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COMMON FIXED POINT THEOREM IN MENGER SPACES USING T-NORM T OF HADŽIĆ-TYPE

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Abstract. In this paper, we prove a common fixed point theorem using t-norm T of Hadžić- type (H-type). In fact our result is a generalization of the result of Choudhury and Das [1] under more general condition, that answer to the open problem of Choudhury and Das [1].

Keywords: Menger spaces; φ - contraction; weakly compatible mappings; t-norm T of Hadžić-type.

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1. Introduction

In 1922, Banach proved an important result which is the mile stone in the fixed point theory and its applications. A new class of fixed point problems in metric spaces was addressed by Khan et al. [4] . They proved fixed point theorem for mappings satisfying certain inequalities involving the altering distances function.

In 1942, Menger [5] introduced the notion of probabilistic metric space or statistical metric space, which is in fact, a generalization of metric space. The idea in probabilistic metric space is to associate a distribution function with a point pairs, say (p,q), denoted by F(p,q;t) where t > 0 and identify this function as the probability that distance between p and q is less than t . Sehgal and Reid A.T.Bharucha [12] initiated the study of contraction mapping theorems in PM-spaces. Subsequently, several contraction mapping theorems for different variants of

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commuting and compatible mappings have been proved in PM-spaces.Various aspects of this theory have been elaborately discussed in the book of Hadžić and Pap[3].

Recently Choudhury et. al. [1] extended the idea of altering distances in probabilistic metric spaces and proved a contraction principle in Menger spaces using t-norm T_M given by T_M (a, b) = min{a,b} and put an open problem that whether contraction Principle is valid for anyother choice of the t-norm.

Now in this paper, we prove a common fixed point theorem using t-norm T of Hadžić-type (H-type for short) that answer to the open problem of Choudhury and Das [1].

2. Preliminaries

First, we recall that a real valued function defined on the set of real numbers is known as a distribution function if it is non-decreasing, left continuous and inf f(x) = 0 , sup f(x) = 1. An example of a distribution function is the Heavy side function H(x), defined by

H(x) = 0 if x ≤ 0 and H(x)=1 if x > 0.

Definition 2.1.[3] A mapping T: [0, 1]×[0, 1] → [0, 1] is called a continuous t-norm if the following conditions are satisfied:

T(a, 1) = a for every a ∈ [0, 1];

T(a, b) = T(b, a) for every a,b∈ [0, 1];

T(T (a, b), c) = T(a, T(b, c));

T(a, b) ≤ T(c, d), for a ≤ c, b ≤ d.

Basic examples of t-norm are the Lukasiewicz t-norm T_L, T_L(a,b) = Max(a+b−1,0) , t-norm T_P, T_P(a,b) = ab, and t-norm T_M, T_M(a,b) = Min{a,b}, $T_D(x,y) = \begin{cases} \min(x,y) \; if \; \max(x,y) = 1, \\ 0 \qquad\qquad\qquad otherwise. \end{cases}$

Definition2.2. [3] A Menger space is a triplet (X, F, T), where X is a non-empty set, F is a function defined on X × X to L_+(setof all distribution functions) which satisfies the following conditions :

(i) $F_{xy}(0) = 0$,

(ii) $F_{xy}(s) = 1$ for all s > 0 iff x = y,

(iii) $F_{xy}(s) = F_{yx}(s)$,

(iv) $F_{xy}(u+v) \geq T(F_{xz}(u), F_{zy}(v))$ for all u, v ≥ 0 and x, y, z ∈ X where T is a t-norm.

For a given metric space (X, d) with usual metric d, one can put $F_{xy}(t) = H(t - d(x, y))$, where H is defined as:

$$H(s) = \begin{cases} 1 & if s > 0, \\ 0 & if s \leq 0 \end{cases}$$

andt-norm T is defined as T(a, b) = min{a, b}.

If (X, F, T) is a Menger space with continuous t-norm then the topology induced by the family $\{S_{\epsilon\lambda}(p) : p \in X, \epsilon > 0, \lambda > 0\}$ is called the (ϵ- λ) – topology, where $S_{\epsilon\lambda}(p) = \{q \in X : F_{pq}(\epsilon) > 1-\lambda\}$ is called the (ϵ- λ) – neighborhood of p.

A sequence $\{x_n\} \subset X$ is said to be

(i) converge to some point x ∈ X in the (ϵ- λ) – topology if and only if given $\epsilon > 0$, $\lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that, for all n >$N_{\epsilon,\lambda}$, $F_{x_n x}(\epsilon) \geq 1-\lambda$.

(ii) a Cauchy sequence in X if given $\epsilon > 0$, $\lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that $F_{x_n x_m}(\epsilon) \geq 1-\lambda$ for all m, n >$N_{\epsilon,\lambda}$.

A Menger space (X, F, T) is said to be complete if every Cauchy sequence is convergent.

In 1979, Hadzic [2] introduced a special class of t-norms (called as a Hadžić- typenorm) as follows:

Definition 2.3.[2]Let T be a t-norm and let $T_n : [0, 1] \to [0, 1]$ ($n \in \mathbb{N}$) be defined in the following way,

$$T_1(x) = T(x, x),\ T_{n+1}(x) = T(T_n(x), x) \quad (n \in \mathbb{N},\ x \in [0, 1]).$$

We say that the t-norm T is of H-type if T is continuous and the family $\{T_n(x), n \in \mathbb{N}\}$ is equicontinuous at x = 1.

The family $\{T_n(x), n \in \mathbb{N}\}$ is equicontinuous at x = 1, if for every $\lambda \in (0, 1)$ there exists $\delta(\lambda) \in (0, 1)$ such that the following implication holds:

$x > 1-\delta(\lambda)$ implies $T_n(x) > 1 - \lambda$ for all $n \in \mathbb{N}$.

A trivial example of t-norm of H-type is $T = T_M$(T_M (a, b) = min{a, b}).

Remark 2.4. Every t-norm $T_M$is of Hadžić-type but converse need not be true, see [3].

There is a nice characterization of continuous t-norm T of H-type[8] as given below:

(i) If there exists a strictly increasing sequence $\{b_n\}_{n \in N}$in [0,1] such that $\lim_{n\to\infty} b_n = 1$ and $T(b_n,b_n) = b_n \forall\ n \in N$, then T is of Hadžić-type.

(ii) If T is continuous and T is of Hadžić-type, then there exists a sequence $\{b_n\}_{n \in N}$as in (i).

Definition 2.5. [3] If T is a t-norm and $(x_1,x_2, \ldots ,x_n) \in [0,1]^n$ ($n \in N$), then $T^n_{i=1} x_i$ is defined recurrently by 1, if n = 0 and $T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i ,x_n)$ for all $n \geq 1$. If $\{x_i\}_{i \in N}$ is a sequence of numbers from[0,1], then $T^\infty_{i=1} x_i$ is defined as $\lim_{n\to\infty} T^n_{i=1} x_i$ (this limit always exists) and $T^\infty_{i=n} x_i$

as $T^{\infty}_{i=1} x_{n+i}$. In fixed point theory in probabilistic metric spaces there are of particular interest t-norms T and sequences $\{x_n\} \subset [0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} T^{\infty}_{i=1} x_{n+i} = 1$.

In 1972, Sehgal and Bharucha-Reid [12] introduced the idea of contraction in PM space.

Definition 2.6. Probabilistic q-contraction [3] Let (X, F) be a probabilistic metric space. A mapping f: X → X is a probabilistic q-contraction ($q \in (0, 1)$) if $F_{fu\,fv}(x) \geq F_{u\,v}(\frac{x}{q})$ for every u, v ∈ X and every x ∈ R.

The following Theorem was proved by Sehgal and Bharucha-Reid [12].

Theorem 2.7. Let (X, F, T_M) be a complete Menger space where $T_M(a, b) = \min\{a, b\}$ and f : X → X is a probabilistic q-contraction. Then there exist a unique fixed point x of the mapping f and $x = lim_{n\to\infty} f^n p$ for every p ∈ X.

In 1984, Khan et al. [4] Introduceda new category of contractive fixed point problems using a control function (altering distance function) that alters the distance between two points in a metric space.

Definition 2.8. [4] An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ such that

(i) which is monotone increasing and continuous and

(ii) $\psi(t) = 0$ if and only if t = 0.

Khan et. al. [4] proved the following result using altering distance function.

Theorem 2.9. [4] Let (X, d) be a complete metric space and ψ be an altering distance function. Let f : X → X be a self mapping which satisfies the following inequality:

$\psi(d(fx, fy)) \leq c\psi(d(x, y))$, for all x, y ∈ X and for some $0 < c < 1$.

Then f has a unique fixed point.

In fact, Khan et. al.[4] proved a more general fixed point theorem (Theorem 2 in [4]) of which the above result is a corollary. This result was further generalized in a different direction by various authors. One can refer to [7], [9] and [10].

Recently, Choudhury et. al. [1] extended the idea of altering distance function in Menger spaces and obtained fixed point results for self-mapping using φ function.

Definition 2.10. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the condition **(φ)** if it satisfies the following conditions:

(i) $\varphi(t) = 0$ if and only if t = 0,

(ii) $\varphi(t)$ is increasing and $\varphi(t) \to \infty$ as $t \to \infty$,

(iii) φ is left continuous in $(0, \infty)$,

(iv) φ is continuous at 0,

(v) $\varphi$is superadditive, that is, $\varphi(x + y) \geq \varphi(x) + \varphi(y)$, for all x, y ≥ 0.

Definition 2.11.[1] Let (X, F, T) be a Menger space. A self map f : X → X is said to be φ-contractive if

(*) $F_{fxfy}\,(\varphi(t)) \geq F_{xy}(\varphi(\frac{t}{c}))$, where 0 < c < 1, x, y ∈ X and t > 0 and the function φ satisfy the condition **(φ).**

Definition 2.12. Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

Example 2.13. Let X = [0, 1] be equipped with the usual metric d(x,y) = |x-y|.

Define f, g: [0, 1] → [0, 1] by

$$f\,x = \begin{cases} 0 \ \ if\ x = 0, \\ 0.15\ if\ x > 0, \end{cases}; \qquad g\,x = \begin{cases} 0 \ \ if\ x = 0, \\ 0.35\ if\ x > 0, \end{cases}$$

Then, 0 is a coincidence point and fg 0 = gf 0, showing that f, g are weakly compatible maps on [0, 1].

Proposition 2.14. [7] Let (x_n, n ∈ℕ) be a sequence of numbers in [0, 1] such that $\lim_{n\to\infty} x_n = 1$ and the t-norm T is of H-type, then

$\lim_{n\to\infty} T_{i=n}^{\infty} x_i = \lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1$.

Throughout this paper, (X, F, T) will denote a Menger space which satisfies the condition $\lim_{t\to\infty} F_{xy}(t) = 1$for all x, y ∈ X and t > 0.

3. Main Result

Recently, Choudhury et. al. [1] proved the following fixed point theorem using continuous t-norm T_M, which is strongest t-norm.

Theorem 3.1. Let (X, F,T_M) be a Menger space with continuous t-norm T_M and f : X → X be φ-contractive satisfying (*).Then f has a fixed point.

Now we prove our main result for a pair of weakly compatible mapsusing continuous t-norm T of H- type.

Theorem 3.2. Let (X, F,T) be a complete Menger space with continuous t-norm T of H- type andlet f, g be two self-mappings on X satisfy the following inequality:

(3.1) f (X) ⊆ g (X),

(3.2) any one of f (X) and g (X) is complete, (3.3) F_{fxfy} $(\varphi(t)) \geq F_{gxgy}(\varphi(\frac{t}{c}))$, where 0 < c < 1, x, y ∈ X and t > 0 and the function φ satisfy the condition **(φ).**

For any x_0∈ X, the sequence $\{y_n\}$ in X be constructed as follows : $y_n = fx_n = gx_{n+1}$, n = 0,1, 2,3,… and for μ ∈ (c, 1) the following condition holds:

$lim_{n\to\infty} T_{i=n}^{\infty} F_{y_0 y_1}(\frac{1}{\mu^i}) = 1.$

Then f and g have a unique common fixed point provided f and g are weakly compatible on X.

Proof: In view of the properties of **(φ)**-function, for u> 0 we can find a positive number r such that u > φ(r). For u > 0, we have

$$\begin{aligned} F_{y_n y_{n+1}}(u) &\geq F_{fx_n fx_{n+1}}(\varphi(r)) \\ &\geq F_{gx_n gx_{n+1}}(\varphi(\tfrac{r}{c})) \\ &= F_{y_{n-1} y_n}(\varphi(\tfrac{r}{c})) \\ &= F_{fx_{n-1} fx_n}(\varphi(\tfrac{r}{c})) \\ &\geq F_{gx_{n-1} gx_n}(\varphi(\tfrac{r}{c^2})) \\ &= F_{y_{n-2} y_{n-1}}(\varphi(\tfrac{r}{c^2})) \\ &\;\;\cdots \\ &\geq F_{y_0 y_1}(\varphi(\tfrac{r}{c^n})). \end{aligned}$$

Therefore,

$$F_{y_n y_{n+1}}(u) \geq F_{y_0 y_1}(\varphi(\tfrac{r}{c^n})).$$

Proceeding limit as n→ ∞,we have $lim_{n\to\infty} F_{y_n y_{n+1}}(u) = 1$.

We claim that the sequence $\{y_n\}$ is a Cauchy sequence.

Let, $\sigma = \frac{c}{\mu}$, where μ ∈ (c, 1) and c ∈ (0, 1), then $0 < \sigma < 1$, therefore the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists m_0∈ N such that $\sum_{i=m_0}^{\infty} \sigma^i < 1$.Now for every m > m_0 and for every s ∈ N and in view of **(φ)**,

$$u > \varphi(r) > \varphi(r\textstyle\sum_{i=m_0}^{\infty} \sigma^i) > \varphi(r \sum_{i=m}^{m+s} \sigma^i) \text{ which implies that}$$

$F_{y_{m+s+1} y_m}(u) > F_{y_{m+s+1} y_m}(\varphi(r))$

$\geq F_{y_{m+s+1} y_m} \varphi(r\sum_{m}^{m+s} \sigma^i))$

$\geq \underbrace{T(T\ldots T}_{s-times} (F_{y_{m+s+1} y_{m+s}} \varphi(r\sigma^{m+s}) ,\ F_{y_{m+s} y_{m+s-1}} \varphi(r\ \sigma^{m+s-1}),$

$. . . , F_{y_{m+1}y_m}\varphi(\text{ r}\sigma^m)))$

$\geq \underbrace{T(T\ldots T}_{s-times} (F_{y_0y_1}\varphi(\frac{r\sigma^{m+s}}{c^{m+s}}), . . ., F_{y_0y_1}\varphi(\frac{r\sigma^m}{c^m})))$

$\geq T_{i=m}^{m+s}F_{y_0y_1}\varphi(\frac{r}{\mu^i})$

$=T_{i=m}^{\infty}F_{y_0,y_1}\varphi(\frac{r}{\mu^i})$.

It is obvious that,

$lim_{n\to\infty}T_{i=n}^{\infty}F_{y_0y_1}(\frac{1}{\mu^i}) = 1$, implies that, $lim_{n\to\infty}T_{i=n}^{\infty}F_{y_0y_1}\varphi(\frac{1}{\mu^i}) = 1$, and this implies that,

$lim_{n\to\infty}T_{i=n}^{\infty}F_{y_0y_1}\varphi(\frac{r}{\mu^i}) = 1$, for every r > 0.

Now for every u > 0, there exists r > 0 such that u >φ(r) > 0, there exist $m_1(\varphi(r), \lambda)$ such that $F_{y_{m+s+1}y_m}(u) > 1-\lambda$, for every m $\geq m_1(\varphi(r), \lambda)$ and every s $\in$ N.

This means that the sequence $\{y_n\}$ is Cauchy sequence.Since either f(X) or g(X) is complete, for definiteness assume that g(X) is complete subspace of X then the subsequence of $\{y_n\}$ must get a limit in g(X). Call it be z. Let $p \in g^{-1}z$. Then g p = z as $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also convergent implying thereby the convergence of subsequence of the convergent sequence.

Which gives, $lim_{n\to\infty}\, y_n = lim_{n\to\infty}\, gx_n = lim_{n\to\infty}\, fx_n = z$.

Now we claim that fp = z.

From the property of ($\boldsymbol{\varphi}$), it follows that given ϵ> 0, we can find ϵ_1> 0 such that $\epsilon>\varphi(\epsilon_1) > 0$.

Then for all n = 0, 1, 2, 3, …,

$$F_{fpz}(\epsilon) \geq T(F_{fpy_n}\varphi(\epsilon_1)), F_{y_nz}(\epsilon-\varphi(\epsilon_1)))$$

$$= T(F_{fpfx_n}\varphi(\epsilon_1)), F_{y_nz}(\epsilon-\varphi(\epsilon_1)))$$

$$\geq T(F_{gpgx_n}(\varphi(\frac{\epsilon_1}{c})), F_{y_nz}(\epsilon-\varphi(\epsilon_1)))$$

$$= T(F_{zy_{n-1}}(\varphi(\frac{\epsilon_1}{c})), F_{y_nz}(\epsilon-\varphi(\epsilon_1))).$$

Since T is continuous, taking limit as n→ ∞ in the above inequality, we have for all ϵ> 0, $F_{fpz}(\epsilon)$ = 1, i.e., fp = z, we get fp = gp = z, since f and g are weakly compatible therefore we have fg p = gf p, i.e., fz = gz.

We claim that fz = z, from (3.3), we have

$F_{fzz}(\varphi(t)) = F_{fzfp}(\varphi(t)) \geq F_{gzgp}(\varphi(\frac{t}{c})) = F_{fzfp}(\varphi(\frac{t}{c})) \geq F_{gzgp}(\varphi(\frac{\epsilon_1}{c^2}))$

Proceeding as above, for any t > 0, $F_{fzz}(\varphi(\mathrm{t})) \geq F_{fzz}(\varphi(\frac{\epsilon_1}{c^n})) \to 1$ as n→ ∞, which gives fz = z = gz. Thus z is a common fixed point of f and g.

Uniqueness.

If possible let w and v be two fixed points of f and g, then in view of **($\boldsymbol{\varphi}$)** for given ϵ> 0, we can find ϵ_1> 0 such that $\epsilon > \varphi(\epsilon_1) > 0$. Then one can see that

$F_{wv}(\epsilon) = F_{fwfv}(\epsilon)$

$\geq F_{fwfv}(\varphi(\epsilon_1))$

$\geq F_{gwgv}(\varphi(\frac{\epsilon_1}{c}))$

$= F_{fwfv}(\varphi(\frac{\epsilon_1}{c}))$

$\geq F_{gwgv}(\varphi(\frac{\epsilon_1}{c^2})$

$= F_{wv}(\varphi(\frac{\epsilon_1}{c^2}))$.

Proceeding as above, for any $\epsilon > 0$, $F_{wv}(\epsilon) \geq F_{wv}(\varphi(\frac{\epsilon_1}{c^n})) \to 1$ as n→ ∞, which gives w = v.

Next we give the following example to validate our result

Example 3.3. Let X {a, b, c, d}, T_M is the t-norm and F be defined as

$$F_{ab}(t) = F_{ac}(t) = F_{ad}(t) = \begin{cases} 0 \; if \; t \leq 0, \\ 0.4 \; if \; 0 < t < 4, \\ 1 \; if \; t \geq 4. \end{cases}$$

$$F_{bc}(t) = F_{bd}(t) = F_{cd}(t) = \begin{cases} 0 \; if \; t \leq 0, \\ 1 \; if \; t > 0. \end{cases}$$

Then (X, F, T_M) is a complete Menger space.

If we define f, g: X →X as follows:

f(a) = d, f(b) = c, f(c) = c, f(d) = d, andg(a) = d, g(b) = c, g(c) = c, g(d) = c, where φ(t) = t and c is the unique common fixed point of f and g, then the mappings f and g satisfy all the conditions of the Theorem 3.2.

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