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FIXED POINT THEOREMS FOR CYCLIC CONTRACTION ON b-METRIC SPACES WITH wt-DISTANCE

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Abstract. In this paper, some fixed point theorems for cyclic generalized φ-contraction and $(\psi,\varphi)-$weakly contraction on b-metric spaces with wt-distance are proved, which extend some results in the literature.

Keywords: cyclic; fixed point; wt-distance; b-metric space.

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1. INTRODUCTION AND PRELIMINARIES

Since the concept of b-metric space as a generalization of metric space was given by Czerwik [1], many fixed point results in metric spaces were generalized in b-metric spaces (see [2, 6], etc.). In 2014, the concept of wt-distance on b-metric spaces was given by N. Hussain et al. [3], we shall use wt-distance on b-metric spaces to extend some results by others.

In the section one, we give some elementary definitions and lemmas. In the section two, inspired by H.K. Nashine and Z. Kadelburg [8] and H.P. Huang [5], we define cyclic generalized φ-contraction and $(\psi,\varphi)-$weakly contraction on b-metric spaces with wt-distance and related fixed point results are proved, which extend some results in the literature.

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Throughout, we denote all natural number by $\mathbb{N}$.

Definition 1.1. [1] Let X be a nonempty set and constant $s \geq 1$ be a fixed real number. Suppose that the mapping $d : X \times X \to [0,\infty)$ satisfies the following conditions:

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $d(x,y) \leq s[d(x,z)+d(z,y)]$ for all $x,y,z \in X$.

Then (X,d) is called a b-metric space with coefficient s.

Definition 1.2. [3, 4] Let (X,d) be a b-metric space with constant $s \geq 1$, then a function $p : X \times X \to [0,\infty)$ is called a wt-distance on X if the following conditions are satisfied:

(1) $p(x,z) \leq s[p(x,y)+p(y,z)]$ for any $x,y,z \in X$;

(2) $p(x,\cdot) : X \to [0,\infty)$ is s-lower semi-continuous for any $x \in X$, if

$$\liminf_{n\to\infty} p(x,x_n) = \infty, \ or \ p(x,x_0) \leq \liminf_{n\to\infty} sp(x,x_n),$$

where $\lim\limits_{n\to\infty} d(x_0,x_n) = 0$;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.

The wt-distance p is called symmetric if $p(x,y) = p(y,x)$ for any $x,y \in X$. We say that

(a) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim\limits_{n\to\infty} d(x_n,x) = 0$, i.e., $x_n \to x$;

(b) The sequence $\{x_n\}$ is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n,x_m) = 0$;

(c) (X,d) is complete if and only if any Cauchy sequence in X is convergent.

Lemma 1.3. [3, 4] Let (X,d) be a b-metric space with constant $s \geq 1$ and p be a wt-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero. Then for any $x,y,z \in X$, the following properties hold:

(1) If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;

(2) If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\lim_{n\to\infty} d(y_n,z) = 0$;

(3) If $p(x_n,x_m) \leq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;

(4) If $p(y,x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

2. MAIN RESULTS

In this part, we will show our lemmas, theorems and corollaries.

Lemma 2.1. Let (X,d) be a b-metric space with constant $s \geq 1$ and p be a wt-distance on X, $\{x_n\}$ be sequence in X, then the inequality

$$p(x_0,x_k) \leq s^n \sum_{i=0}^{k-1} p(x_i,x_{i+1}) \tag{2.1}$$

is valid for every $n \in \mathbb{N}$ and every $k \in \{1,2,\cdots,2^{n-1},2^n\}$.

Proof. Let us use mathematical induction, denote (2.1) by $P(n)$, then we have

$$P(0): p(x_0,x_1) \leq p(x_0,x_1) = s^0 \sum_{i=0}^{0} p(x_i,x_{i+1}),$$

$$P(1): p(x_0,x_2) \leq s[p(x_0,x_1)+p(x_1,x_2)] = s^1 \sum_{i=0}^{1} p(x_i,x_{i+1})$$

Now, we assume that

$$P(n): p(x_0,x_k) \leq s^n \sum_{i=0}^{k-1} p(x_i,x_{i+1}) \tag{2.2}$$

is valid for every $x_0,x_1,\cdots,x_{2^n} \in X$ for every $k \in \{1,2,\cdots,2^{n-1},2^n\}$, then we will prove that $P(n+1)$ is also valid.

Indeed, for $k \in \{2^n+1,2^n+2,\cdots,2^{n+1}-1,2^{n+1}\}$, by (2.2), we have

$$\begin{aligned} p(x_0,x_k) &\leq s[p(x_0,x_{2^n})+p(x_{2^n},x_k)] \\ &\leq s\left[s^n \sum_{i=0}^{2^n-1} p(x_i,x_{i+1}) + s^n \sum_{i=2^n}^{k-1} p(x_i,x_{i+1})\right] \\ &= s^{n+1} \sum_{i=0}^{k-1} p(x_i,x_{i+1}). \quad \square \end{aligned}$$

Lemma 2.2. Let (X,d) be a b-metric space with constant $s \geq 1$ and p be a wt-distance on X, $\{x_n\}$ be sequence in X, we say the $\{x_n\}$ is a Cauchy sequence if there exists $c \in [0,1)$, such that $p(x_n,x_{n+1}) \leq cp(x_{n-1},x_n)$ for every $n \in \mathbb{N}$.

Proof. We note that $p(x_n,x_{n+1}) \le c^n p(x_0,x_1)$ for every $n\in\mathbb{N}$. For all $m,k\in\mathbb{N}$ with $r=[\log_2^k]$, we have

$$\begin{aligned}
p(x_{m+1},x_{m+k}) \le& s[p(x_{m+1},x_{m+2})+p(x_{m+2},x_{m+k})]\\
\le& sp(x_{m+1},x_{m+2})+s^2p(x_{m+2},x_{m+2^2})+s^2p(x_{m+2^2},x_{m+k})\\
\le& sp(x_{m+1},x_{m+2})+s^2p(x_{m+2},x_{m+2^2})\\
&+s^3p(x_{m+2^2},x_{m+2^3})+s^3p(x_{m+2^3},x_{m+k})\\
&\cdots\\
\le& \sum_{n=1}^{r} s^n p(x_{m+2^{n-1}},x_{m+2^n})+s^{r+1}p(x_{m+2^r},x_{m+k}) \qquad (2.3)
\end{aligned}$$

Then by (2.3) and Lemma 2.1, we have

$$\begin{aligned}
p(x_{m+1},x_{m+k}) \le& \sum_{n=1}^{r} s^{2n}\left(\sum_{i=m}^{m+2^{n-1}-1} p(x_{2^{n-1}+i},x_{2^{n-1}+i+1})\right)\\
&+s^{2(r+1)}\left(\sum_{i=m}^{m+k-2^r-1} p(x_{2^r+i},x_{2^r+i+1})\right)\\
\le& \sum_{n=1}^{r+1} s^{2n}\left(\sum_{i=m}^{m+2^{n-1}-1} p(x_{2^{n-1}+i},x_{2^{n-1}+i+1})\right)\\
\le& p(x_0,x_1)\sum_{n=1}^{r+1} s^{2n}\left(\sum_{i=0}^{2^{n-1}-1} c^{m+2^{n-1}+i}\right)\\
\le& \frac{p(x_0,x_1)}{1-c}\sum_{n=1}^{r+1} s^{2n}c^{m+2^{n-1}}\\
=& \frac{p(x_0,x_1)}{1-c}c^m\sum_{n=1}^{r+1} s^{2n}c^{2^{n-1}}\\
=& \frac{p(x_0,x_1)}{1-c}c^m\sum_{n=1}^{r+1} c^{2n\log_c s+2^{n-1}} \to 0\ (m\to\infty), \qquad (2.4)
\end{aligned}$$

where $0<c<1$ and $\sum_{n=1}^{\infty} c^{2n\log_c s+2^{n-1}}$ is convergent.

Then by lemma 1.3, the proof is immediate. □

Now, we denote by Φ the set of functions $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<\frac{t}{as}$ for each $t>0, a>1$ and $\varphi(0)=0$.

Definition 2.3. [7, 9] Let (X,d) be a b-metric space, k be a positive integer, $A_1,A_2,\cdots,A_k$ be nonempty subsets of X, $V=\bigcup_{i=1}^{k}A_i$, $f:V\to V$, then f is called a cyclic operator if

(1) $A_i, i=1,2,\cdots,k$ are nonempty subsets;

(2) $f(A_1)\subseteq A_2,\cdots,f(A_{p-1})\subseteq A_p, f(A_p)\subseteq A_1$.

Definition 2.4. Let (X,d) be a b-metric space with constant $s\geq 1$ and p be a wt-distance on X, k be a positive integer, $A_1,A_2,\cdots,A_k$ be nonempty subsets of X, $V=\bigcup_{i=1}^{k}A_i$, $f:V\to V$ satisfies a cyclic generalized φ-contraction for some $\varphi\in\Phi$, if

(1) $V=\bigcup_{i=1}^{k}A_i$ is a cyclic representation of V with respect to f;

(2) for any $(x,y)\in A_i\times A_{i+1}$, $i=1,2,\cdots,k$, $(A_{k+1}=A_1)$, there exist $L\geq 0$ and constant $\lambda_1>0$, $0<\lambda_2<\frac{as}{2}$, $a>1$ and $a>\lambda_2$ such that

$$p(fx,fy)\leq M_s(x,y)+L\min\{\varphi(p(x,fx)),\varphi(p(y,fy)),\varphi(p(x,fy)),\varphi(p(y,fx))\} \tag{2.5}$$

where

$$\begin{aligned}M_s(x,y)=&\max\{\varphi(p(x,y)),\varphi(p(x,fx)),\varphi(\lambda_1 p(x,fx)+(1-\lambda_1)p(y,fy)),\\&\varphi(\frac{\lambda_2 p(x,fy)+(1-\lambda_2)p(fx,y)}{s})\}.\end{aligned}$$

Theorem 2.5. Let (X,d) be a complete b-metric space with $s\geq 1$ and p be a wt-distance on X, $p(x,x)=0$ for any $x\in X$, $V=\bigcup_{i=1}^{k}A_i$ and $A_1,A_2,\cdots,A_k$ be nonempty closed subsets of X, k be a positive integer, $f:V\to V$ is a cyclic generalized φ-contraction mapping for some $\varphi\in\Phi$.

Suppose that either

(1) $inf\{p(x,w)+p(x,fx):x\in X\}>0$ for every $w\in X$ with $w\neq fw$;

or

(2) the mapping f is continuous.

Then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^{k}A_i$.

Proof. For any $x_0\in A_1$ (such a point exists since $A_1\neq\emptyset$), we can construct the sequence $\{x_n\}$ in X by $x_{n+1}=fx_n$ $(n\in\mathbb{N}\cup\{0\})$. If $x_n=x_{n+1}$ for some $n\in\mathbb{N}\cup\{0\}$, then f has fixed point. Now, suppose that $x_n\neq x_{n+1}$ for any $n\in\mathbb{N}\cup\{0\}$.

Next, we shall prove that

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0.$$

Indeed, if not, suppose that $p(x_n,x_{n+1}) > 0$ for all $n \in \mathbb{N}\cup\{0\}$, there exists $i = i(n) \in \{1,2,\cdots,k\}$ such that $(x_n,x_{n+1}) \in A_i \times A_{i+1}$, then we claim that $\xi_n \leq \acute{c}\xi_{n-1}$ for all $n \in \mathbb{N}$ $(0 < \acute{c} < 1)$, where $\xi_n = p(x_n,x_{n+1})$.

By (2.5), we have

$$\begin{aligned}
p(x_n,x_{n+1}) =& p(fx_{n-1},fx_n)\\
\leq& M_s(x_{n-1},x_n) + L\min\{\varphi(p(x_{n-1},fx_{n-1})),\varphi(p(x_n,fx_n)),\\
&\varphi(p(x_{n-1},fx_n)),\varphi(p(x_n,fx_{n-1}))\}\\
=& M_s(x_{n-1},x_n) + L\min\{\varphi(p(x_{n-1},x_n)),\varphi(p(x_n,x_{n+1})),\\
&\varphi(p(x_{n-1},x_{n+1})),\varphi(p(x_n,x_n))\}\\
=& M_s(x_{n-1},x_n), \qquad (2.6)
\end{aligned}$$

from Definition 2.4, we have

$$\begin{aligned}
M_s(x_{n-1},x_n) \;=\;& \max\{\varphi(p(x_{n-1},x_n)),\varphi(\lambda_1 p(x_{n-1},x_n)+(1-\lambda_1)p(x_n,x_{n+1})),\\
&\varphi(\frac{\lambda_2 p(x_{n-1},x_{n+1})}{s})\}.
\end{aligned}$$

Consider the following possibilities.

If $M_s(x_{n-1},x_n) = \varphi(p(x_{n-1},x_n))$, then by (2.6) and $\varphi(t) < \frac{t}{as}$, we have

$$\xi_n = p(x_n,x_{n+1}) \leq M_s(x_{n-1},x_n) = \varphi(p(x_{n-1},x_n)) < \frac{p(x_{n-1},x_n)}{as} = r_1\xi_{n-1},$$

where $r_1 \doteq \frac{1}{as} \in (0,1)$.

If $M_s(x_{n-1},x_n) = \varphi(\lambda_1 p(x_{n-1},x_n)+(1-\lambda_1)p(x_n,x_{n+1}))$, then by (2.6) and $\varphi(t) < \frac{t}{as}$, we have

$$\begin{aligned}
\xi_n \;=\;& p(x_n,x_{n+1}) \leq M_s(x_{n-1},x_n)\\
=\;& \varphi(\lambda_1 p(x_{n-1},x_n)+(1-\lambda_1)p(x_n,x_{n+1}))\\
<\;& \frac{\lambda_1 p(x_{n-1},x_n)+(1-\lambda_1)p(x_n,x_{n+1})}{as}
\end{aligned}$$

i.e.,

$$\xi_n = p(x_n,x_{n+1}) < r_2 p(x_{n-1},x_n) = r_2\xi_{n-1},$$

where $r_2 \doteq \frac{\lambda_1}{as+\lambda_1-1} \in (0,1)$.

If $M_s(x_{n-1},x_n) = \varphi(\frac{\lambda_2 p(x_{n-1},x_{n+1})}{s})$, then by (2.6) and $\varphi(t) < \frac{t}{as}$, we have

$$\begin{aligned}\xi_n &= p(x_n,x_{n+1}) \le M_s(x_{n-1},x_n) = \varphi\left(\frac{\lambda_2 p(x_{n-1},x_{n+1})}{s}\right)\\ &< \frac{\lambda_2 p(x_{n-1},x_{n+1})}{as^2} \le \frac{\lambda_2}{as}[p(x_{n-1},x_n)+p(x_n,x_{n+1})]\end{aligned}$$

i.e.,

$$\xi_n = p(x_n,x_{n+1}) < r_3 p(x_{n-1},x_n) = r_3\xi_{n-1}.$$

where $r_3 \doteq \frac{\lambda_2}{as-\lambda_2} \in (0,1)$.

Let $\acute{c} = \max\{r_1,r_2,r_3\}$, then we have

$$0 < \xi_n < \acute{c}\xi_{n-1} < (\acute{c})^2\xi_{n-2} < \cdots < (\acute{c})^n\xi_0. \tag{2.7}$$

Since $\acute{c} \in (0,1)$, then we have

$$\lim_{n\to\infty}\xi_n = \lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{2.8}$$

By (2.7) and Lemma 2.2, then $\{x_n\}$ is a Cauchy sequence.

Since X is a complete space, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{2.9}$$

We shall prove that $u \in \bigcap_{i=1}^k A_i$. By Definition, we have $x_0 \in A_1$ and $\{x_{nk}\} \subseteq A_1$. Since A_1 is closed, we get that $u \in A_1$. Similarly, we have $\{x_{nk+1}\} \subseteq A_2$ and $u \in A_2$. By mathematical induction, we get that $u \in \bigcap_{i=1}^k A_i$.

By (2.4), we obtain that $\lim\limits_{n,m\to\infty} p(x_n,x_m) = 0$. Then for any $\varepsilon > 0$, there exists a $n > N_\varepsilon \in \mathbb{N}$ such that $p(x_{N_\varepsilon},x_n) < \frac{\varepsilon}{s}$.

By (2.9) and $p(x,\cdot)$ is s-lower semi-continuous, thus we have

$$p(x_{N_\varepsilon},u) \le \liminf_{n\to\infty} sp(x_{N_\varepsilon},x_n) \le \varepsilon$$

Let $\varepsilon = \frac{1}{t}$ and $N_\varepsilon = n_t$ $(t \in \mathbb{N})$, then we have

$$\lim_{t\to\infty} p(x_{n_t},u) = 0. \tag{2.10}$$

Next, we shall prove that the u is a fixed point of f.

Case (1), suppose that $fu \neq u$, then by (2.8) and (2.10), we have

$$0 < \inf\{p(x,u)+p(x,fx) : x \in X\} \leq \inf\{p(x_n,u)+p(x_n,x_{n+1}) : n \in N\} \to 0 \ (n \to \infty)$$

which is a contradiction, thus $fu = u$.

Case (2), suppose that there exists a $w \in X$ with $fw \neq w$ such that $\inf\{p(x,w)+p(x,fx) : x \in X\} = 0$, then there exists a sequence $\{y_n\} \subset X$ such that $p(y_n,w)+p(y_n,fy_n) \to 0$ as $n \to \infty$, thus we have

$$\lim_{n\to\infty} p(y_n,w) = \lim_{n\to\infty} p(y_n,fy_n) = 0 \ \ \textit{and} \ \ \lim_{n\to\infty} d(fy_n,w) = 0 \ (\textit{by Lemma } 1.3). \tag{2.11}$$

Since

$$\begin{aligned} M_s(y_n,fy_n) &= \max\{\varphi(p(y_n,fy_n)),\varphi(p(y_n,fy_n)),\varphi(\lambda_1 p(y_n,fy_n) \\ &\quad +(1-\lambda_1)p(fy_n,f^2y_n)),\varphi(\frac{\lambda_2 p(y_n,f^2y_n)}{s})\} \\ &\leq \frac{1}{as}\max\{p(y_n,fy_n),\lambda_1 p(y_n,fy_n)+(1-\lambda_1)p(fy_n,f^2y_n),\frac{\lambda_2 p(y_n,f^2y_n)}{s}\} \\ &\leq \frac{1}{as}\max\{p(y_n,fy_n),\lambda_1 p(y_n,fy_n)+(1-\lambda_1)p(fy_n,f^2y_n), \\ &\quad \lambda_2(p(y_n,fy_n)+p(fy_n,f^2y_n))\} \\ &\quad \to \max\{\frac{1-\lambda_1}{as},\frac{\lambda_2}{as}\}\lim_{n\to\infty} p(fy_n,f^2y_n), \end{aligned}$$

and

$$\begin{aligned} p(fy_n,f^2y_n) \leq & M_s(y_n,fy_n)+L\min\{\varphi(p(y_n,fy_n)),\varphi(p(fy_n,f^2y_n)), \\ & \varphi(p(y_n,f^2y_n)),\varphi(p(fy_n,fy_n))\} \\ = & M_s(y_n,fy_n) \to \max\{\frac{1-\lambda_1}{as},\frac{\lambda_2}{as}\}\lim_{n\to\infty} p(fy_n,f^2y_n), \end{aligned} \tag{2.12}$$

which is contradictive with $\max\{\frac{1-\lambda_1}{as},\frac{\lambda_2}{as}\} \in (0,1)$. Thus we have

$$\lim_{n\to\infty} p(fy_n,f^2y_n) = 0. \tag{2.13}$$

By (2.11) and (2.13), we have

$$p(y_n,f^2y_n) \leq s(p(y_n,fy_n)+p(fy_n,f^2y_n)) \to 0 \ (n \to \infty). \tag{2.14}$$

Thus by (2.11), (2.14) and Lemma 1.3, we obtain that $\lim_{n\to\infty} d(f^2y_n, w) = 0$.

By the continuity of f, we have $fw = f(\lim_{n\to\infty} fy_n) = \lim_{n\to\infty} f^2y_n = w$, which is a contradiction with the hypothesis. So case (1) always holds, and by case (1), $u = fu$.

Finally, we shall prove the uniqueness of fixed point u of f.

Assume that there exists $v \in X$ such that $fv = v$ with $v \neq u$, then we have

$$\begin{aligned} p(u,v) &= p(fu, fv) \\ &\leq M_s(u,v) + L\min\{\varphi(p(u,fu)), \varphi(p(v,fv)), \varphi(p(u,fv)), \varphi(p(v,fu))\} \\ &= M_s(u,v) + L\min\{\varphi(p(u,u)), \varphi(p(v,v)), \varphi(p(u,v)), \varphi(p(v,u)) \\ &= M_s(u,v), \end{aligned}$$

where

$$\begin{aligned} M_s(u,v) &= \max\{\varphi(p(u,v)), \varphi(p(u,fu)), \varphi(\lambda_1 p(u,fu) \\ &\quad +(1-\lambda_1)p(v,fv)), \varphi(\frac{\lambda_2 p(u,fv) + (1-\lambda_2)p(fu,v)}{s})\} \\ &= \max\{\varphi(p(u,v)), \varphi(p(u,u)), \varphi(\lambda_1 p(u,u) \\ &\quad +(1-\lambda_1)p(v,v)), \varphi(\frac{\lambda_2 p(u,v) + (1-\lambda_2)p(u,v)}{s})\} \\ &= \max\{\varphi(p(u,v)), \varphi(\frac{p(u,v)}{s})\} \\ &\leq \frac{1}{as}p(u,v) \end{aligned}$$

Then, we get that $p(u,v) \leq \frac{1}{as}p(u,v)$ $(as > 1)$, a contradiction. Thus we have $p(u,v) = 0$.

Similarly, we get that $p(u,u) = 0$, and by Lemma 1.3, we have $u = v$. $\square$

We can get a more comfortable theorem if wt-distance p is symmetric.

Theorem 2.6. Let (X,d) be a complete b-metric space with constant $s \geq 1$ and p be a symmetric wt-distance on X, $p(x,x) = 0$ for $x \in X$, $V = \bigcup_{i=1}^{k} A_i$ and $A_1, A_2, \cdots, A_k$ be nonempty closed subsets of X, k be a positive integer, $f : V \to V$ is a cyclic generalized φ-contraction mapping for some $\varphi \in \Phi$, then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^{k} A_i$.

Proof. By comparing Theorem 2.6 with Theorem 2.5, we find that we can omit the condition "case (1) and case (2)" by the condition that wt-distance p is symmetric. By observing the proof of Theorem 2.5, we find that the condition "case (1) and case (2)" is only used to prove the existence of fixed point u. So we continue using the similar notations in Theorem 2.5 to prove the existence of fixed point u by the condition that wt-distance p is symmetric.

Next, we shall prove that the u is a fixed point of f.

Since Cauchy sequence $\{x_n\} \subset X$ with $x_{n+1} = fx_n$ converges to $u \in X$. And by the symmetry of wt-distance p and (2.10), we have

$$(*) \qquad \lim_{n\to\infty} p(u,x_n) = 0.$$

Then by (*), (2.8) and (2.10), we have

$$\begin{aligned}
p(u,fu) &\leq s(p(u,fx_n)+p(fx_n,fu)) \leq sp(u,fx_n)+sM_s(x_n,u)+\\
&\quad sL\min\{\varphi(p(x_n,fx_n)),\varphi(p(u,fu)),\varphi(p(x_n,fu)),\varphi(p(u,fx_n))\}\\
&= sp(u,x_{n+1})+s\max\{\varphi(p(x_n,u)),\varphi(p(x_n,x_{n+1})),\varphi(\lambda_1 p(x_n,x_{n+1})+\\
&\quad (1-\lambda_1)p(u,fu)),\varphi(\frac{\lambda_2 p(x_n,fu)+(1-\lambda_2)p(x_{n+1},u)}{s})\}+\\
&\quad sL\min\{\varphi(p(x_n,x_{n+1})),\varphi(p(u,fu)),\varphi(p(x_n,fu)),\varphi(p(u,x_{n+1}))\}\\
&\leq sp(u,x_{n+1})+\frac{1}{a}max\{p(x_n,u),p(x_n,x_{n+1}),\lambda_1 p(x_n,x_{n+1})+\\
&\quad (1-\lambda_1)p(u,fu),\frac{\lambda_2 p(x_n,fu)+(1-\lambda_2)p(x_{n+1},u)}{s}\}+\\
&\quad \frac{1}{a}L\min\{p(x_n,x_{n+1}),p(u,fu),p(x_n,fu),p(u,x_{n+1})\}\\
&= \frac{1}{a}\max\{(1-\lambda_1)p(u,fu),\lim_{n\to\infty}\frac{\lambda_2 p(x_n,fu)}{s}\}(n\to\infty)\\
&\leq \frac{1}{a}\max\{(1-\lambda_1)p(u,fu),\lim_{n\to\infty}\lambda_2[p(x_n,u)+p(u,fu)]\}\\
&= \max\{\frac{1-\lambda_1}{a},\frac{\lambda_2}{a}\}p(u,fu)
\end{aligned}$$

which is contradictive with $\max\{\frac{1-\lambda_1}{a},\frac{\lambda_2}{a}\} \in (0,1)$. So $p(u,fu)=0$. Similarly, we obtain that $p(u,u)=0$. By Lemma 1.3 again, we have that $u=fu$. $\square$

Since b-metric d is also a wt-distance on (X,d), then we obtain the following corollary.

Corollary 2.7. Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $V=\bigcup_{i=1}^{k} A_i$ and $A_1,A_2,\cdots,A_k$ be nonempty closed subsets of X, k be a positive integer, $f:V\to V$ is a cyclic generalized φ-contraction mapping for some $\varphi \in \Phi$ (where let $p = d$ in (2.5)), then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^{k} A_i$.

If let $\lambda_1 = \lambda_2 = \frac{1}{2}$ in Corollary 2.7, we obtain the Theorem 2.2 by given by H.K. Nashine and Z. Kadelburg [8].

Corollary 2.8. [8] Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $V = \bigcup_{i=1}^{k} A_i$ and $A_1, A_2, \cdots, A_k$ be nonempty closed subsets of X, k be a positive integer, $f : V \to V$ is a cyclic generalized φ-contraction mapping for some $\varphi \in \Phi$ (where let $p = d$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$ in (2.5)), then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^{k} A_i$.

Definition 2.9. Let (X,d) be a complete b-metric space with constant $s \geq 1$, p be a *wt*-distance on X and $p(x,x) = 0$ for any $x \in X$. $V = \bigcup_{i=1}^{k} A_i$ and $A_1, A_2, \cdots, A_k$ be nonempty closed subsets of X, k be a positive integer. If there exists $f : V \to V$ with $fA_i = A_{i+1}$ and $A_{k+1} = A_1$ such that

$$\psi(s^{\alpha} p(fx, fy)) \leq \psi\left(\frac{p(x, fy) + p(fx, y)}{s^{\varepsilon}}\right) - \varphi(p(fx, y)), \ \forall\, x, y \in V, \tag{2.15}$$

where $s^{\alpha+\varepsilon-1} > 2$, $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(x) = 0$ implies $x = 0$, then f is called the $(\psi, \varphi)-$weakly contractive.

Theorem 2.10. Let (X,d) be a complete b-metric space with constant $s \geq 1$ and p be a *wt*-distance on X, $p(x,x) = 0$ for any $x \in X$. $V = \bigcup_{i=1}^{k} A_i$ and $A_1, A_2, \cdots, A_k$ be nonempty closed subsets of X, k be a positive integer. If $f : V \to V$ is $(\psi, \varphi)-$weakly contractive, and suppose that either

(1) $inf\{p(x,w) + p(x, fx) : x \in X\} > 0$ for every $w \in X$ with $w \neq fw$;

or

(2) the mapping f is continuous.

Then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^{k} A_i$.

Proof. For any $x_0 \in A_1$ (such a point exists since $A_1 \neq \emptyset$), we can construct the sequence $\{x_n\}$ in X by $x_{n+1} = fx_n$, $n \in \mathbb{N} \cup \{0\}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then f has fixed point. Now, suppose that $x_n \neq x_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, we shall prove that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

Indeed, if not, we have that $p(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$, there exists $i = i(n) \in \{1, 2, \cdots, k\}$, such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$, then we have

$$\begin{aligned}\psi(s^{\alpha}p(fx_{n-1},fx_n)) &\leq \psi(\frac{p(x_{n-1},fx_n)+p(fx_{n-1},x_n)}{s^{\varepsilon}})-\varphi(p(fx_{n-1},x_n)) \\ &= \psi(\frac{p(x_{n-1},x_{n+1})}{s^{\varepsilon}})-\varphi(0) \\ &\leq \psi(\frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})}{s^{\varepsilon-1}})\end{aligned}$$

since ψ is nondecreasing, we have

$$s^{\alpha}p(fx_{n-1},fx_n)=s^{\alpha}p(x_n,x_{n+1}) \leq \frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})}{s^{\varepsilon-1}}$$

i.e.,

$$p(x_n,x_{n+1}) \leq \acute{c}p(x_{n-1},x_n)$$

where $\acute{c}=\frac{1}{s^{\alpha+\varepsilon-1}-1}<1$, then we have

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0 \tag{2.16}$$

by Lemma 2.2 we have that $\{x_n\}$ is a Cauchy sequence.

Since X is a complete space, there exists $u\in X$ such that

$$\lim_{n\to\infty}x_n=u. \tag{2.17}$$

We shall prove that $u\in\bigcap_{i=1}^{k}A_i$. By Definition, we have $x_0\in A_1$ and $\{x_{nk}\}\subseteq A_1$. Since A_1 is closed, we get that $u\in A_1$. Similarly, we have $\{x_{nk+1}\}\subseteq A_2$ and $u\in A_2$. By mathematical induction, we get that $u\in\bigcap_{i=1}^{k}A_i$.

By (2.4), we obtain that $\lim\limits_{n,m\to\infty}p(x_n,x_m)=0$. Then for any $\varepsilon>0$, there exists a $n>N_{\varepsilon}\in\mathbb{N}$ such that $p(x_{N_{\varepsilon}},x_n)<\frac{\varepsilon}{s}$.

By (2.17) and $p(x,\cdot)$ is s-lower semi-continuous, thus we have

$$p(x_{N_{\varepsilon}},u)\leq\liminf_{n\to\infty}sp(x_{N_{\varepsilon}},x_n)\leq\varepsilon$$

Let $\varepsilon=\frac{1}{t}$ and $N_{\varepsilon}=n_t$, then we have

$$\lim_{t\to\infty}p(x_{n_t},u)=0. \tag{2.18}$$

Next, we shall prove that the u is a fixed point of f.

Case (1), suppose that $fu \neq u$, then by (2.16) and (2.18), we have

$$0 < \inf\{p(x,u) + p(x,fx) : x \in X\} \leq \inf\{p(x_n,u) + p(x_n,x_{n+1}) : n \in N\} \to 0 \ (n \to \infty)$$

which is a contradiction, thus $fu = u$.

Case (2), suppose that there exists a $w \in X$ with $fw \neq w$ such that $\inf\{p(x,w) + p(x,fx) : x \in X\} = 0$, then there exists a sequence $\{y_n\} \subset X$ such that $p(y_n,w) + p(y_n,fy_n) \to 0$ as $n \to \infty$, thus we have

$$\lim_{n\to\infty} p(y_n,w) = 0 \ and \ \lim_{n\to\infty} p(y_n,fy_n) = 0. \tag{2.19}$$

Then by Lemma 1.3 (2), $fy_n \to w$ as $n \to \infty$.

Since

$$\begin{aligned} \psi(s^{\alpha} p(fy_n,f^2y_n)) &\leq \psi(\frac{p(y_n,f^2y_n) + p(fy_n,fy_n)}{s^{\varepsilon}}) - \varphi(p(fy_n,fy_n)) \\ &= \psi(\frac{p(y_n,f^2y_n)}{s^{\varepsilon}}) \end{aligned}$$

and by the condition that ψ is nondecreasing, then we have

$$\begin{aligned} s^{\alpha} p(fy_n,f^2y_n) &\leq \frac{p(y_n,f^2y_n)}{s^{\varepsilon}} \\ &\leq \frac{p(y_n,fy_n) + p(fy_n,f^2y_n)}{s^{\varepsilon-1}}, \end{aligned}$$

i.e.,

$$p(fy_n,f^2y_n) \leq \frac{1}{(s^{\alpha+\varepsilon-1} - 1)} p(y_n,fy_n) \to 0 \ (n \to \infty) \ (by \ (2.19)) \tag{2.20}$$

and by (2.19) and (2.20), we have

$$p(y_n,f^2y_n) \leq s(p(y_n,fy_n) + p(fy_n,f^2y_n)) \to 0 \ (n \to \infty). \tag{2.21}$$

Thus by (2.19), (2.21) and Lemma 1.3 (2), we obtain that $\lim_{n\to\infty} f^2y_n = w$.

By the continuity of f, we have $fw = f(\lim_{n\to\infty} fy_n) = \lim_{n\to\infty} f^2y_n = w$, which is a contradiction with the hypothesis. So case (1) always holds, and $u = fu$.

Finally, we shall prove the uniqueness of fixed point u of f.

Assume that there exists $v \in X$ such that $fv = v$ with $v \neq u$, then we have

$$\begin{aligned}\psi(s^{\alpha}p(u,v)) = \psi(s^{\alpha}p(fu,fv)) &\leq \psi(\frac{p(u,fv)+p(fu,v)}{s^{\varepsilon}}) - \varphi(p(fu,v)) \\ &= \psi(\frac{p(u,v)+p(u,v)}{s^{\varepsilon}}) - \varphi(p(u,v)) \\ &\leq \psi(\frac{2p(u,v)}{s^{\varepsilon}})\end{aligned}$$

then we have

$$\frac{s^{\alpha+\varepsilon}}{2}p(u,v) \leq p(u,v)$$

thus we get that

$$p(u,v) = 0,$$

where $s^{\alpha+\varepsilon} > 2s \geq 2$.

Similarly, we get that $p(u,u) = 0$, and by Lemma 1.3, we have $u = v$. □

We can get a more comfortable theorem if wt-distance p is symmetric.

Theorem 2.11. Let (X,d) be a complete b-metric space with constant $s \geq 1$ and p be a symmetric wt-distance on X, $p(x,x) = 0$ for any $x \in X$. $V = \bigcup_{i=1}^{k} A_i$ and $A_1, A_2, \cdots, A_k$ be nonempty closed subsets of X, k be a positive integer. If $f : V \to V$ is the $(\psi,\varphi)-$weakly contractive, then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^{k} A_i$.

Proof. By comparing Theorem 2.11 with Theorem 2.10, we find that we can omit the condition ”case (1) and case (2)” by the condition that wt-distance p is symmetric. By observing the proof of Theorem 2.10, we find that the condition ”case (1) and case (2)” is only used to prove the existence of fixed point u. So we continue using the similar notations in Theorem 2.10 to prove the existence of fixed point u by the condition that wt-distance p is symmetric.

Next, we shall prove that the u is a fixed point of f.

Since Cauchy sequence $\{x_n\} \subset X$ with $x_{n+1} = fx_n$ converges to $u \in X$. And by the symmetry of wt-distance p and (2.18), we have

$$\lim_{n\to\infty} p(u,x_n) = 0. \tag{2.22}$$

By (2.15), we have

$$\begin{aligned}
\psi(s^{\alpha}p(x_n,fu)) &= \psi(s^{\alpha}p(fx_{n-1},fu))\\
&\leq \psi(\frac{p(x_{n-1},fu)+p(fx_{n-1},u)}{s^{\varepsilon}})-\varphi(p(fx_{n-1},u))\\
&= \psi(\frac{p(x_{n-1},fu)+p(x_n,u)}{s^{\varepsilon}})-\varphi(p(x_n,u))\\
&\leq \psi(\frac{p(x_{n-1},fu)+p(x_n,u)}{s^{\varepsilon}}),
\end{aligned}$$

and by the condition that ψ is nondecreasing, then we have

$$\begin{aligned}
s^{\alpha}p(x_n,fu) &\leq \frac{p(x_{n-1},fu)+p(x_n,u)}{s^{\varepsilon}}\\
&\leq \frac{p(x_{n-1},x_n)+p(x_n,fu)}{s^{\varepsilon-1}}+\frac{p(x_n,u)}{s^{\varepsilon}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
p(x_n,fu) \leq& \frac{sp(x_{n-1},x_n)+p(x_n,u)}{s^{\alpha+\varepsilon}-s}\\
&\to 0\ (n\to\infty)\ (by\ (2.16),(2.18))
\end{aligned} \tag{2.23}$$

where $s^{\alpha+\varepsilon}-s>s\geq 1$.

By (2.22) and (2.23), we have

$$p(u,fu)\leq s[p(u,x_n)+p(x_n,fu)]\to 0\ (n\to\infty).$$

Since $p(u,u)=0$ and by Lemma 1.3 again, we have that $u=fu$. $\square$

Similarly, let $p=d$ in Theorem 2.11, we have the following corollary.

Corollary 2.12. Let (X,d) be a complete b-metric space with constant $s\geq 1$, $V=\bigcup_{i=1}^{k}A_i$ and $A_1,A_2,\cdots,A_k$ be nonempty closed subsets of X, k be a positive integer. If $f:V\to V$ is the $(\psi,\varphi)-$weakly contractive, then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^{k}A_i$.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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