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COMMON FIXED POINTS FOR SOME VARIANTS OF WEAKLY CONTRACTION MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

BAPURAO C. DHAGE[1,*], HEMANT K. NASHINE[2] AND VILAS S. PATIL[1]

[1]Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India
[2]Department of Mathematics, Disha Institute of Management and Technology, Satya Vihar, Vidhansabha-Chandrakhuri Marg, Mandir Hasaud, Raipur-492101(Chhattisgarh), India

Abstract. In this paper, we introduce two new variants of weakly contraction mappings for two and three maps in metric spaces and derive a couple of common fixed point theorems for $\mathcal{T}$-strictly weakly isotone increasing mappings and relatively weakly increasing mapping in partially ordered complete metric spaces. Our results are illustrated by giving some examples.

Keywords: Partially ordered set, asymptotically regular map, orbitally complete metric space, orbital continuity, weakly increasing map and fixed point theorem etc.

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1. Introduction

The Banach contraction mapping principle [6] is a very popular tool in solving existence problems in many branches of mathematical analysis. This famous theorem can be stated as follows.

Theorem 1.1. [6]. Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ be a mapping of $\mathcal{X}$ into itself satisfying:

$$d(\mathcal{T}x, \mathcal{T}y) \leq k\, d(x, y),\ \forall\, x, y \in \mathcal{X}, \tag{1.1}$$

where $k \in (0, 1)$. Then, $\mathcal{T}$ has a unique fixed point.

*Corresponding author

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There are great number of generalizations of the Banach maping contraction principle. In this connection, Chatterjea [9] introduced the notion of $\mathcal{C}$-contraction as follows.

Definition 1.1. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, where $(\mathcal{X}, d)$ is a metric space is said to be a $\mathcal{C}$-contraction if there exists a $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$, we have

$$d(\mathcal{T}x, \mathcal{T}y) \leq \alpha(d(x, \mathcal{T}y) + d(y, \mathcal{T}x)).$$

Chatterjea [9] proved that if $\mathcal{X}$ is complete, then every $\mathcal{C}$-contraction has a unique fixed point. Choudhury [10] introduced a notion of weakly $\mathcal{C}$-contraction as a generalization of $\mathcal{C}$-contraction.

Definition 1.2. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, where $(\mathcal{X}, d)$ is a metric space is said to be weakly $\mathcal{C}$-contractive (or a weak $\mathcal{C}$-contraction) if for all $x, y \in \mathcal{X}$,

$$d(\mathcal{T}x, \mathcal{T}y) \leq \frac{1}{2}[d(x, \mathcal{T}y) + d(y, \mathcal{T}x)] - \psi(d(x, \mathcal{T}y), d(y, \mathcal{T}x)),$$

where $\psi : [0, +\infty)^2 \to [0, +\infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

In [10], the author proves that if $\mathcal{X}$ is complete then every weakly $\mathcal{C}$-contraction has a unique fixed point.

Next, Browder and Petryshyn introduced the following concept of asymptotic regularity of a self-map at a point in a metric space.

Definition 1.3. [8] A self-map $\mathcal{T}$ of a metric space $(\mathcal{X}, d)$ is said to be asymptotically regular at a point $x \in \mathcal{X}$ if $\lim_{n\to\infty} d(\mathcal{T}^n x, \mathcal{T}^{n+1}x) = 0$.

Recall that the set $\mathcal{O}(x_0; \mathcal{T}) = \{\mathcal{T}^n x_0 : n = 0, 1, 2, \dots\}$ is called the orbit of the self-map $\mathcal{T}$ at the point $x_0 \in \mathcal{X}$.

Definition 1.4. [11] A metric space $(\mathcal{X}, d)$ is said to be $\mathcal{T}$-orbitally complete if every Cauchy sequence contained in $\mathcal{O}(x; \mathcal{T})$ (for some x in $\mathcal{X}$) converges to a point in $\mathcal{X}$.

Here, it can be pointed out that every complete metric space is $\mathcal{T}$-orbitally complete, but a $\mathcal{T}$-orbitally complete metric space need not be complete.

Definition 1.5. [8] A self-map $\mathcal{T}$ defined on a metric space $(\mathcal{X}, d)$ is said to be orbitally continuous at a point z in $\mathcal{X}$ if for any sequence $\{x_n\} \subset \mathcal{O}(x; \mathcal{T})$ (for some $x \in \mathcal{X}$), $x_n \to z$ as $n \to \infty$ implies $\mathcal{T}x_n \to \mathcal{T}z$ as $n \to \infty$.

Clearly, every continuous self-mapping of a metric space is orbitally continuous, but not conversely.

Sastry et al. [28] extended the above concepts to two and three mappings and employed them to prove common fixed point results for commuting mappings. In what follows, we collect such definitions for three maps.

Definition 1.6. Let $\mathcal{S}, \mathcal{T}, \mathcal{R}$ be three self-mappings defined on a metric space $(\mathcal{X}, d)$.

(1) If for a point $x_0 \in \mathcal{X}$, there exits a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{R}x_{2n+1} = \mathcal{S}x_{2n}$, $\mathcal{R}x_{2n+2} = \mathcal{T}x_{2n+1}$, $n = 0, 1, 2, \dots$, then the set $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R}) = \{\mathcal{R}x_n : n = 1, 2, \dots\}$ is called the orbit of $(\mathcal{S}, \mathcal{T}, \mathcal{R})$ at x_0.

(2) The space $(\mathcal{X}, d)$ is said to be $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0 if every Cauchy sequence in $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$ converges in $\mathcal{X}$.

(3) The map $\mathcal{R}$ is said to be orbitally continuous at x_0 if it is continuous on $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$.

(4) The pair $(\mathcal{S}, \mathcal{T})$ is said to be asymptotically regular (in short a.r.) with respect to $\mathcal{R}$ at x_0 if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{R}x_{2n+1} = \mathcal{S}x_{2n}$, $\mathcal{R}x_{2n+2} = \mathcal{T}x_{2n+1}$, $n = 0, 1, 2, \dots$, and $d(\mathcal{R}x_n, \mathcal{R}x_{n+1}) \to 0$ as $n \to \infty$.

(5) If $\mathcal{R}$ is the identity mapping on $\mathcal{X}$, we omit "$\mathcal{R}$" in the respective definitions.

On the other hand, fixed point theory has developed rapidly in metric spaces endowed with a partial order. The first result in this direction was given by Ran and Reurings [27] who presented its applications to matrix equations. Subsequently, Nieto and Rodríguez-López [25] extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed point theorems in ordered metric spaces. For more details see [1, 3, 7, 4, 5, 14, 19, 20, 22, 23, 26, 29, 30] and the references cited therein.

In this paper we generalize the results of Harjani and Sadarangani [14] (and, hence, some other related common fixed point results) in two directions. The first is treated in Section 3, where the notion of generalized weakly $(\mathcal{T}, \mathcal{S})$-contractive condition is introduced in metric spaces. The existence and (under additional assumptions) uniqueness of their common fixed point is obtained under assumptions that these mappings are $\mathcal{T}$-strictly weakly isotone increasing and they satisfy a generalized weakly $(\mathcal{T}, \mathcal{S})$-contractive condition.

In Section 4 we consider the case of three self-mappings $\mathcal{S}, \mathcal{T}, \mathcal{R}$ where the pair $\mathcal{S}, \mathcal{T}$ is $\mathcal{R}$-relatively asymptotically regular and relatively weakly increasing, while the new contractive condition, known as generalized weakly $(\mathcal{T}, \mathcal{S}, \mathcal{R})$-contractive is introduced. We also furnish suitable examples to verify the hypotheses of our results.

2. Notation and definitions

First, we introduce some further notation and definitions that will be used later.

If $(\mathcal{X}, \preceq)$ is a partially ordered set then $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be totally ordered if every two elements of $\mathcal{K}$ are comparable. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is such that, for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$, then the mapping $\mathcal{T}$ is said to be non-decreasing.

Definition 2.1. Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$.

(1) [12, 13] The pair $(\mathcal{S}, \mathcal{T})$ is called weakly increasing if $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x$ and $\mathcal{T}x \preceq \mathcal{S}\mathcal{T}x$ for all $x \in \mathcal{X}$.

(2) [12, 13, 31] The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-weakly isotone increasing if for all $x \in \mathcal{X}$ we have $\mathcal{S}x \preceq \mathcal{T}\mathcal{S}x \preceq \mathcal{S}\mathcal{T}\mathcal{S}x$.

(3) [24] The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-strictly weakly isotone increasing if, for all $x \in \mathcal{X}$ such that $x \prec \mathcal{S}x$, we have $\mathcal{S}x \prec \mathcal{T}\mathcal{S}x \prec \mathcal{S}\mathcal{T}\mathcal{S}x$.

(4) [22] Let $\mathcal{R} : \mathcal{X} \to \mathcal{X}$ be such that $\mathcal{T}\mathcal{X} \subseteq \mathcal{R}\mathcal{X}$ and $\mathcal{S}\mathcal{X} \subseteq \mathcal{R}\mathcal{X}$, and denote $\mathcal{R}^{-1}(x) := \{u \in \mathcal{X} : \mathcal{R}u = x\}$, for $x \in \mathcal{X}$. We say that $\mathcal{T}$ and $\mathcal{S}$ are weakly increasing with respect to $\mathcal{R}$ if and only if for all $x \in \mathcal{X}$, we have:

$$\mathcal{T}x \preceq \mathcal{S}y, \ \forall\, y \in \mathcal{R}^{-1}(\mathcal{T}x)$$

and

$$\mathcal{S}x \preceq \mathcal{T}y, \ \forall\, y \in \mathcal{R}^{-1}(\mathcal{S}x).$$

Remark 2.1. (1) None of two weakly increasing mappings need be non-decreasing. There exist some examples to illustrate this fact in [2].

(2) If $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are weakly increasing, then $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing.

(3) $\mathcal{S}$ can be $\mathcal{T}$-strictly weakly isotone increasing, while some of these two mappings can be not strictly increasing (see the following example).

(4) If $\mathcal{R}$ is the identity mapping ($\mathcal{R}x = x$ for all $x \in \mathcal{X}$), then $\mathcal{T}$ and $\mathcal{S}$ are weakly increasing with respect to $\mathcal{R}$ if and only if they are weakly increasing mappings.

Example 2.1. Let $\mathcal{X} = [0, +\infty)$ be endowed with the usual ordering and define $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{S}x = \begin{cases} 2x, & \text{if } x \in [0,1], \\ 3x, & \text{if } x > 1; \end{cases} \qquad \mathcal{T}x = \begin{cases} 2, & \text{if } x \in [0,1], \\ 2x, & \text{if } x > 1. \end{cases}$$

Clearly, we have $x \prec \mathcal{S}x \prec \mathcal{T}\mathcal{S}x \prec \mathcal{S}\mathcal{T}\mathcal{S}x$ for all $x \in \mathcal{X}$, and so, $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing; $\mathcal{T}$ is not strictly increasing.

Definition 2.2. [16, 17]. Let $(\mathcal{X}, d)$ be a metric space and $f, g : \mathcal{X} \to \mathcal{X}$.

(1) If $w = fx = gx$, for some $x \in \mathcal{X}$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. If $w = x$, then x is a common fixed point of f and g.

(2) The mappings f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Definition 2.3. Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called a partially ordered metric space if

(i) $(\mathcal{X}, d)$ is a metric space,

(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

The space $(\mathcal{X}, d, \preceq)$ is called regular if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in $\mathcal{X}$ with respect to $\preceq$ such that $z_n \to z \in \mathcal{X}$ as $n \to \infty$, then $z_n \preceq z$.

Definition 2.4. [1] Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$. The mapping $\mathcal{T}$ is called dominating if $x \preceq \mathcal{T}x$ for each $x \in \mathcal{X}$.

3. Common fixed points for Generalized weakly $(\mathcal{T}, \mathcal{S})$-contraction mappings

In the following we introduce the notion of generalized weakly $(\mathcal{T}, \mathcal{S})$-contraction in metric space. For convenience, we denote by $\mathbf{F}$ the class of functions $\psi : [0, +\infty)^4 \to [0, +\infty)$ lower semicontinuous satisfying $\psi(x, y, z, t) = 0$ if and only if $x = y = z = t = 0$.

Definition 3.1. Let $(\mathcal{X}, d)$ be a metric space. Two mappings $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are called a generalized weakly $(\mathcal{S}, \mathcal{C})$-contraction if

$$\begin{aligned} d(\mathcal{T}x, \mathcal{S}y) \leq & \frac{1}{4}[d(x, \mathcal{T}x) + d(y, \mathcal{S}y) + d(x, \mathcal{S}y) + d(y, \mathcal{T}x)] \\ & - \psi(d(x, \mathcal{T}x), d(y, \mathcal{S}y), d(x, \mathcal{S}y), d(y, \mathcal{T}x)) \end{aligned} \tag{3.1}$$

for any $x, y \in \mathcal{X}$ and $\psi \in \mathbf{F}$.

Now, we state and prove our first result.

Theorem 3.1. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Suppose $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are two mappings satisfying generalized weakly $(\mathcal{T}, \mathcal{S})$-contractions conditions for all comparable $x, y \in \mathcal{X}$.*

We assume the following hypotheses:

(i) *$\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing;*

(ii) *there exists an $x_0 \in \mathcal{X}$ such that $x_0 \prec \mathcal{S}x_0$;*

(iii) *$\mathcal{S}$ or $\mathcal{T}$ is continuous at x_0;*

Then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$ is totally ordered if and only if $\mathcal{S}$ and $\mathcal{T}$ have one and only one common fixed point.

Proof. First of all we show that, if $\mathcal{S}$ or $\mathcal{T}$ has a fixed point, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Indeed, let z be a fixed point of $\mathcal{S}$. Now assume $d(z, \mathcal{T}z) > 0$. If we use the inequality (3.1), for

$x = y = z$, we have

$$\begin{aligned} d(\mathcal{T}z, z) = d(\mathcal{T}z, \mathcal{S}z)) &\leq \frac{1}{4}[d(z, \mathcal{T}z) + d(z, \mathcal{S}z) + d(z, \mathcal{S}z) + d(z, \mathcal{T}z)] \\ &- \psi(d(z, \mathcal{T}z), d(z, \mathcal{S}z), d(x, \mathcal{S}z), d(z, \mathcal{T}z)) \\ &= \frac{1}{2}d(z, \mathcal{T}z) - \psi(d(z, \mathcal{T}z), 0, 0, d(z, \mathcal{T}z)) \end{aligned}$$

whence $\psi(d(z, \mathcal{T}z), 0, 0, d(z, \mathcal{T}z)) \leq 0$, which is a contradiction. Thus by the property of ψ, we have $d(z, \mathcal{T}z) = 0$ and so z is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Analogously, one can observe that if z is a fixed point of $\mathcal{T}$, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Let x_0 be such that $x_0 \prec \mathcal{S}x_0$. We can define a sequence $\{x_n\}$ in $\mathcal{X}$ as follows:

$$x_{2n+1} = \mathcal{S}x_{2n} \text{ and } x_{2n+2} = \mathcal{T}x_{2n+1} \text{ for } n \in \{0, 1, \ldots\}. \tag{3.2}$$

Since $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing, we have

$$\begin{aligned} x_1 &= \mathcal{S}x_0 \prec \mathcal{T}\mathcal{S}x_0 = \mathcal{T}x_1 = x_2 \prec \mathcal{S}\mathcal{T}\mathcal{S}x_0 = \mathcal{S}\mathcal{T}x_1 = \mathcal{S}x_2 = x_3, \\ x_3 &= \mathcal{S}x_2 \prec \mathcal{T}\mathcal{S}x_2 = \mathcal{T}x_3 = x_4 \prec \mathcal{S}\mathcal{T}\mathcal{S}x_2 = \mathcal{S}\mathcal{T}x_3 = \mathcal{S}x_4 = x_5. \end{aligned}$$

Continuing this process we get

$$x_1 \prec x_2 \prec \ldots \prec x_n \prec x_{n+1} \prec \ldots. \tag{3.3}$$

Now we claim that for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}). \tag{3.4}$$

From (3.2) we have that $x_n \prec x_{n+1}$ for all $n \in \mathbb{N}$. Then from (3.1) with $x = x_{2n+1}$ and $y = x_{2n}$, we get

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(\mathcal{T}x_{2n-1}, \mathcal{S}x_{2n}) \\ &\leq \frac{1}{4}[d(x_{2n-1}, \mathcal{T}x_{2n-1}) + d(x_{2n}, \mathcal{S}x_{2n}) + d(x_{2n-1}, \mathcal{T}x_{2n}) + d(x_{2n}, \mathcal{S}x_{2n-1})] \\ &- \psi(d(x_{2n-1}, \mathcal{T}x_{2n-1}), d(x_{2n}, \mathcal{S}x_{2n}), d(x_{2n-1}, \mathcal{T}x_{2n}), d(x_{2n}, \mathcal{S}x_{2n-1})) \\ &= \frac{1}{4}[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n+1})] \qquad (3.5) \\ &- \psi(d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n+1}), 0) \\ &\leq \frac{1}{4}[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n+1})] \\ &\leq \frac{1}{2}[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]. \end{aligned}$$

Thus we have $d(x_{2n}, x_{2n+1}) < d(x_{2n-1}, x_{2n})$ for all $n \in \mathbb{N}$. Similarly, we can prove that $d(x_{2n-1}, x_{2n}) < d(x_{2n-2}, x_{2n-1})$ for all $n \geq 1$. Therefore, we conclude that (3.4) holds.

Now, from (3.4) it follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing. Therefore, there is some $\gamma \geq 0$ such that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = \gamma. \tag{3.6}$$

We prove that $\gamma = 0$. Assume that $\gamma > 0$. On passing to the limit as $n \to \infty$ in (3.5) we have ?

$$\gamma \leq \lim_{n\to\infty} \frac{1}{4}[2\gamma + d(x_{n-1}, x_{n+1}))] \leq \gamma,$$

or

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2\gamma. \tag{3.7}$$

Passing to the limit as $n \to \infty$ in (3.5) and using (3.6), (3.7) and the lower semi-continuity of ψ, we have

$$\begin{aligned}\gamma &\leq \frac{1}{4}(4\gamma) - \liminf_{n\to\infty} \psi(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0)\\ &\leq \gamma - \psi(\gamma, \gamma, 2\gamma, 0)\end{aligned}$$

or, $\psi(\gamma, \gamma, 2\gamma, 0) < 0$ which is a contradiction unless $\gamma = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. To this end, it is sufficient to verify that $\{x_{2n}\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon > 0$ such that for each even integer $2k$ there are even integers $2n(k)$, $2m(k)$ with $2m(k) > 2n(k) > 2k$ such that

$$r_k = d(x_{2n(k)}, x_{2m(k)}) \geq \varepsilon \ \ for \ \ k \in \{1, 2, 3, ...\}. \tag{3.9}$$

For every even integer $2k$, let $2m(k)$ be the smallest number exceeding $2n(k)$ satisfying condition (3.9) for which

$$d(x_{2n(k)}, x_{2m(k)-2}) < \varepsilon. \tag{3.10}$$

From (3.9), (3.10) and the triangular inequality, we have

$$\begin{aligned}\varepsilon \leq r_k &\leq d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)})\\ &\leq \varepsilon + d(x_{2m(k)-2}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)})..\end{aligned}$$

Hence by (3.8), it follows that

$$\lim_{k\to+\infty} r_k = \varepsilon. \tag{3.11}$$

Now, from the triangular inequality, we have

$$| \ d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)}) \ | \leq d(x_{2m(k)-1}, x_{2m(k)}).$$

Passing to the limit as $k \to +\infty$ and using (3.8) and (3.11), we get

$$\lim_{k\to+\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}
&d(x_{2n(k)}, x_{2m(k)})\\
&\leq d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)})\\
&\leq d(x_{2n(k)}, x_{2n(k)+1}) + d(\mathcal{S}x_{2n(k)}, \mathcal{T}x_{2m(k)-1})\\
&\leq d(x_{2n(k)}, x_{2n(k)+1}) + \frac{1}{4}\Big[d(x_{2m(k)-1}, \mathcal{T}x_{2m(k)-1}) + d(x_{2n(k)}, \mathcal{S}x_{2n(k)})\\
&\qquad + d(x_{2m(k)-1}, \mathcal{T}x_{2n(k)}) + d(x_{2n(k)}, \mathcal{S}x_{2m(k)-1})\Big] \qquad (3.13)\\
&- \psi(d(x_{2m(k)-1}, \mathcal{T}x_{2m(k)-1}), d(x_{2n(k)}, \mathcal{S}x_{2n(k)}), d(x_{2m(k)-1}, \mathcal{T}x_{2n(k)}), d(x_{2n(k)}, \mathcal{S}x_{2m(k)-1}))\\
&= d(x_{2n(k)}, x_{2n(k)+1})\\
&+ \frac{1}{4}[d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2m(k)-1}, x_{2n(k)+1}) + d(x_{2n(k)}, x_{2m(k)})]\\
&- \psi(d(x_{2m(k)-1}, x_{2m(k)}), d(x_{2n(k)}, x_{2n(k)+1}), d(x_{2m(k)-1}, x_{2n(k)+1}), d(x_{2n(k)}, x_{2m(k)})).
\end{aligned}$$

Taking into account (3.8) and (3.11) and the lower semi-continuity of ψ, letting $k \to \infty$ in the last inequality, we obtain ?

$$\epsilon \leq \frac{1}{4}[0 + 0 + \epsilon + \epsilon] - \psi(0, 0, \epsilon, \epsilon) \leq \frac{1}{2}\epsilon$$

and from the last inequality, $\psi(0, 0, \epsilon, \epsilon) \leq -\frac{1}{2}\epsilon < 0$. Therefore, $\varphi(0, 0\epsilon, \epsilon) = 0$. From the fact that $\psi(x, y, z, t) = 0 \Leftrightarrow x = y = z = t = 0$, we have $\epsilon = 0$, a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence.

From the completeness of $\mathcal{X}$, there exists $z \in \mathcal{X}$ such that $x_n \to z$ as $n \to +\infty$. Now we show that z is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Clearly, if $\mathcal{S}$ or $\mathcal{T}$ is continuous then $z = \mathcal{S}z$ or $z = \mathcal{T}z$. Thus, it is immediate to conclude that $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Now, suppose that the set of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is totally ordered. We claim that there is a unique common fixed point of $\mathcal{T}$ and $\mathcal{S}$. Assume to the contrary that $\mathcal{S}u = \mathcal{T}u = u$ and $\mathcal{S}v = \mathcal{T}v = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (3.1) and the lower semi-continuity of ψ, we obtain ?

$$\begin{aligned}
d(u, v) = d(\mathcal{S}u, \mathcal{T}v) &\leq \frac{1}{4}[d(v, \mathcal{T}v) + d(u, \mathcal{S}u) + d(v, \mathcal{S}u) + d(u, \mathcal{T}v)]\\
&\quad - \psi(d(v, \mathcal{T}v), d(u, \mathcal{S}u), d(v, \mathcal{S}u), d(u, \mathcal{T}v)),
\end{aligned}$$

a contradiction. Hence, $u = v$. The converse is trivial.

Remark 3.1. Theorem 3.1 remains true if condition (3.1) is replaced by

$$\begin{aligned}
\varphi(d(\mathcal{T}x, \mathcal{S}y)) &\leq \varphi(\frac{1}{4}[d(x, \mathcal{T}x) + d(y, \mathcal{S}y) + d(x, \mathcal{S}y) + d(y, \mathcal{T}x)])\\
&\quad - \psi(d(x, \mathcal{T}x), d(y, \mathcal{S}y), d(x, \mathcal{S}y), d(y, \mathcal{T}x))
\end{aligned}$$

for some continuous and nondecreasing function $\varphi : [0, +\infty) \to [0, +\infty)$. The proof is essentially the same, hence, for the sake of simplicity, we stay within the given version. The same remark applies to all other results in the rest of the paper. See also paper of Jachymski [15] where it is shown that practically each weak contractive condition with function φ can be replaced by an equivalent condition without φ.

Now, we are also able to prove the existence of a common fixed point of two mappings without using the continuity of $\mathcal{S}$ or $\mathcal{T}$. More precisely, we have the following theorem.

Theorem 3.2. *Let $(\mathcal{X}, d, \preceq)$ and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfy all the conditions of Theorem 3.1, except that condition* (iii) *is substituted by*

(iii') $\mathcal{X}$ *is regular.*

Then the conclusion of Theorem 3.1 holds.

Proof. Following the proof of Theorem 3.1, we have that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$ which is complete at x_0. Then, there exists $z \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now suppose that $d(z, \mathcal{S}z) > 0$. From regularity of $\mathcal{X}$, we have $x_{2n} \preceq z$ for all $n \in \mathbb{N}$. Hence, we can apply the considered contractive condition. Then, setting $x = x_{2n}$ and $y = z$ in (3.1), we obtain:

$$\begin{aligned} d(x_{2n+2}, \mathcal{S}z) = {} & d(\mathcal{T}x_{2n+1}, \mathcal{S}z) \\ & \le \frac{1}{4}[d(x_{2n+1}, \mathcal{T}x_{2n+1}) + d(z, \mathcal{S}z) + d(x_{2n+1}, \mathcal{S}z) + d(z, \mathcal{T}x_{2n+1})] \\ & - \psi(d(x_{2n+1}, \mathcal{T}x_{2n+1}), d(x_{2n+1}, \mathcal{S}z), d(x_{2n+1}, \mathcal{S}z), d(z, \mathcal{T}x_{2n+1})) \\ & = \frac{1}{4}[d(x_{2n+1}, x_{2n+2}) + d(z, \mathcal{S}y) + d(x_{2n+1}, \mathcal{S}z) + d(z, x_{2n+2})] \\ & - \psi(d(x_{2n+1}, x_{2n+z}), d(z, \mathcal{S}z), d(x_{2n+1}, \mathcal{S}z), d(z, x_{2n+z})) \end{aligned}$$

Passing to the limit as $n \to \infty$ and using $x_n \to z$, lower semi-continuity of ψ, we have

$$d(z, \mathcal{S}z) \le \frac{1}{2}d(z, \mathcal{S}z) - \psi(0, d(z, \mathcal{S}z), d(z, \mathcal{S}z), 0) \le \frac{1}{2}d(z, \mathcal{S}z)$$

a contradiction. Therefore $d(z, \mathcal{S}z) = 0$ and consequently, $z = \mathcal{S}z$. Analogously, for $x = z$ and $y = x_{2n}$, one can prove that $\mathcal{T}z = z$. It follows that $z = \mathcal{S}z = \mathcal{T}z$, that is, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Corollary 3.1. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Suppose $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are two mappings satisfying*

$$d(\mathcal{T}x, \mathcal{S}y) \le \beta[d(x, \mathcal{T}x) + d(y, \mathcal{S}y) + d(x, \mathcal{S}y) + d(y, \mathcal{T}x)] \tag{3.15}$$

for all comparable $x, y \in \mathcal{X}$, where $\beta \in [0, \frac{1}{4})$.

We assume the following hypotheses:

(i) *$\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing;*

(ii) *there exists an $x_0 \in \mathcal{X}$ such that $x_0 \prec \mathcal{S}x_0$;*

(iii) *$\mathcal{S}$ or $\mathcal{T}$ is continuous at x_0 or $\mathcal{X}$ is regular.*

Then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$ is totally ordered if and only if $\mathcal{S}$ and $\mathcal{T}$ have one and only one common fixed point.

Proof. Let $\beta \in [0, \frac{1}{4})$. Here, it suffices to take the function $\psi : [0, +\infty)^4 \to [0, +\infty)$ defined by $\psi(a, b, c, e) = (\frac{1}{4} - \beta)(a + b + c + e)$. Obviously, ψ satisfies that $\psi(a, b, c, e) = 0$ if and only if $a = b = c = e = 0$, and $\psi(x, y, z, t) = (\frac{1}{4} - \beta)(x + y + z + t) = \psi(x + y + z + t, 0)$. Then, we can apply Theorems 3.1.

Putting $\mathcal{S} = \mathcal{T}$ in Theorem 3.2, we obtain easily the following result.

Corollary 3.2. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{X}$ such that $(\mathcal{X}, d)$ is a complete metric space. Suppose $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying generalized weakly $\mathcal{T}$-contractions conditions, that is,*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \leq \frac{1}{4}&[d(x, \mathcal{T}x) + d(y, \mathcal{T}y) + d(x, \mathcal{T}y) + d(y, \mathcal{T}x)] \\ &- \psi(d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)), \end{aligned}$$

for all comparable $x, y \in \mathcal{X}$.

Also suppose that $\mathcal{T}x \prec \mathcal{T}(\mathcal{T}x)$ for all $x \in \mathcal{X}$ such that $x \prec \mathcal{T}x$. If there exists an $x_0 \in \mathcal{X}$ such that $x_0 \prec \mathcal{T}x_0$ and the condition

$$\begin{cases} \{x_n\} \subset \mathcal{X} \text{ is a non-decreasing sequence with } x_n \to z \text{ in } \mathcal{X}, \\ \text{then } x_n \preceq z \text{ for all } n \end{cases}$$

holds, then $\mathcal{T}$ has a fixed point. Moreover, the set of fixed points of $\mathcal{T}$ is totally ordered if and only if it is singleton.

To conclude this section, we provide a sufficient condition to ensure the uniqueness of the fixed point in the above Theorems 3.1 and 3.2.

Firstly, we recall that the usual definition of the diameter of a set $\mathcal{A}$ in a metric space $(\mathcal{X}, d)$, is

$$\mathrm{diam}(\mathcal{A}) := \sup\{d(x, y) : x, y \in \mathcal{A}\},$$

for any subset $\mathcal{A}$ of $\mathcal{X}$. Then, we obtain the following fixed point theorem.

Theorem 3.3. *Adding to the hypotheses of Theorems 3.1 (resp. Theorem 3.2) the following condition:*

$$\lim_{n \to +\infty} diam((\mathcal{T} \circ \mathcal{S})^n(\mathcal{X})) = 0,$$

where $\circ$ denotes the composition of mappings, we obtain the uniqueness of the common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Proof. Let z and z' be two common fixed points of $\mathcal{S}$ and $\mathcal{T}$, that is,

$$z = \mathcal{T}z = \mathcal{S}z$$

and

$$z' = \mathcal{T}z' = \mathcal{S}z'.$$

It is immediate to show that for all $n \in \mathbb{N}$, we have:

$$(\mathcal{T} \circ \mathcal{S})^n x = x, \text{ for all } x \in \{z, z'\}.$$

Then

$$\begin{aligned} d(z, z') &= d((\mathcal{T} \circ \mathcal{S})^n z, (\mathcal{T} \circ \mathcal{S})^n z') \\ &\leq \operatorname{diam}((\mathcal{T} \circ \mathcal{S})^n (\mathcal{X})) \\ &\to 0 \text{ as } n \to +\infty. \end{aligned}$$

Hence $z = z'$ and the proof is complete.

We present an example showing how our results can be used.

Example 3.1. Let the set $\mathcal{X} = [0, +\infty)$ be equipped with the usual metric d and the order defined by

$$x \preceq y \iff x \geq y.$$

Consider the following self-mappings on $\mathcal{X}$:

$$\mathcal{T}x = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq \frac{1}{2}, \\ 2x, & x > \frac{1}{2}, \end{cases} \qquad \mathcal{S}x = \begin{cases} \frac{1}{3}x, & 0 \leq x \leq \frac{1}{3}, \\ 3x, & x > \frac{1}{3}. \end{cases}$$

Take $x_0 = \frac{1}{3}$. Then it is easy to show that all the conditions (i)–(iii) of Theorem 3.1 are fulfilled. Take $\psi(x, y, z, t) = \frac{x+y+z+t}{6}$. Then contractive condition (3.1) takes the form

$$\left| \frac{1}{2}x - \frac{1}{3}y \right| \leq \frac{1}{12} \left[\frac{1}{2}x + \frac{2}{3}y + \left| x - \frac{1}{3}y \right| + \left| y - \frac{1}{2}x \right| \right],$$

for $x, y \in \mathcal{X}$. Using substitution $y = tx$, $t > 0$, the last inequality reduces to

$$|3 - 2t| \leq \frac{1}{12}[3 + 4t + 2|3 - t| + 3|2t - 1|],$$

and can be checked by discussion on possible values for $t > 0$. Hence, all the conditions of Theorem 3.1 are satisfied and $\mathcal{S}, \mathcal{T}$ have a unique common fixed point (which is 0). Note that $\mathcal{S}$ and $\mathcal{T}$ do not satisfy the contractive condition for arbitrary $x, y \in \mathcal{X}$.

4. Common fixed points for Generalized $(\mathcal{T}, \mathcal{S}, \mathcal{R})$-contraction mappings

Here, we introduce the notion of a generalized weakly $(\mathcal{T}, \mathcal{S}, \mathcal{R})$-contraction in metric spaces.

Definition 4.1. Let $(\mathcal{X}, d)$ be a metric space. Three mappings $\mathcal{T}, \mathcal{S}, \mathcal{R} : \mathcal{X} \to \mathcal{X}$ are called a generalized weakly $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-contraction if

$$\begin{aligned} d(\mathcal{T}x, \mathcal{S}y) \leq & \frac{1}{4}[d(\mathcal{R}x, \mathcal{T}x) + d(\mathcal{R}y, \mathcal{S}y) + d(\mathcal{R}x, \mathcal{S}y) + d(\mathcal{R}y, \mathcal{T}x)] \\ & - \psi(d(\mathcal{R}x, \mathcal{T}x), d(\mathcal{R}y, \mathcal{S}y), d(\mathcal{R}x, \mathcal{S}y), d(\mathcal{R}y, \mathcal{T}x)), \end{aligned} \tag{4.1}$$

for all $x, y \in \mathcal{X}$ and $\psi \in \mathbf{F}$.

Now, we state and prove our second main result concerning the existence of common fixed point of three mappings $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{R}$ on $\mathcal{X}$ into itself.

Theorem 4.1. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{R}$ be self-maps of $\mathcal{X}$ such that $\mathcal{S}(\mathcal{X}) \cup \mathcal{T}(\mathcal{X}) \subseteq \mathcal{R}(X)$ satisfying generalized weakly $((\mathcal{T}, \mathcal{S}, \mathcal{R}))$-contractions conditions for every pair $(x, y) \in \mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R}) \times \mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$ (for some x_0) such that $\mathcal{R}x$ and $\mathcal{R}y$ are comparable.*

Assume that the following hypotheses hold in $\mathcal{X}$.

(i) *$(\mathcal{S}, \mathcal{T})$ is a.r. with respect to $\mathcal{R}$ at $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0;*
(iii) *$\mathcal{T}$ and $\mathcal{S}$ are weakly increasing with respect to $\mathcal{R}$;*
(iv) *$\mathcal{T}$ and $\mathcal{S}$ are dominating.*

Assume further that either

(a) *$\mathcal{S}$ and $\mathcal{R}$ are compatible; $\mathcal{S}$ or $\mathcal{R}$ is orbitally continuous at x_0 or*
(b) *$\mathcal{T}$ and $\mathcal{R}$ are compatible; $\mathcal{T}$ or $\mathcal{R}$ is orbitally continuous at x_0.*

Then $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ is well ordered if and only if $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ have one and only one common fixed point.

Proof. Since $(\mathcal{S}, \mathcal{T})$ is a.r. with respect to $\mathcal{R}$ at x_0 in $\mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\mathcal{R}x_{2n+1} = \mathcal{T}x_{2n}, \qquad \mathcal{R}x_{2n+2} = \mathcal{S}x_{2n+1}, \qquad \forall\, n \in \mathbb{N}^*, \tag{4.2}$$

and

$$\lim_{n\to\infty} d(\mathcal{R}x_n, \mathcal{R}x_{n+1}) = 0 \tag{4.3}$$

holds. We claim that

$$\mathcal{R}x_n \preceq \mathcal{R}x_{n+1},\ \forall\, n \in \mathbb{N}. \tag{4.4}$$

To this aim, we will use the increasing property with respect to $\mathcal{R}$ satisfied by the mappings $\mathcal{T}$ and $\mathcal{S}$. From (4.2), we have

$$\mathcal{R}x_1 = \mathcal{T}x_0 \preceq \mathcal{S}y,\ \forall\, y \in \mathcal{R}^{-1}(\mathcal{T}x_0).$$

Since $\mathcal{R}x_1 = \mathcal{T}x_0$, then $x_1 \in \mathcal{R}^{-1}(\mathcal{T}x_0)$, and we get

$$\mathcal{R}x_1 = \mathcal{T}x_0 \preceq \mathcal{S}x_1 = \mathcal{R}x_2.$$

Again,

$$\mathcal{R}x_2 = \mathcal{S}x_1 \preceq \mathcal{T}y, \ \forall\, y \in \mathcal{R}^{-1}(\mathcal{S}x_1).$$

Since $x_2 \in \mathcal{R}^{-1}(\mathcal{S}x_1)$, we get

$$\mathcal{R}x_2 = \mathcal{S}x_1 \preceq \mathcal{T}x_2 = \mathcal{R}x_3.$$

Hence, by induction, (4.4) holds. Therefore, we can apply (4.1) for $x = x_p$ and $y = x_q$ for all p and q.

Now, we assert that $\{\mathcal{R}x_n\}$ is a Cauchy sequence in the metric space $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$. We proceed by negation and suppose that $\{\mathcal{R}x_{2n}\}$ is not Cauchy. Then, there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \ d(\mathcal{R}x_{2m(k)}, \mathcal{R}x_{2n(k)}) \geq \varepsilon, \ d(\mathcal{R}x_{2m(k)}, \mathcal{R}x_{2n(k)-2}) < \varepsilon. \tag{4.5}$$

From (4.5) and using the triangular inequality, we get

$$\begin{aligned}
\varepsilon &\leq d(\mathcal{R}x_{2m(k)}, \mathcal{R}x_{2n(k)}) \\
&\leq d(\mathcal{R}x_{2m(k)}, \mathcal{R}x_{2n(k)-2}) + d(\mathcal{R}x_{2n(k)-2}, \mathcal{R}x_{2n(k)-1}) + d(\mathcal{R}x_{2n(k)-1}, \mathcal{R}x_{2n(k)}) \\
&< \varepsilon + d(\mathcal{R}x_{2n(k)-2}, \mathcal{R}x_{2n(k)-1}) + d(\mathcal{R}x_{2n(k)-1}, \mathcal{R}x_{2n(k)}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (4.3), we obtain

$$\lim_{k\to\infty} d(\mathcal{R}x_{2m(k)}, \mathcal{R}x_{2n(k)}) = \varepsilon. \tag{4.6}$$

Again, the triangular inequality gives us

$$|d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-1}) - d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)})| \leq d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)}).$$

Letting $k \to \infty$ in the above inequality and using (4.3) and (4.6), we get:

$$\lim_{k\to+\infty} d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-1}) = \varepsilon. \tag{4.7}$$

On the other hand, we have

$$
\begin{aligned}
& d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)}) \\
& \leq d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}) + d(\mathcal{R}x_{2n(k)+1}, \mathcal{R}x_{2m(k)}) \\
& = d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}) + d(\mathcal{T}x_{2n(k)}, \mathcal{S}x_{2m(k)-1}) \\
& \leq d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}) \\
& \qquad + \frac{1}{4}\Big[d(\mathcal{R}x_{2m(k)-1}, \mathcal{T}x_{2m(k)-1}) + d(\mathcal{R}x_{2n(k)}, \mathcal{S}x_{2n(k)}) \\
& \qquad\qquad + d(\mathcal{R}x_{2m(k)-1}, \mathcal{T}x_{2n(k)}) + d(\mathcal{R}x_{2n(k)}, \mathcal{S}x_{2m(k)-1})\Big] \\
& \qquad - \psi\Big(d(\mathcal{R}x_{2m(k)-1}, \mathcal{T}x_{2m(k)-1}), d(\mathcal{R}x_{2n(k)}, \mathcal{S}x_{2n(k)}), \\
& \qquad\qquad d(\mathcal{R}x_{2m(k)-1}, \mathcal{T}x_{2n(k)}), d(\mathcal{R}x_{2n(k)}, \mathcal{S}x_{2m(k)-1})\Big) \\
& \leq d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}) \\
& + \frac{1}{4}\Big[d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)}) + d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}) \\
& \qquad\qquad + d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2n(k)+1}) + d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)})\Big] \\
& \qquad - \psi\Big(d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)}), d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}), \\
& \qquad\qquad d(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2n(k)+1}), d(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)})\Big).
\end{aligned}
\tag{4.8}
$$

Passing to the limit as $k \to \infty$ in the above inequality (4.8) and using (4.3), (4.6) and the lower semi-continuity of ψ, we obtain ?

$$\epsilon \leq \frac{1}{4}[0 + 0 + \epsilon + \epsilon] - \psi(0, 0, \epsilon, \epsilon) \leq \frac{1}{2}\epsilon$$

and from the last inequality, $\psi(0, 0.\epsilon, \epsilon) \leq -\frac{1}{2}\epsilon < 0$. Therefore $\varphi(0, 0, \epsilon, \epsilon) = 0$. From the fact that $\psi(x, y, z, t) = 0 \Leftrightarrow x = y = z = t = 0$, we have $\epsilon = 0$, a contradiction. Hence, we deduce that $\{\mathcal{R}x_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$. Since $\mathcal{X}$ is $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0, there exists some $z \in \mathcal{X}$ such that

$$\mathcal{R}x_n \to z \text{ as } n \to \infty. \tag{4.9}$$

We will prove that z is a common fixed point of the three mappings $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$.

We have

$$\mathcal{R}x_{2n+1} = \mathcal{S}x_{2n} \to z \text{ as } n \to \infty \tag{4.10}$$

and

$$\mathcal{R}x_{2n+2} = \mathcal{T}x_{2n+1} \to z \text{ as } n \to \infty. \tag{4.11}$$

Suppose that (a) holds; e.g., let $\mathcal{R}$ be continuous on $\mathcal{X}$. Since $\mathcal{S}$ and $\mathcal{R}$ are compatible, we have

$$\lim_{n\to\infty} \mathcal{S}\mathcal{R}x_{2n+2} = \lim_{n\to\infty} \mathcal{R}\mathcal{S}x_{2n+2} = \mathcal{R}z. \tag{4.12}$$

From (4.8) and the continuity of $\mathcal{R}$, we have

$$\mathcal{R}(\mathcal{R}x_n) \to \mathcal{R}z \text{ as } n \to \infty. \tag{4.13}$$

Now, using (iv), $x_{2n+1} \preceq \mathcal{T}x_{2n+1} = \mathcal{R}x_{2n+2}$ and since $\mathcal{R}$ is monotone, $\mathcal{R}x_{2n+1}$ and $\mathcal{R}\mathcal{R}x_{2n+2}$ are comparable. Thus, we can apply (4.1) to get

$$\begin{aligned} & d(\mathcal{S}\mathcal{R}x_{2n+2}, \mathcal{T}x_{2n+1}) \\ & \le \frac{1}{4}\Big[d(\mathcal{R}x_{2n+1}, \mathcal{T}x_{2n+1}) + d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{S}\mathcal{R}x_{2n+2}) \\ & \qquad + d(\mathcal{R}x_{2n+1}, \mathcal{S}\mathcal{R}x_{2n+2}) + d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{T}x_{2n+1})\Big] \\ & - \psi\Big(d(\mathcal{R}x_{2n+1}, \mathcal{T}x_{2n+1}), d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{S}\mathcal{R}x_{2n+2}), \\ & \qquad d(\mathcal{R}x_{2n+1}, \mathcal{S}\mathcal{R}x_{2n+2}), d(\mathcal{R}\mathcal{R}x_{2n+2}, \mathcal{T}x_{2n+1})\Big). \end{aligned} \tag{4.14}$$

Passing to the limit as $n \to \infty$ in (4.14), using (4.12) and (4.13), and the lower semi-continuity of ψ, we obtain ?

$$d(\mathcal{R}z, z) \le \frac{1}{2}d(\mathcal{R}z, z)) - \psi(0, 0, d(\mathcal{R}z, z), d(\mathcal{R}z, z)) < \frac{1}{2}d(\mathcal{R}z, z)),$$

and from the last inequality, $\psi(0, 0, d(\mathcal{R}z, z), d(\mathcal{R}z, z)) \le -\frac{1}{2}d(\mathcal{R}z, z) \le 0$. Therefore

$$\psi(0, 0, d(\mathcal{R}z, z), d(\mathcal{R}z, z)) = 0.$$

From the fact that $\psi(x, y, z, t) = 0 \Leftrightarrow x = y = z = t = 0$, we have $d(\mathcal{R}z, z) = 0$, that is,

$$\mathcal{R}z = z. \tag{4.15}$$

Now, $x_{2n+1} \preceq \mathcal{T}x_{2n+1}$ and $\mathcal{T}x_{2n+1} \to z$ as $n \to \infty$, so by the assumption we have $x_{2n+1} \preceq z$ and (4.1) gives

$$\begin{aligned} d(\mathcal{S}z, \mathcal{T}x_{2n+1}) \le & \frac{1}{4}[d(\mathcal{R}x_{2n+1}, \mathcal{T}x_{2n+1}) + d(\mathcal{R}z, \mathcal{S}z) + d(\mathcal{R}x_{2n+1}, \mathcal{S}z) + d(\mathcal{R}z, \mathcal{T}x_{2n+1})] \\ & - \psi(d(\mathcal{R}x_{2n+1}, \mathcal{T}x_{2n+1}), d(\mathcal{R}z, \mathcal{S}z), d(\mathcal{R}x_{2n+1}, \mathcal{S}z), d(\mathcal{R}z, \mathcal{T}x_{2n+1})), \end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality and using (4.14) and the lower semi-continuity of ψ, it follows that

$$d(\mathcal{S}z, z) \le \frac{1}{2}d(\mathcal{S}z, z)) - \psi(0, d(\mathcal{S}z, z), d(\mathcal{S}z, z), 0) < \frac{1}{2}d(\mathcal{R}z, z)), \tag{4.16}$$

which holds unless

$$\mathcal{S}z = z. \tag{4.17}$$

Now, since $x_{2n} \preceq \mathcal{S}x_{2n}$ and $\mathcal{S}x_{2n} \to z$ as $n \to \infty$, we have that $x_{2n} \preceq z$ for all $n \in \mathbb{N}$. From (4.1),

$$\begin{aligned} d(\mathcal{S}x_{2n}, \mathcal{T}z) \le & \frac{1}{4}[d(\mathcal{R}z, \mathcal{T}z) + d(\mathcal{R}x_{2n}, \mathcal{S}x_{2n}) + d(\mathcal{R}z, \mathcal{S}x_{2n}) + d(\mathcal{R}x_{2n}, \mathcal{T}z)] \\ & - \psi(d(\mathcal{R}z, \mathcal{T}z), d(\mathcal{R}z, \mathcal{S}x_{2n}), d(\mathcal{R}z, \mathcal{S}x_{2n}), d(\mathcal{R}x_{2n}, \mathcal{T}z)), \end{aligned}$$

Passing to the limit as $n \to \infty$, and the lower semi-continuity of ψ, we obtain ?

$$d(z,\mathcal{T}z) \leq \frac{1}{2}d(z,\mathcal{T}z) - \psi(d(z,\mathcal{T}z),0,0,d(z,\mathcal{T}z)) < \frac{1}{2}d(z,\mathcal{T}z)$$

which gives that

$$z = \mathcal{T}z. \tag{4.18}$$

Therefore, $\mathcal{S}z = \mathcal{T}z = \mathcal{R}z = z$, hence z is a common fixed point of $\mathcal{R},\mathcal{S}$ and $\mathcal{T}$. The proof is similar when $\mathcal{S}$ is orbitally continuous.

Similarly, the result follows when condition (b) holds.

Now, suppose that the set of common fixed points of $\mathcal{S},\mathcal{T}$ and $\mathcal{R}$ is totally ordered. We claim that there is a unique common fixed point of $\mathcal{S},\mathcal{T}$ and $\mathcal{R}$. Assume to the contrary that $\mathcal{S}u = \mathcal{T}u = \mathcal{R}u = u$ and $\mathcal{S}v = \mathcal{T}v = \mathcal{R}v = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (4.1) and the lower semi-continuity of ψ, we obtain ?

$$\begin{aligned} d(u,v) = d(\mathcal{S}u,\mathcal{T}v) &\leq \frac{1}{4}[d(\mathcal{R}v,\mathcal{T}v) + d(\mathcal{R}u,\mathcal{S}u) + d(\mathcal{R}v,\mathcal{S}u) + d(\mathcal{R}u,\mathcal{T}v)] \\ &\quad - \psi(d(\mathcal{R}v,\mathcal{T}v), d(\mathcal{R}u,\mathcal{S}u), d(\mathcal{R}v,\mathcal{S}u), d(\mathcal{R}u,\mathcal{T}v)), \end{aligned}$$

a contradiction. Hence, $u = v$. The converse is trivial.

As a consequence of Theorem 4.1, we obtain the following corollaries.

Corollary 4.1. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{R}$ be self-maps of $\mathcal{X}$ such that $\mathcal{S}(\mathcal{X}) \cup \mathcal{T}(\mathcal{X}) \subseteq \mathcal{R}$ satisfying*

$$d(\mathcal{T}x,\mathcal{S}y) \leq \beta[d(\mathcal{R}x,\mathcal{T}x) + d(\mathcal{R}y,\mathcal{S}y) + d(\mathcal{R}x,\mathcal{S}y) + d(\mathcal{R}y,\mathcal{T}x)] \tag{4.19}$$

for every pair $(x,y) \in \mathcal{O}(x_0;\mathcal{S},\mathcal{T},\mathcal{R}) \times \mathcal{O}(x_0;\mathcal{S},\mathcal{T},\mathcal{R})$ (for some x_0) such that $\mathcal{R}x$ and $\mathcal{R}y$ are comparable, where $\beta \in [0, \frac{1}{4})$.

Assume that the following hypotheses hold in $\mathcal{X}$.

(i) *$(\mathcal{S},\mathcal{T})$ is a.r. with respect to $\mathcal{R}$ at $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $(\mathcal{S},\mathcal{T},\mathcal{R})$-orbitally complete at x_0;*
(iii) *$\mathcal{T}$ and $\mathcal{S}$ are weakly increasing with respect to $\mathcal{R}$;*
(iv) *$\mathcal{T}$ and $\mathcal{S}$ are dominating maps.*

Assume further that either

(a) *$\mathcal{S}$ and $\mathcal{R}$ are compatible; $\mathcal{S}$ or $\mathcal{R}$ is orbitally continuous at x_0 or*
(b) *$\mathcal{T}$ and $\mathcal{R}$ are compatible; $\mathcal{T}$ or $\mathcal{R}$ is orbitally continuous at x_0.*

Then $\mathcal{S},\mathcal{T}$ and $\mathcal{R}$ have a unique common fixed point. Moreover, the set of common fixed points of $\mathcal{S},\mathcal{T}$ and $\mathcal{R}$ is totally ordered if and only if $\mathcal{S},\mathcal{T}$ and $\mathcal{R}$ have one and only one common fixed point.

Corollary 4.2. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{T}$ and $\mathcal{S}$ be self-maps on $\mathcal{X}$ satisfying generalized weakly $(\mathcal{T}, \mathcal{S})$-contractions conditions for every pair $(x, y) \in \mathcal{O}(x_0; \mathcal{S}, \mathcal{T}) \times \mathcal{O}(x_0; \mathcal{S}, \mathcal{T})$ (for some x_0) such that x and y are comparable.*

Assume that the following hypotheses hold in $\mathcal{X}$.

(i) *$(\mathcal{S}, \mathcal{T})$ is a.r. at some point $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $(\mathcal{S}, \mathcal{T})$-orbitally complete at x_0;*
(iii) *$\mathcal{T}$ and $\mathcal{S}$ are weakly increasing;*
(iv) *$\mathcal{T}$ and $\mathcal{S}$ are dominating maps;*
(v) *$\mathcal{S}$ or $\mathcal{T}$ is orbitally continuous at x_0.*

Then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{T}$ and $\mathcal{S}$ is totally ordered if and only if $\mathcal{T}$ and $\mathcal{S}$ have one and only one common fixed point.

Corollary 4.3. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{T}$ and $\mathcal{R}$ be self-maps of $\mathcal{X}$ such that $\mathcal{T}(X) \subseteq \mathcal{R}(X)$ satisfying*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}x) \leq \frac{1}{4}[&d(\mathcal{R}x, \mathcal{T}x) + d(\mathcal{R}y, \mathcal{T}y) + d(\mathcal{R}x, \mathcal{T}y) + d(\mathcal{R}y, \mathcal{T}x)] \\ &- \psi(d(\mathcal{R}x, \mathcal{T}x), d(\mathcal{R}y, \mathcal{T}y), d(\mathcal{R}x, \mathcal{T}y), d(\mathcal{R}y, \mathcal{T}x)), \end{aligned} \tag{4.20}$$

for every pair $(x, y) \in \mathcal{O}(x_0; \mathcal{T}, \mathcal{R}) \times \mathcal{O}(x_0; \mathcal{T}, \mathcal{R})$ (for some x_0) such that $\mathcal{R}x$ and $\mathcal{R}y$ are comparable.

Assume that the following hypotheses hold in $\mathcal{X}$.

(i) *$\mathcal{T}$ is a.r. with respect to $\mathcal{R}$ at $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $(\mathcal{T}, \mathcal{R})$-orbitally complete at x_0;*
(iii) *$\mathcal{T}$ is weakly increasing with respect to $\mathcal{R}$;*
(iv) *$\mathcal{T}$ is dominating maps;*
(v) *$\mathcal{T}$ or $\mathcal{R}$ is orbitally continuous at x_0.*

Then $\mathcal{T}$ and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{T}$ and $\mathcal{R}$ is totally ordered if and only if $\mathcal{T}$ and $\mathcal{R}$ have one and only one common fixed point.

Corollary 4.4. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let $\mathcal{T}$ be a self-map of $\mathcal{X}$ satisfying*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}x) \leq \frac{1}{4}[&d(x, \mathcal{T}x) + d(y, \mathcal{T}y) + d(x, \mathcal{T}y) + d(y, \mathcal{T}x)] \\ &- \psi(d(x, \mathcal{T}x), d(y, \mathcal{T}y), d(x, \mathcal{T}y), d(y, \mathcal{T}x)), \end{aligned} \tag{4.21}$$

for every pair $(x, y) \in \mathcal{O}(x_0; \mathcal{T}) \times \mathcal{O}(x_0; \mathcal{T})$ (for some x_0) such that x and y are comparable.

Assume that the following hypotheses hold in $\mathcal{X}$.

(i) *$\mathcal{T}$ is a.r. at some point $x_0 \in \mathcal{X}$;*
(ii) *$\mathcal{X}$ is $\mathcal{T}$-orbitally complete at x_0;*

(iii) *$x \preceq \mathcal{T}x$ for all $x \in \mathcal{O}(x_0; \mathcal{T})$;*

(iv) *$\mathcal{T}$ is orbitally continuous at x_0.*

Then $\mathcal{T}$ has a fixed point. Moreover, the set of fixed points of $\mathcal{T}$ is totally ordered if and only if it is singleton.

We illustrate Theorem 4.1 by an example which is obtained by modifying the one that given in [21].

Example 4.1. Let the set $\mathcal{X} = [0, +\infty)$ be equipped with the usual metric d and the order defined by

$$x \preceq y \iff x \geq y.$$

Consider the following self-mappings of $\mathcal{X}$:

$$\mathcal{R}x = 6x, \qquad \mathcal{S}x = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq \frac{1}{2}, \\ x, & x > \frac{1}{2}, \end{cases} \qquad \mathcal{T}x = \begin{cases} \frac{1}{3}x, & 0 \leq x \leq \frac{1}{3}, \\ x, & x > \frac{1}{3}. \end{cases} \tag{4.22}$$

Take $x_0 = \frac{1}{2}$. Then it is easy to show that all the conditions (i)–(iv) and (a)-(b) of Theorem 4.1 are fulfilled (condition (iii) on $O(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$). Take $\psi(t) = \frac{1}{6}t$. Then contractive condition (4.1) takes the form

$$\left|\frac{1}{2}x - \frac{1}{3}y\right| \leq \frac{1}{12}\left[\frac{11}{2}x + \frac{17}{3}y + \left|6x - \frac{1}{3}y\right| + \left|6y - \frac{1}{2}x\right|\right],$$

for $x, y \in O(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$. Using substitution $y = tx$, $t > 0$, the last inequality reduces to

$$|3 - 2t| \leq \frac{1}{12}[33 + 34t + 2|18 - t| + 3|12t - 1|],$$

and can be checked by discussion on possible values for $t > 0$. Hence, all the conditions of Theorem 4.1 are satisfied and $\mathcal{S}, \mathcal{T}, \mathcal{R}$ have a common fixed point (which is 0).

Remark 4.2. Finally we remark that the results of this paper also remain true if we replace the condition of generalized weakly $(\mathcal{S}, \mathcal{C})$-contraction with

$$\begin{aligned} d(\mathcal{T}x, \mathcal{S}y) \leq \frac{1}{5}\big[&d(x,y) + d(x, \mathcal{T}x) + d(y, \mathcal{S}y) + d(x, \mathcal{S}y) + d(y, \mathcal{T}x)\big] \\ &- \psi\big(d(x,y), d(x, \mathcal{T}x), d(y, \mathcal{S}y), d(x, \mathcal{S}y), d(y, \mathcal{T}x)\big) \end{aligned} \tag{4.23}$$

for any $x, y \in \mathcal{X}$ and $\psi \in \Psi$, where Ψ is a class of lower semi-continuous functions $\psi : [0, \infty)^5 \to [0, \infty)$ satisfying

$$\psi(x, y, z, w, t) = 0 \quad \text{if and only if} \quad x = y = z = w = t = 0.$$

The common fixed point theorems related to the inequality (4.23) can be proved on the similar lines to that of Theorems 14 and 19 with appropriate modifications.

References

[1] M. Abbas, T. Nazir, S. Radenović, Common fixed point of four maps in partially ordered metric spaces, Appl. Math. Lett. 24 (2011), 1520–1526.

[2] I. Altun, H. Simsek, Some fixed point theorems on ordered metric spaces and applications, Fixed Point Theory Appl 2010, Article ID 621492, 17 pages.

[3] R. P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal. 87 (2008), 1–8.

[4] H. Aydi, E. Karapinar, M. Postolache, Tripled coincidence point theorems for weak φ-contractions in partially ordered metric spaces, Fixed Point Theory and Applications 2012, 2012:44 doi:10.1186/1687-1812-2012-44.

[5] H. Aydi, H. K. Nashine, B. Samet, H. Yazidi, Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations, Nonlinear Anal. 74(17) (2011) 6814-6825.

[6] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integerales, Fund. Math. 3(1922), 133-181.

[7] I. Beg and A.R. Butt, Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces, Nonlinear Analysis 71 (9) (2009), 3699-3704.

[8] F. E. Browder, W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967), 197–228.

[9] S. K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci., 25(1972), 727-730.

[10] B. S. Choudhury, Unique fixed point theorem for weak C-contractive mappings, Kathmandu University J. Sci., Engng. Technology, 5(1)(2009), 6-13.

[11] Lj. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267–273.

[12] B. C. Dhage, Condensing mappings and applications to existence theorems for common solution of differential equations, Bull. Korean Math. Soc. 36 (1999), 565-?78.

[13] B. C. Dhage, D. ORegan, R. P. Agarwal, Common fixed point theorems for a pair of countably condensing mappings in ordered Banach spaces, J. Appl. Math. Stoch. Anal. 16 (2003), 243-?48.

[14] J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. 71 (2009), 3403–3410.

[15] J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces, Nonlinear Analysis 74 (2011), 768-774.

[16] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly 83 (1976), 261–263.

[17] G. Jungck, Compatible mappings and common fixed points, Inter. J. Math. Math. Sci. 9 (1986), 771–779.

[18] H. K. Nashine, New fixed point theorems for mappings satisfying generalized weakly contractive condition with weaker control functions, Ann. Polonici Math. (2011), in press.

[19] H. K. Nashine, I. Altun, Fixed point theorems for generalized weakly contractive condition in ordered metric spaces, Fixed Point Theory and Appl. 2011 (2011) Article ID 132367, 20 pages.

[20] H. K. Nashine, I. Altun, A common fixed point theorem on ordered metric spaces, to appear in Bull. Iranian Math. Soc.

[21] H. K. Nashine, Z. Kadelburg, Z. Golubovic, Common fixed point results using generalized altering distances on orbitally complete ordered metric spaces, J. Appl. Math. 2012, Article ID 382094, 12 pages.

[22] H. K. Nashine, B. Samet, Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces, Nonlinear Anal. 74 (2011), 2201–2209.

[23] H. K. Nashine, B. Samet, C. Vetro, Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces, Math. Comput. Modelling 54 (2011), 712–720.

[24] H. K. Nashine, B. Samet, C. Vetro, Common fixed points of strictly weakly isotone increasing mappings in ordered metric spaces, submitted to Filomat.

[25] J. J. Nieto, R. R. López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005), 223–239.

[26] D. O'Regan, A. Petrusel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341 (2008), 1241–1252.

[27] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435–1443.

[28] K. P. R. Sastry, S. V. R. Naidu, I. H. N. Rao, K. P. R. Rao, Common fixed points for asymptotically regular mappings, Indian J. Pure Appl. Math. 15 (1984), 849–854.

[29] W. Shatanawi, B. Samet, On (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces, Computers and Mathematics with Applications (2011), doi:10.1016/j.camwa.2011.08.033.

[30] W. Shatanawi, Fixed point theorems for nonlinear weakly C-contractive mapping in metric spaces, Mathematical and Computer Modelling, 54(11?2)(2011), 2816-2826.

[31] C. Vetro, Common fixed points in ordered Banach spaces, Le Matematiche 63 (2008), 93-?00.