ is a Cauchy sequence. Now X_ρ is complete and A is a ρ-closed subset of X_ρ, there exists $x^*\in A$ such that $\lim_{n\to\infty}x_n=x^*$. Since T is ρ-continuous, Tx_n is ρ-convergent to Tx^*. Hence the continuity of the modular ρ implies that $\rho(x_{n+1}-Tx_n)$ $\rho-$converges to $\rho(x^*-Tx^*)$ and by (10) ,we have

$$\rho(x^*-Tx^*)=d_\rho(A,B).$$

That is, x^* is a best proximity point of T.

Next, we show the uniqueness of the best proximity point. Let us suppose that T has two best proximity points x_1 and $x_2\in A$, such that $x_1\neq x_2$ and $\rho(x_1-Tx_1)=\rho(x_2-Tx_2)=d_\rho(A,B)$. Then by the $P-$property of (A,B), we have $\rho(x_1-x_2)=\rho(Tx_1-Tx_2)$. Note that $\rho(x_1-x_2)>0$ as $x_1\neq x_2$, T is F_ρ-contraction and ρ is an increasing function, thus

$$\begin{aligned}F(\rho(c(x_1-x_2))) &= F(\rho(c(Tx_1-Tx_2)))\leq F(\rho(l(x_1-x_2)))-\tau\\ &\leq F(\rho(c(x_1-x_2)))-\tau<F(\rho(c(x_1-x_2))),\end{aligned}$$

which is a contradiction. Hence the best proximity point is unique. □

By assumption there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by letting $x_1 \in X$ such that $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$, $x_4 = Tx_3$, continue this process we get

$$x_{2i+1} = Sx_{2i} \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0, 1, 2, \ldots.$$

Since (S,T) is a pair of (α, φ)-admissible, so

$$\alpha(x_0, Sx_0) = \alpha(x_0, x_1) \geq 1,$$

$$\alpha(Sx_0, Tx_1) = \alpha(x_1, x_2) \geq 1 \text{ and } \alpha(Tx_1, Sx_2) = \alpha(x_2, x_3) \geq 1$$

continuing this manner, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \geq 0.$$

Similarly, we can get

$$\varphi(x_n, x_{n+1}) \geq 1 \text{ for all } n \geq 0.$$

From (1), we have

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &\leq \alpha(x_{2i}, Sx_{2i})\, \varphi(x_{2i+1}, Tx_{2i+1})\, s^3 p(Sx_{2i}, Tx_{2i+1}) \qquad (2)\\ &\leq \beta(\mathscr{F}(x_{2i}, x_{2i+1}))\, \mathscr{F}(x_{2i}, x_{2i+1}) \\ &< \mathscr{F}(x_{2i}, x_{2i+1}). \end{aligned}$$

Where

$$\begin{aligned} \mathscr{F}(x_{2i}, x_{2i+1}) &= p(x_{2i}, x_{2i+1}) + |p(x_{2i}, Sx_{2i}) - p(x_{2i+1}, Tx_{2i+1})| \qquad (3)\\ &= p(x_{2i}, x_{2i+1}) + |p(x_{2i}, x_{2i+1}) - p(x_{2i+1}, x_{2i+2})|. \end{aligned}$$

Assume that there exists $i > 0$ such that $p(x_{2i}, x_{2i+1}) \leq p(x_{2i+1}, x_{2i+2})$, then by definition of absolute value in (3), we obtain

$$\begin{aligned} \mathscr{F}(x_{2i}, x_{2i+1}) &= p(x_{2i}, x_{2i+1}) - [p(x_{2i}, x_{2i+1}) - p(x_{2i+1}, x_{2i+2})]. \\ &= p(x_{2i+1}, x_{2i+2}) \end{aligned}$$

by (2), we get

$$\begin{aligned} p(x_{2i+1},x_{2i+2}) &\leq \beta(p(x_{2i+1},x_{2i+2}))\,p(x_{2i+1},x_{2i+2}) \\ &< p(x_{2i+1},x_{2i+2}), \end{aligned}$$

implies that $p(x_{2i+1},x_{2i+2}) < p(x_{2i+1},x_{2i+2})$, which is a contradiction. Thus, for all $i > 0$, $p(x_{2i},x_{2i+1}) > p(x_{2i+1},x_{2i+2})$. Therefore (2) becomes

$$\begin{aligned} p(x_{2i+1},x_{2i+2}) &\leq \beta(2p(x_{2i},x_{2i+1}) - p(x_{2i+1},x_{2i+2})) \times \\ &\quad (2p(x_{2i},x_{2i+1}) - p(x_{2i+1},x_{2i+2})). \end{aligned} \tag{4}$$

Now from (4), we get

$$p(x_{2i+1},x_{2i+2}) < 2p(x_{2i},x_{2i+1}) - p(x_{2i+1},x_{2i+2}),$$

therefore, $2p(x_{2i+1},x_{2i+2}) < 2p(x_{2i},x_{2i+1})$ that is $p(x_{2i+1},x_{2i+2}) < p(x_{2i},x_{2i+1})$ implies that $p(x_{n+1},x_{n+2}) < p(x_n,x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, the sequence $\{p(x_n,x_{n+1})\}$ is decreasing and bounded from below, so there exists a real number r such that $\lim_{n\to\infty} p(x_n,x_{n+1}) = r$. Suppose that $r > 0$, we prove that $r = \lim_{n\to\infty} p(x_n,x_{n+1}) = 0$. Now for all $n \in \mathbb{N} \cup \{0\}$ using the formula (4) and taking limit as $n \to \infty$, we conclude

$$\lim_{n\to\infty} \frac{p(x_{n+1},x_{n+2})}{(2p(x_n,x_{n+1}) - p(x_{n+1},x_{n+2}))} \leq \lim_{n\to\infty} \beta(2p(x_n,x_{n+1}) - p(x_{n+1},x_{n+2})) \leq 1,$$

this implies

$$1 \leq \lim_{n\to\infty} \beta(2p(x_n,x_{n+1}) - p(x_{n+1},x_{n+2})) \leq 1,$$

that is

$$\lim_{n\to\infty} \beta(2p(x_n,x_{n+1}) - p(x_{n+1},x_{n+2})) = 1.$$

Since $\beta \in \xi$, we get

$$\lim_{n\to\infty} (2p(x_n,x_{n+1}) - p(x_{n+1},x_{n+2})) = 0,$$

which yields that

$$r = \lim_{n\to\infty} p(x_n,x_{n+1}) = 0,$$

which is a contradiction. Thus

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{5}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X. Suppose on the contrary that $\{x_n\}$ is not Cauchy sequence. Then there exists $\varepsilon > 0$ and the subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$p(x_{n_k}, x_{m_k}) \geq \varepsilon, \tag{6}$$

we choose n_k to be the smallest number such that (6) holds, then we have

$$p\left(x_{n_k-1}, x_{m_k}\right) < \varepsilon \tag{7}$$

By triangular inequality and using (6) and (7), we obtain

$$\begin{aligned} \varepsilon &\leq p(x_{n_k}, x_{m_k}) \\ &\leq s[p\left(x_{n_k}, x_{n_k-1}\right) + p\left(x_{n_k-1}, x_{m_k}\right) - p\left(x_{n_k-1}, x_{n_k-1}\right)] \\ &< s[p\left(x_{n_k}, x_{n_k-1}\right) + p\left(x_{n_k-1}, x_{m_k}\right)] \\ &< s[p\left(x_{n_k}, x_{n_k-1}\right) + \varepsilon]. \end{aligned} \tag{8}$$

Taking the limit as $k \to \infty$ in (8) and using (5), we get

$$\varepsilon \leq \lim_{k\to\infty} p(x_{n_k}, x_{m_k}) < s\varepsilon. \tag{9}$$

Again by triangular inequality

$$\begin{aligned} p(x_{n_k}, x_{m_k}) &\leq sp\left(x_{n_k}, x_{n_k+1}\right) + sp\left(x_{n_k+1}, x_{m_k}\right) - s\lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k+1}\right) \\ &\leq sp\left(x_{n_k}, x_{n_k+1}\right) + sp\left(x_{n_k+1}, x_{m_k}\right), \end{aligned} \tag{10}$$

and

$$\begin{aligned} p\left(x_{n_k+1}, x_{m_k}\right) &\leq sp\left(x_{n_k+1}, x_{n_k}\right) + sp(x_{n_k}, x_{m_k}) - sp(x_{n_k}, x_{n_k}) \\ &\leq sp\left(x_{n_k+1}, x_{n_k}\right) + sp(x_{n_k}, x_{m_k}). \end{aligned} \tag{11}$$

Taking the limit as $k \to \infty$ in (10) and applying (5) and (9),

$$\varepsilon \leq \lim_{k\to\infty} p(x_{n_k}, x_{m_k}) \leq s\lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k}\right).$$

Again by taking the limit as $k \to \infty$ in (11) and applying (5) and (9),

$$\lim_{k\to\infty} p\left(x_{n_{k+1}}, x_{m_k}\right) \leq s\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) \leq s.s\varepsilon = s^2\varepsilon.$$

Thus, we conclude

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k}\right) \leq s^2\varepsilon. \tag{12}$$

Similarly,

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} p\left(x_{n_k}, x_{m_k+1}\right) = \lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k+2}\right) \leq s^2\varepsilon. \tag{13}$$

Also by triangular inequality, we find

$$p\left(x_{n_k+1}, x_{m_k}\right) \leq s[p\left(x_{n_k+1}, x_{m_k+1}\right) + p\left(x_{m_k+1}, x_{m_k}\right)], \tag{14}$$

taking the limit as $k \to \infty$ in (14), from (5) and (12), we get

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k}\right) \leq s \lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k+1}\right),$$

implies that

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k+1}\right). \tag{15}$$

By triangular inequality again

$$p\left(x_{n_k+1}, x_{m_k+1}\right) \leq s[p\left(x_{n_k+1}, x_{n_k}\right) + p\left(x_{n_k}, x_{m_k+1}\right)],$$

taking the limit as $k \to \infty$ from (5) and (12), we obtain

$$\lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k+1}\right) \leq s^3\varepsilon, \tag{16}$$

from (15) and (16), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k+1}\right) \leq s^3\varepsilon \tag{17}$$

Since $\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k+1}\right)$ by multiplying by s^3 and from (1), we have

$$\begin{aligned} s\varepsilon &\leq s^3 \lim_{k\to\infty} p\left(x_{n_k+1}, x_{m_k+1}\right) = s^3 \lim_{k\to\infty} p\left(Sx_{n_k}, Tx_{m_k}\right) \\ &\leq \alpha\left(x_{n_k}, Sx_{n_k}\right)\varphi\left(x_{m_k}, Tx_{m_k}\right) s^3 \lim_{k\to\infty} p\left(Sx_{n_k}, Tx_{m_k}\right) \\ &\leq \lim_{k\to\infty} \beta\left(\mathscr{F}\left(x_{n_k}, x_{m_k}\right)\right)\mathscr{F}\left(x_{n_k}, x_{m_k}\right) \\ &< \lim_{k\to\infty} \mathscr{F}\left(x_{n_k}, x_{m_k}\right), \end{aligned} \tag{18}$$

where

$$\mathscr{F}(x_{n_k}, x_{m_k}) = p(x_{n_k}, x_{m_k}) + \left| p\left(x_{n_k}, x_{n_k+1}\right) - p\left(x_{m_k}, x_{m_k+1}\right) \right|.$$

By taking limit as $k \to \infty$ and from (5) and (9), we obtain

$$\varepsilon \leq \lim_{k\to\infty} \mathscr{F}(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} p(x_{n_k}, x_{m_k}) \leq s\varepsilon. \tag{19}$$

Now from (18) and (19), we conclude

$$\begin{aligned} s\varepsilon &\leq \lim_{k\to\infty} \beta\left(\mathscr{F}(x_{n_k}, x_{m_k})\right) \mathscr{F}(x_{n_k}, x_{m_k}) < \lim_{k\to\infty} \mathscr{F}(x_{n_k}, x_{m_k}) \\ &\leq \lim_{k\to\infty} \beta\left(\mathscr{F}(x_{n_k}, x_{m_k})\right) s\varepsilon \leq s\varepsilon. \end{aligned}$$

Hence, we deduce

$$1 \leq \lim_{k\to\infty} \beta\left(\mathscr{F}(x_{n_k}, x_{m_k})\right) \leq 1,$$

implies that

$$\lim_{k\to\infty} \beta\left(\mathscr{F}(x_{n_k}, x_{m_k})\right) = 1.$$

Since $\beta \in \xi$, so

$$\lim_{k\to\infty} \mathscr{F}(x_{n_k}, x_{m_k}) = 0,$$

which is a contradiction with respect to (19). Therefore $\{x_n\}$ is a Cauchy sequence in the complete partial b-metric-like space X. So there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, this implies $x_{2k+1} \to u$ and $x_{2k+2} \to u$ as $k \to \infty$. Now we show that $Sx = Tu = u$. From (2), we have

$$\begin{aligned} p(x_{2k+1}, Tu) &\leq \alpha(x_{2k}, Tx_{2k})\, \varphi(u, Tu)\, s^3 p(Sx_{2k}, Tu) \\ &\leq \beta\left(\mathscr{F}(x_{2k}, u)\right) \mathscr{F}(x_{2k}, u), \end{aligned} \tag{20}$$

where

$$\mathscr{F}(x_{2k}, u) = p(x_{2k}, u) + |p(x_{2k}, x_{2k+1}) - p(u, Tu)|.$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} \mathscr{F}(x_{2k}, u) = p(u, Tu). \tag{21}$$

Taking limit as $k \to \infty$ again in (20) gives

$$p(u,Tu) \leq \lim_{k\to\infty} \beta(\mathscr{F}(x_{2k},u))\, p(u,Tu) \leq p(u,Tu),$$

by dividing by $p(u,Tu)$, we deduce

$$\lim_{k\to\infty} \beta(\mathscr{F}(x_{2k},u)) = 1,$$

implies that

$$\lim_{k\to\infty} \mathscr{F}(x_{2k},u) = 0. \tag{22}$$

From (21) and (22), we get

$$p(u,Tu) = 0, \tag{23}$$

implies that $Tu = u$. Similarly, we can find that $Su = u$. Hence, the pair (S,T) has a common fixed point $u = Su = Tu$. Assume that u,w are two common fixed points of the pair (S,T) with $u \neq w$ such that $u = Su$, $w = Tw$ and $p(u,u) = p(w,w) = 0$. Then from (1), we have

$$\begin{aligned} p(u,w) &\leq \alpha(u,Su)\,\varphi(w,Tw)\, s^3 p(Su,Tw) \\ &\leq \beta(\mathscr{F}(u,w))\,\mathscr{F}(u,w) \\ &\leq \beta(p(u,w))\, p(u,w) \\ &< p(u,w), \end{aligned}$$

which is a contradiction. Therefore $u = w$ implies that the pair (S,T) has a unique common fixed point $u \in X$ such that $u = Su = Tu$ with $p(u,u) = 0$.

Corollary 2.4. Let (X,p) be a complete partial b-metric-like space and $T : X \longrightarrow X$ be a mapping such that for all $x,y \in X$ and the functions $\alpha,\varphi : X\times X \to [0,\infty)$, $\beta \in \xi$, then we have

$$\alpha(x,Tx)\,\varphi(y,Ty)\, s^3 p(Tx,Ty) \leq \beta(\mathscr{F}(x,y))\,\mathscr{F}(x,y),$$

where

$$\mathscr{F}(x,y) = p(x,y) + |p(x,Tx) - p(y,Ty)|.$$

Assume that

(A) T is (α,φ)-admissible,

(B) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\varphi(x_0, Tx_0) \geq 1$,

(C) for every sequence $\{x_n\}$ in X such that $\alpha(x_n, Tx_n) \geq 1$ and $\varphi(x_n, Tx_n) \geq 1 \ \forall\, n \in \mathbb{N} \cup \{0\}$ and x_n converges to x, then $\alpha(x, Tx) \geq 1$ and $\varphi(x, Tx) \geq 1$.

Then T has a unique fixed point $u \in X$ with $d(u,u) = 0$.

Corollary 2.5. [10] Let (X,d) be a metric-like space, $T : X \longrightarrow X$ be a mapping and $\beta \in \xi$ such that

$$d(Tx, Ty) \leq \beta(\mathscr{F}(x,y))\, \mathscr{F}(x,y),$$

for all $x, y \in X$, where

$$\mathscr{F}(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty)|.$$

Then T has a unique fixed point $u \in X$ with $d(u,u) = 0$.

Example 2.6. Let $X = [0,1]$. Define $p : X \times X \to [0,\infty)$ by $p(x,y) = \max\left\{x^2, y^2\right\}$ for all $x, y \in X$. Then (X,p) is complete partial b-metric-like space with constant $s = 2$. Let $S, T : X \to X$ be two mappings defined by

$$Sx = \begin{cases} \frac{x}{10}, & \text{if } x \in [0, \frac{1}{2}] \\ \frac{x}{\sqrt{2}}, & \text{otherwise,} \end{cases} \quad \text{and } Tx = \begin{cases} \frac{x}{10}, & \text{if } x \in [0, \frac{1}{2}] \\ 4x, & \text{otherwise} \end{cases}.$$

Let $\beta : [0,\infty) \to [0,1)$ be a function such that $\beta(t) = \frac{1}{2}$ and $\alpha, \varphi : X \times X \to [0,\infty)$ defined as

$$\alpha(x,y) = \begin{cases} e^{xy}, & \text{if } x, y \in [0, \frac{1}{2}] \\ 4, & \text{otherwise} \end{cases} \quad \text{and } \varphi(x,y) = \begin{cases} 2e^{xy}, & \text{if } x, y \in [0, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}$$

If $x, y \in [0, \frac{1}{2}]$, then

$$\begin{aligned} \alpha(x,Sx)\,\varphi(y,Ty)\, s^3 p(Sx,Ty) &= 16 \times e^{\frac{x^2+y^2}{10}} \times \max\left\{\frac{x^2}{100}, \frac{y^2}{100}\right\} \\ &= \frac{4 \times e^{\frac{x^2+y^2}{10}}}{25} \max\left\{x^2, y^2\right\} \\ &= \frac{4 \times e^{\frac{x^2+y^2}{10}}}{25} p(x,y) \\ &\leq \frac{1}{2} p(x,y) \\ &\leq \beta(\mathscr{F}(x,y))\, \mathscr{F}(x,y). \end{aligned}$$

Otherwise, we have

$$\alpha\left(x,Sx\right)\varphi\left(y,Ty\right)s^{3}p\left(Sx,Ty\right)=0\leq\beta\left(\mathscr{F}\left(x,y\right)\right)\mathscr{F}\left(x,y\right).$$

Now we show that (S,T) is a pair of (α,φ)-admissible mapping. For $x,y\in[0,\frac{1}{2}]$, then $\alpha\left(x,y\right)\geq 1$, $\varphi\left(x,y\right)\geq 1$, $Sx\leq 1$, $Sy\leq 1$, $Tx\leq 1$, and $Ty\leq 1$, so it follows that $\alpha\left(Sx,Ty\right)\geq 1$, $\alpha\left(Tx,Sy\right)\geq 1$ and $\varphi\left(Sx,Ty\right)\geq 1$, $\varphi\left(Tx,Sy\right)\geq 1$. Furthermore, if $\{x_n\}$ is a sequence in X such that $\alpha\left(x_n,x_{n+1}\right)\geq 1$ and $\varphi\left(x_n,x_{n+1}\right)\geq 1$, for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to x$ as $n\to\infty$, then $x_n\subseteq[0,\frac{1}{2}]$ and hence $x\in[0,\frac{1}{2}]$. This implies that $\alpha\left(x,Sx\right)\geq 1$ and $\varphi\left(x,Tx\right)\geq 1$. Therefore, all conditions of Theorem 2.3 are satisfied and the pair (S,T) has a common fixed point $x=0$. We note that Corollary 2.5 is not satisfied for $x,y\in(\frac{1}{2},1]$.

Example 2.7. Let $X=[0,1]$. Define $p:X\times X\to[0,\infty)$ by $p\left(x,y\right)=\max\left\{x^2,y^2\right\}$ for all $x,y\in X$. Then (X,p) is a complete partial b-metric-like space with constant $s=2$. Let $T:X\to X$ be a mapping defined by

$$Tx=\begin{cases}\frac{x}{8}, & \text{if } x\in[0,\frac{1}{2}]\\ 5, & \text{otherwise,}\end{cases}.$$

Define the functions $\alpha,\varphi:X\times X\to[0,\infty)$ as

$$\alpha\left(x,y\right)=\begin{cases}3, & \text{if } x,y\in[0,\frac{1}{2}]\\ 1, & \text{otherwise.}\end{cases}, \text{ and } \varphi\left(x,y\right)=\begin{cases}1 & \text{if } x,y\in[0,\frac{1}{2}]\\ 0 & \text{otherwise}\end{cases}.$$

And $\beta:[0,\infty)\to[0,1)$ be a function such that $\beta\left(t\right)=\frac{2}{3}$. If $x,y\in[0,\frac{1}{2}]$ with $x\geq y$ or $y\geq x$, then

$$\begin{aligned}\alpha\left(x,Tx\right)\varphi\left(y,Ty\right)s^{3}p\left(Tx,Ty\right) &= 24\max\left\{\frac{x^2}{64},\frac{y^2}{64}\right\}\\ &= \frac{24}{64}\max\left\{x^2,y^2\right\}\\ &= \frac{3}{8}p\left(x,y\right)\\ &\leq \frac{2}{3}p\left(x,y\right)\\ &\leq \beta\left(\mathscr{F}\left(x,y\right)\right)\mathscr{F}\left(x,y\right).\end{aligned}$$

Otherwise, we have

$$\alpha(x,Tx)\,\varphi(y,Ty)\,s^3p(Tx,Ty) = 0 \leq \beta(\mathscr{F}(x,y))\,\mathscr{F}(x,y).$$

For all $x,y \in [0,\frac{1}{2}]$ $\alpha(x,y) \geq 1$, $\varphi(x,y) \geq 1$ and $Tx \leq 1$, $Ty \leq 1$ implies that $\alpha(Tx,Ty) \geq 1$, $\varphi(Tx,Ty) \geq 1$ and $\alpha(x,Tx) \geq 1$, $\varphi(x,Tx) \geq 1$. Furthermore, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ and $\varphi(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $x_n \subseteq [0,\frac{1}{2}]$ and hence $x \in [0,\frac{1}{2}]$. This implies that $\alpha(x,Tx) \geq 1$ and $\varphi(x,Tx) \geq 1$. Hence in all cases, Corollary (2.4) holds for all $x,y \in X$ and $x = 0$ is a fixed point of T. But we note that Corollary 2.5 is not satisfied for $x,y \in (\frac{1}{2},1]$.

3. Consequences

In this section we introduce some consequences considering X is a b-metric-like space and $\varphi = 1$ from previous results.

Definition 3.1. Let (X,d) be a b-metric-like space, $S,T : X \to X$ be two mappings. Suppose there exist functions $\beta \in \xi$ and $\alpha : X \times X \to [0,\infty)$. Then (S,T) is said to be a pair of generalized $\mathscr{F}$-Geraghty type contraction mappings, if for all $x,y \in X$ such that

$$\alpha(x,y)\,s^3d(Sx,Ty) \leq \beta(\mathscr{F}(x,y))\,\mathscr{F}(x,y), \tag{24}$$

where

$$\mathscr{F}(x,y) = d(x,y) + |d(x,Sx) - d(y,Ty)|.$$

Theorem 3.2. Let (X,d) be a complete b-metric-like space and $S,T : X \to X$ be two mappings. Suppose there exist functions $\beta \in \xi$ and $\alpha : X \times X \to [0,\infty)$, such that the following conditions hold:

(i) (S,T) is a pair of generalized $\mathscr{F}$-Geraghty type contraction mappings;

(ii) (S,T) is a pair of α-admissible mappings;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$;

(iv) for every sequence $\{x_n\}$ in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, we have $\alpha(x,Sx) \geq 1$.

Then the pair (S,T) has a unique common fixed point $u \in X$.

Proof. Let $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, there exists x_1 in A_0 such that $\rho(x_1 - Tx_0) = d_\rho(A,B)$. Moreover, $Tx_1 \in T(A_0) \subseteq B_0$ implies the existence of an $x_2 \in A_0$ such that $\rho(x_2 - Tx_1) = d_\rho(A,B)$. Continuing in this way, we obtain a sequence $\{x_n\}$ in A_0 such that

$$\rho(x_{n+1} - Tx_n) = d_\rho(A,B), \text{ for all } n \in \mathbb{N}. \tag{10}$$

Since pair (A,B) has the $P-$property, from (10) we have,

$$\rho(x_n - x_{n+1}) = \rho(Tx_{n-1} - Tx_n), \text{ for all } n \in \mathbb{N}. \tag{11}$$

We now prove that the sequence $\{x_n\}$ is ρ-convergent in A_0.If there exists $n_0 \in \mathbb{N}$ such that $\rho(Tx_{n_0-1} - Tx_{n_0}) = 0$, then $\rho(x_{n_0} - x_{n_0+1}) = 0 \Leftrightarrow x_{n_0} - x_{n_0+1} = 0 \Leftrightarrow x_{n_0} = x_{n_0+1}$ by (11). Thus

$$Tx_{n_0} = Tx_{n_0+1} \Leftrightarrow Tx_{n_0} - Tx_{n_0+1} = 0 \Leftrightarrow \rho(Tx_{n_0} - Tx_{n_0+1}) = 0. \tag{12}$$

From (11) and (12), we obtain

$$\rho(x_{n_0+2} - x_{n_0+1}) = \rho(Tx_{n_0+1} - Tx_{n_0}) = 0 \Rightarrow x_{n_0+2} = x_{n_0+1}.$$

Thus $x_n = x_{n_0}$ for all $n \geq n_0$ and hence $\{x_n\}$ is ρ-convergent in A_0.

Next let $\rho(Tx_{n-1} - Tx_n) \neq 0$ for all $n \in \mathbb{N}$. Then, for any positive integer n, using (11), we have

$$\tau + F(\rho(c(Tx_n - Tx_{n-1}))) \leq F(\rho(l(x_n - x_{n-1}))).$$

because T is an F_ρ-contraction and this implies that

$$\begin{aligned} F(\rho(c\ (x_{n+1} -\ x_n))) &\leq F(\rho(l(x_n - x_{n-1}))) - \tau \\ F(\rho(c\ (x_{n+1} -\ x_n))) &\leq F(\rho(c(x_n - x_{n-1}))) - \tau \\ F(\rho(c(x_{n+1} - x_n))) &\leq F(\rho(l(x_{n-1} - x_{n-2}))) - 2\tau \\ &\leq F(\rho(c(x_{n-2} - x_{n-3}))) - 3\tau \leq \ldots \leq F(\rho(c(x_1 - x_0))) - n\tau. \end{aligned} \tag{13}$$

Denote $\beta_n := (\rho(c(x_{n+1} - x_n)))$. Then by $(\ 13)$, $\lim_{n\to\infty} F(\beta_n) = -\infty$. Appealing to (C_2), we get

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \rho(\ x_{n+1} - x_n) = 0. \tag{14}$$

A use of (C_3) guarantees the existence of a $k \in (0,1)$ such that

$$\lim_{n\to\infty} \beta_n^k F(\beta_n) = 0, \tag{15}$$

and so by (13), for all $n \in \mathbb{N}$, we have

$$\beta_n^k (F(\beta_n) - F(\beta_0)) \leq -\beta_n^k n\tau \leq 0.$$

Reading (14) and (15) together, we get

$$\lim_{n\to\infty} n\beta_n^k = 0.$$

Hence there exists $n_1 \in \mathbb{N}$ such that $n\beta_n^k \leq 1$ for all $n \geq n_1$. That is, for all $n \geq n_1$,

$$\beta_n \leq \frac{1}{n^{\frac{1}{k}}}, \tag{16}$$

or

$$\rho\left(x_n - x_{n+1} \right) \leq \frac{1}{n^{\frac{1}{k}}}. \tag{17}$$

Similarly, there exists $n_2 \in \mathbb{N}$ such that

$$\begin{aligned} \rho\left(x_n - x_{n+2} \right) &\leq \omega(2)\left[\rho\left(x_n - x_{n+1} \right) + \rho\left(x_{n+1} - x_{n+2} \right)\right] \\ &\leq \omega(2)\left(\frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}}\right) \\ &\leq \frac{\omega(2)}{n^{\frac{1}{k}}}. \end{aligned}$$

This implies that

$$\rho\left(x_n - x_{n+2} \right) \leq \frac{\omega(2)}{n^{\frac{1}{k}}}. \tag{18}$$

Now we have the following two cases.

CASE 1: If $m > 2$ is odd, then $m = 2L+1$, $L \geq 1$, using (17) for all $n \geq h$, $h = \max(n_0, n_1)$

$$\begin{aligned} \rho(x_n - x_{n+m}) &\leq \omega(2L+1)\left[\rho(x_n - x_{n+1}) + \rho(x_{n+1} - x_{n+2}) + \ldots + \rho(x_{n+2L} - x_{n+2L+1})\right] \\ &\leq \omega(2L+1)\left[\frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \ldots + \frac{1}{(n+2L)^{\frac{1}{k}}}\right] \\ &\leq \omega(2L+1)\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

CASE 2: If $m > 2$ is even, then $m = 2L$,$L \geq 2$, using (17) and (18) for all $n \geq h$, $h = \max(n_0, n_1)$

$$\begin{aligned}\rho(x_n - x_{n+m}) &\leq \omega(2L)\left[\rho(x_n - x_{n+2}) + \rho(x_{n+2} - x_{n+3}) + \ldots + \rho(x_{n+2L-1} - x_{n+2L})\right] \\ &\leq \omega(2L)\left[\frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+2)^{\frac{1}{k}}} + \ldots + \frac{1}{(n+2L-1)^{\frac{1}{k}}}\right] \\ &\leq \omega(2L)\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

Combining these two cases, we have

$$\rho(x_n - x_{n+m}) \leq \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}} \text{ for all } n \geq h, m \in \mathbb{N}.$$

Since the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ is convergent $\left(\text{as } \frac{1}{k} > 1\right)$, we deduce that $\{x_n\}$ is a Cauchy sequence. Now X_ρ is complete and A is a ρ-closed subset of X_ρ, there exists $x^* \in A$ such that $\lim_{n\to\infty} x_n = x^*$. Since T is ρ-continuous, Tx_n is ρ-convergent to Tx^*. Hence the continuity of the modular ρ implies that $\rho(x_{n+1} - Tx_n)$ $\rho-$converges to $\rho(x^* - Tx^*)$ and by (10) ,we have

$$\rho(x^* - Tx^*) = d_\rho(A, B).$$

That is, x^* is a best proximity point of T.

Next, we show the uniqueness of the best proximity point. Let us suppose that T has two best proximity points x_1 and $x_2 \in A$, such that $x_1 \neq x_2$ and $\rho(x_1 - Tx_1) = \rho(x_2 - Tx_2) = d_\rho(A, B)$. Then by the $P-$property of (A, B), we have $\rho(x_1 - x_2) = \rho(Tx_1 - Tx_2)$. Note that $\rho(x_1 - x_2) > 0$ as $x_1 \neq x_2$, T is F_ρ-contraction and ρ is an increasing function, thus

$$\begin{aligned}F(\rho(c(x_1 - x_2))) &= F(\rho(c(Tx_1 - Tx_2))) \leq F(\rho(l(x_1 - x_2))) - \tau \\ &\leq F(\rho(c(x_1 - x_2))) - \tau < F(\rho(c(x_1 - x_2))),\end{aligned}$$

which is a contradiction. Hence the best proximity point is unique. □

By assumption (iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by letting $x_1 \in X$ such that $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$, $x_4 = Tx_3$, continue this process we get

$$x_{2i+1} = Sx_{2i} \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0, 1, 2, \ldots.$$

Since (S,T) is a pair of $\alpha-$admissible, so

$$\alpha(x_0,Sx_0)=\alpha(x_0,x_1)\geq 1,$$

$$\alpha(Sx_0,Tx_1)=\alpha(x_1,x_2)\geq 1 \text{ and } \alpha(Tx_1,Sx_2)=\alpha(x_2,x_3)\geq 1.$$

Continuing this manner, we obtain,

$$\alpha(x_n,x_{n+1})\geq 1 \text{ for all } n\geq 0.$$

Now by the analogous way of proof of Theorem 2.3, we conclude that $\{x_n\}$ is a Cauchy sequence in the complete b-metric-like space X. So there exists $u\in X$ such that $x_n\to u$ as $n\to\infty$, this implies $x_{2k+1}\to u$ and $x_{2k+2}\to u$ as $k\to\infty$. Now we show that $Su=Tu=u$. From (24), we have

$$\begin{aligned} d(x_{2k+1},Tu) &\leq \alpha(x_{2k},u)s^3d(Sx_{2k},Tu) \\ &\leq \beta(\mathscr{F}(x_{2k},u))\mathscr{F}(x_{2k},u) \\ &< \mathscr{F}(x_{2k},u), \end{aligned} \tag{25}$$

where

$$\mathscr{F}(x_{2k},u)=d(x_{2k},u)+|d(x_{2k},x_{2k+1})-d(u,Tu)|.$$

Taking limit as $k\to\infty$, we get

$$\lim_{k\to\infty}\mathscr{F}(x_{2k},u)=d(u,Tu). \tag{26}$$

By dividing by $\mathscr{F}(x_{2k},u)$ and taking limit as $k\to\infty$ again in (25) gives

$$\lim_{k\to\infty}\frac{d(x_{2k+1},Tu)}{\mathscr{F}(x_{2k},u)}\leq\lim_{k\to\infty}\beta(\mathscr{F}(x_{2k},u))\leq 1.$$

Hence, we deduce

$$1\leq\lim_{k\to\infty}\beta(\mathscr{F}(x_{2k},u))\leq 1,$$

implies that

$$\lim_{k\to\infty}\beta(\mathscr{F}(x_{2k},u))=1.$$

As $\beta \in \xi$, then

$$\lim_{k\to\infty} \mathscr{F}(x_{2k}, u) = 0. \tag{27}$$

From (26) and (27), we get

$$d(u, Tu) = 0, \tag{28}$$

implies that $Tu = u$. Similarly, we can find that $Su = u$. Hence, the pair (S, T) has a common fixed point $u \in X$ such that $u = Su = Tu$. Assume that u, w are two common fixed points of the pair (S, T) with $u \neq w$ such that $u = Su$, $w = Tw$ and $d(u, u) = d(w, w) = 0$. Then from (24), we have

$$\begin{aligned} d(u, w) &\leq \alpha(u, w) s^3 d(Su, Tw) \\ &\leq \beta(\mathscr{F}(u, w)) \mathscr{F}(u, w) \\ &\leq \beta(d(u, w)) d(u, w) \\ &< d(u, w), \end{aligned}$$

which is a contradiction. therefore $u = w$ implies that the pair (S, T) has a unique common fixed point $u \in X$ such that $u = Su = Tu$ with $d(u, u) = 0$.

Corollary 3.3. Let (X, d) be a complete b-metric-like space and $T : X \longrightarrow X$ be a mapping such that for all $x, y \in X$ there exist a real number $s \geq 1$,and two functions $\alpha : X \times X \to [0, \infty)$ and $\beta \in \xi$, then we have

$$\alpha(x, y) s^3 d(Tx, Ty) \leq \beta(\mathscr{F}(x, y)) \mathscr{F}(x, y),$$

where

$$\mathscr{F}(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|.$$

Also assume that the following conditions hold:

(i) T is α-admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) for every sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, we have $\alpha(x, Tx) \geq 1$.

Then T has a unique fixed point $u \in X$ with $d(u,u) = 0$.

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Example 3.4. Let $X = [0, \frac{3}{2}]$. Define $d : X \times X \to [0,\infty)$ by $d(x,y) = x^2 + y^2 + (x-y)^2$ for all $x, y \in X$. Then (X,d) is a complete b-metric-like space with constant $s = 2$, but is not b-metric space since $d(1,1) = 2$ nor metric-like space. Let $S, T : X \to X$ be two mappings defined by

$$Sx = \begin{cases} \frac{x}{10\sqrt{2}}, & \text{if } x \in [0,1] \\ 2x, & \text{otherwise,} \end{cases} \quad \text{and } Tx = \begin{cases} \frac{x}{10\sqrt{2}}, & \text{if } x \in [0,1] \\ 3x, & \text{otherwise} \end{cases}.$$

Let $\beta : [0,\infty) \to [0,1)$ be a function such that $\beta(t) = \frac{2}{5}$ and $\alpha : X \times X \to [0,\infty)$ defined as

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1] \\ 0, & \text{otherwise} \end{cases}.$$

If $x, y \in [0,1]$, then

$$\begin{aligned} \alpha(x,y) s^3 d(Sx,Ty) &= 8[\frac{x^2}{200} + \frac{y^2}{200} + \left(\frac{x}{10\sqrt{2}} - \frac{y}{10\sqrt{2}}\right)^2] \\ &= \frac{1}{25}[x^2 + y^2 + (x-y)^2] \\ &\leq \frac{2}{5} d(x,y) \\ &\leq \beta(\mathscr{F}(x,y)) \mathscr{F}(x,y). \end{aligned}$$

Otherwise, we have

$$\alpha(x,y) s^3 d(Sx,Ty) = 0 \leq \beta(\mathscr{F}(x,y)) \mathscr{F}(x,y).$$

Now we show that (S,T) is a pair of α-admissible mapping. For $x, y \in [0,1]$, then $\alpha(x,y) \geq 1$, $Sx \leq 1$, $Sy \leq 1$, $Tx \leq 1$, and $Ty \leq 1$, so it follows that $\alpha(Sx,Ty) \geq 1$ and $\alpha(Tx,Sy) \geq 1$. Furthermore, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $x_n \subseteq [0,1]$ and hence $x \in [0,1]$. This implies that $\alpha(x,Sx) \geq 1$. Therefore, all conditions of theorem 3.2 are satisfied and the pair (S,T) has a common fixed point $u = 0$. But we note that Corollary 2.5 is not satisfied for $x, y \in (1, \frac{3}{2}]$.

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Example 3.5. Let $X=[0,\infty)$. Define $d:X\times X\to[0,\infty)$ by $d(x,y)=(x+y)^2$ for all $x,y\in X$. Then (X,d) is a complete b-metric-like space with constant $s=2$, but is not b-metric space since $d(1,1)=4$ nor metric-like space. Let $T:X\to X$ be a mapping defined by

$$Tx=\begin{cases}\frac{x}{9}, & \text{if } x\in[0,1]\\ 2x, & \text{otherwise,}\end{cases}.$$

Define $\alpha:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}2, & \text{if } x,y\in[0,1]\\ 0, & \text{otherwise}\end{cases},$$

and $\beta:[0,\infty)\to[0,1)$ be a function such that $\beta(t)=\frac{1}{4}$. If $x,y\in[0,1]$, then

$$\begin{aligned}\alpha(x,y)s^3d(Tx,Ty) &= 16\left(\frac{x}{9}+\frac{y}{9}\right)^2\\ &= \frac{16}{81}(x+y)^2\\ &= \frac{16}{81}d(x,y)\\ &\leq \frac{1}{4}d(x,y)\\ &\leq \beta(\mathscr{F}(x,y))\mathscr{F}(x,y).\end{aligned}$$

Otherwise, we have

$$\alpha(x,y)s^3d(Tx,Ty)=0\leq\beta(\mathscr{F}(x,y))\mathscr{F}(x,y).$$

Also we note that T is α-admissible mapping. Furthermore, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$, for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to x$ as $n\to\infty$, then $x_n\subseteq[0,1]$ and hence $x\in[0,1]$. This implies that $\alpha(x,Tx)\geq 1$. for all $x,y\in[0,1]$. Hence in all cases, Corollary (3.3) holds for all $x,y\in X$ and $x=0$ is a fixed point of T. But for $x,y\in(1,\infty)$, $d(Tx,Ty)=4(x+y)^2=4d(x,y)\geq\beta(\mathscr{F}(x,y))\mathscr{F}(x,y)$. Therefore, Corollary 2.5 is not satisfied.

4. Applications to Nonlinear Integral Equations

We consider existence of a solution for the following family of Volterra type integral equations given by.

$$x(t) = \int_0^T h(t,s)\, g(s, x(s))\, ds, \tag{29}$$

$t \in I = [0,T]$, $T > 0$. Let $X = C(I,\mathbb{R})$ be the space of all continuous real functions defined from I to $\mathbb{R}$, also let X be endowed with the b-metric-like $d(x,y) = \max\limits_{t\in[0,T]} (|x(t)| + |y(t)|)^2$ for all $x, y \in X$. Obviously, (X,d) is a complete b-metric-like space with the constant $s = 2$.

Theorem 3.7. Assume that the following conditions hold:

(i) $g : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, such that there exists a constant $0 \leq \lambda < 1$ and for all $x, y \in X$

$$|g(t,x(t))| + |g(t,y(t)| < \lambda(|x(t)| + |y(t)|);$$

(ii) $h : I \times I \to R$ is a continuous at $t \in I$ and suppose that $\int_0^T h(t,s)ds \leq L$;

(iii) $\lambda^2 L^2 \leq \frac{1}{32}$;

(iv) Define two functions $\beta : [0,\infty) \to [0,1)$ by $\beta(t) = \frac{1}{2}$, and $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = 2$.

Then the integral equation (29) has a unique solution in X.

Proof. Let $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, and $A_0 \subseteq g(A_0)$, there exists x_1 in A_0 such that $\rho(gx_1 - Tx_0) = d_\rho(A,B)$. If $x_0 = x_1$ then put $x_n := x_1$ for all $n \geq 2$. Also, since $Tx_1 \in T(A_0) \subseteq B_0$, and $A_0 \subseteq g(A_0)$, there exists x_2 in A_0 such that $\rho(gx_2 - Tx_1) = d_\rho(A,B)$. If $x_1 = x_2$, then put $x_n := x_2$ for all $n \geq 3$. Going on in this way, we get a sequence $\{x_n\}$ in A_0 such that

$$\rho(gx_{n+1} - Tx_n) = d_\rho(A,B) \text{ for all } n \in \mathbb{N}. \tag{19}$$

We now prove that the sequence $\{x_n\}$ is ρ-convergent in A_0. Without loss of real generality, we can assume that $\rho(gx_n - gx_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Since T is a ρ-continuous F_ρ-proximal contraction of the first kind, for any positive integer n, by(19), we have

$$\tau + F(\rho(c(gx_n - gx_{n+1}))) \leq F(\rho(l(x_{n-1} - x_n))),$$

or

$$\begin{aligned} F\left(\rho\left(c\left(x_n - x_{n+1}\right)\right)\right) &\leq F\left(\rho\left(l\left(x_{n-1} - x_n\right)\right)\right) - \tau \\ &\leq F\left(\rho\left(c\left(x_{n-1} - x_n\right)\right)\right) - \tau \\ &\leq F\left(\rho\left(l\left(x_{n-1} - x_n\right)\right)\right) - 2\tau. \end{aligned}$$

Inductively, we reach at

$$F\left(\rho\left(c\left(x_n - x_{n+1}\right)\right)\right) \leq F\left(\rho\left(l\left(x_0 - x_1\right)\right)\right) - n\tau. \tag{20}$$

Following the techniques similar to Theorem **??**, it follows that $\{x_n\}$ is a $\rho-$Cauchy sequence in A. Thus $\lim_{n\to\infty} x_n = x$ for some $x \in A$ from the assumptions on X_ρ and A. Now continuity of ρ, T and g implies that $\rho\left(gx_{n+1} - Tx_n\right)$ $\rho-$converges to $\rho\left(gx - Tx\right)$. Thus from (19), we achieve

$$\rho\left(gx - Tx\right) = d_\rho\left(A, B\right).$$

That is, x is the coincidence best proximity point of T and g.

To show the uniqueness of the coincidence best proximity point, suppose that T and g has two coincidence best proximity points x_1 and $x_2 \in A$. Let $x_1 \neq x_2$ so $\rho\left(x_1 - x_2\right) > 0$. Exploiting the facts that T is an F_ρ-proximal contraction of first kind and g is an isometry, we can write

$$F\left(\rho\left(x_1 - x_2\right)\right) = F\left(\rho\left(gx_1 - gx_2\right)\right) \leq F\left(\rho\left(\, x_1 - \, x_2\right)\right) - \tau < F\left(\rho\left(\, x_1 - \, x_2\right)\right).$$

This is a contradiction. Hence the coincidence best proximity point of T and g is unique. □

Define an operator $T : X \to X$ by

$$Tx\left(t\right) = \int_0^T h(t,s)g(s,x(s))ds,$$

$t \in I = [0,T]$, $T > 0$, and for all $x,y \in X$, we have

$$\begin{aligned}
(|Tx(t)|+|Ty(t)|)^2 &= (\left|\int_0^T h(t,s)g(s,x(s))ds\right| + \left|\int_0^T h(t,s)g(s,y(s))ds\right|)^2 \\
&\leq (\int_0^T |h(t,s)g(s,x(s))|\,ds + \int_0^T |h(t,s)g(s,y(s))|\,ds)^2 \\
&= (\int_0^1 (|h(t,s)g(s,x(s))| + |h(t,s)g(s,y(s))|)ds)^2 \\
&\leq (\int_0^T h(s,t)\lambda(|x(s)|+|y(s)|)ds)^2 \\
&= (\int_0^T h(s,t)\lambda((|x(s)|+|y(s)|)^2)^{\frac{1}{2}}ds)^2 \\
&\leq \lambda^2 d(x(t),y(t))(\int_0^T h(s,t)ds)^2 \\
&\leq \lambda^2 L^2 d(x,y) \\
&\leq \frac{1}{32}\mathscr{F}(x,y),
\end{aligned}$$

Therefore

$$16d(Tx,Ty) \leq \frac{1}{2}\mathscr{F}(x,y),$$

which implies that

$$\alpha(x,y)s^3 d(Tx,Ty) \leq \beta(\mathscr{F}(x,y))\mathscr{F}(x,y).$$

Hence Corollary 3.3 is satisfied and the equation (29) has a unique solution in X.

5. Conclusion

We conclude that the above Theorem 2.3 improves and extends Theorem 2.1 of [10] in Partial b-Metric-Like Space with respect to (α,φ)-admissible mappings. Examples are included to satisfy our results, we show that Theorem 2.1 in paper of Aydi et. al is not applicable with such example. Indeed, we obtained some consequences with examples and discussing an application for a family of Volterra type integral equations.

Author's Contribution

The second author made the first draft of this paper. All authors read and approved the final manuscript.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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