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A NEW COMMON FIXED POINT THEOREM FOR ALTMAN TYPE MAPPINGS IN G-METRIC SPACES

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Abstract. In the framework of complete G-metric spaces, using the concept of R-weakly commuting self-mapping pairs, we establish a new common fixed point theorem for two self-mappings of Altman type. An example is provided to support our results. The results obtained in this paper differ from the recent relative results in literature.

Keywords: R-weakly commuting mappings; Altman type mappings; complete G-metric space; common fixed point.

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1. Introduction and Preliminaries

Metric fixed point theory is an important mathematical discipline because of its applications in areas as variational and linear inequalities, optimization theory. Many results have been obtained by many authors considering different contractive conditions for self-mappings in metric space. In 1975, Altman [1] proved a fixed point theorem for a mapping which satisfies the condition $d(fx, fy) \leq Q(d(x, y))$, where $Q : [0, +\infty) \to [0, +\infty)$ is an increasing function satisfying the following conditions:

(i) $0 < Q(t) < t, t \in (0, \infty)$;

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(ii) $p(t) = t/(t - Q(t))$ is a decreasing function;

(iii) for some positive number t_1, there holds $\int_0^{t_1} p(t)dt < +\infty$.

Remark 1.1. By condition (i) and that Q is increasing, we know that $Q(0) = 0$ and $Q(t) = t \Leftrightarrow t = 0$.

Liu [2] and Zhang [3] discussed common fixed point theorems for Altman type mappings in metric space. In 2006, a new structure of generalized metric space was introduced by Mustafa and Sims [4] as an appropriate notion of generalized metric space called G-metric space. Abbas and Rhoades [5] initiated the study of common fixed point in generalized metric space. Recently, many fixed point and common fixed point theorems for certain contractive conditions have been established in G-metric spaces, and for more details, one can refer to [6-25]. However, no one has discussed the common fixed point problems for the Altman type mappings recently.

Inspired by that, the purpose of this paper is to study common fixed point problem of Altman type for two self-mappings in G-metric space. we using the concept of R-weakly commuting self-mapping pairs in G-metric space, prove a new common fixed point theorem for two self-mappings. The results obtained in this paper differ from the recent relative results in literature.

Throughout the paper, we mean by $\mathbb{N}$ the set of all natural numbers.

Definition 1.1 ([4]). Let X be a nonempty set, and let $G : X \times X \times X \longrightarrow R^+$ be a function satisfying the following axioms:

$(G1)$ $G(x, y, z)$=0 if $x = y = z$;

$(G2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

$(G3)$ $G(x, x, y) \leq G(x, y, z)$,for all $x, y, z \in X$ with $z \neq$ y;

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables);

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality),

then the function G is called a generalized metric, or, more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2 ([4]). Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Proposition 1.1 ([4]). Let (X, G) be a G-metric space, then the followings are equivalent:

(1) $\{x_n\}$ is G-convergent to x.

(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.

(3) $G(x_n, x, x) \to 0$ as $n \to \infty$.

(4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Definition 1.3 ([4]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if, for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 1.4 ([4]). A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 1.2 ([4]). Let (X, G) be a G-metric space. Then the followings are equivalent.

(1) The sequence $\{x_n\}$ is G-Cauchy.

(2) For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq k$.

Proposition 1.3 ([4]). Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 1.5 ([4]). Let (X, G) and (X', G') be G-metric space, and $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.4 ([4]). Let (X, G) and (X', G') be G-metric space. Then $f : X \to X'$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.

Proposition 1.5 ([4]). Let (X, G) be a G-metric space. Then, for any x, y, z, a in X it follows that :

(i) if $G(x, y, z) = 0$, then $x = y = z$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(y, x, x)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

In 2010, Manro [6] introduced the concept of weakly commuting mappings, R-weakly commuting mappings into G-metric space as follows:

Definition 1.6 ([6]). A pair of self-mappings (f, g) of a G-metric space is said to be weakly commuting if $G(fgx, gfx, gfx) \leq G(fx, gx, gx), \forall x \in X$.

Definition 1.7 ([6]). A pair of self-mappings (f, g) of a G-metric space is said to be R-weakly commuting if there exists some positive real number R such that $G(fgx, gfx, gfx) \leq RG(fx, gx, gx), \forall x \in X$.

Remark 1.2. If $R \leq 1$, then R-weakly commuting mappings are weakly commuting.

Definition 1.8. Let f and g be self-mappings of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

2. Main results

Theorem 2.1. *Let (T, f) be a pairs of continuous self mappings in complete G-metric spaces (X, G). If there exists an increasing function $Q : [0, +\infty) \to [0, +\infty)$ satisfying the condition $(i) \sim (iii)$ and the following conditions:*

(iv) $T(X) \subseteq f(X)$;

(v) $G(Tx,Ty,Tz) \le Q\left(\max\{G(fx,fy,fz), G(fx,Tx,Tx), G(fy,Ty,Ty), G(fz,Tz,Tz)\}\right)$, *for all* $x,y,z \in X$.

Then the pairs (T,f) *has a coincidence point in* X. *Furthermore, if the pairs* (T,f) *be* R*-weakly commuting, then* T *and* f *have a unique common fixed point in* X.

Proof. Let x_0 be an arbitrary point in X. From the condition (iv), there exists $x_1 \in X$ such that $y_0 = Tx_0 = fx_1$. By induction, there exist two sequences $\{x_n\}, \{y_n\}$ in X, such that

$$y_n = Tx_n = fx_{n+1}, \quad \forall n \in \mathbb{N}. \tag{2.1}$$

If there exists $n \in \mathbb{N}$, such that $y_n = y_{n+1}$, then $Tx = fx$ with $x = x_{n+1}$, which implies that the pairs (T,f) has a coincidence point $x = x_{n+1}$. Without loss of generality, we may assume $\forall n \in \mathbb{N}$, $y_n \neq y_{n+1}$. Let $t_n = G(y_n, y_{n+1}, y_{n+2})$, now we show that

$$t_{n+1} \le Q(t_n) < t_n, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Actually, from the condition (v), (2.1) and (G3), we have

$$\begin{aligned}
t_{n+1} &= G(y_{n+1}, y_{n+2}, y_{n+3}) = G(Tx_{n+1}, Tx_{n+2}, Tx_{n+3}) \\
&\le Q\left(\max\left\{\begin{array}{c} G(fx_{n+1}, fx_{n+2}, fx_{n+3}), G(fx_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ G(fx_{n+2}, Tx_{n+2}, Tx_{n+2}), G(fx_{n+3}, Tx_{n+3}, Tx_{n+3}) \end{array}\right\}\right) \\
&= Q\left(\max\left\{\begin{array}{c} G(y_n, y_{n+1}, y_{n+2}), G(y_n, y_{n+1}, y_{n+1}), \\ G(y_{n+1}, y_{n+2}, y_{n+2}), G(y_{n+2}, y_{n+3}, y_{n+3}) \end{array}\right\}\right) \\
&\le Q\left(\max\left\{\begin{array}{c} G(y_n, y_{n+1}, y_{n+2}), G(y_n, y_{n+1}, y_{n+2}), \\ G(y_{n+1}, y_{n+2}, y_{n+3}), G(y_{n+1}, y_{n+2}, y_{n+3}) \end{array}\right\}\right) \\
&= Q\left(\max\{G(y_n, y_{n+1}, y_{n+2}), G(y_{n+1}, y_{n+2}, y_{n+3})\}\right) \\
&= Q(t_n).
\end{aligned}$$

By the property (i) of Q, we have $t_{n+1} \le Q(t_n) < t_n$, this is the (2.2) hold. This implies that $\{t_n\}$ is a nonnegative sequence which is strictly decreasing, hence $\{t_n\}$ is convergent and $t_{n+1} \le Q(t_n) < t_n, \forall n \in \mathbb{N}$.

Next, we prove $\{y_n\}$ is a G-Cauchy sequence in X. In fact, for any $n, m \in \mathbb{N}$, $m > n$, by combining $(G5), (G3)$ and (2.2), we have

$$\begin{aligned} G(y_n, y_m, y_m) &\leq \sum_{i=n}^{m-1} G(y_i, y_{i+1}, y_{i+1}) \\ &\leq \sum_{i=n}^{m-1} G(y_i, y_{i+1}, y_{i+2}) \\ &= \sum_{i=n}^{m-1} t_i \\ &= \sum_{i=n}^{m-1} \frac{t_i(t_i - t_{i+1})}{t_i - t_{i+1}} \\ &\leq \sum_{i=n}^{m-1} \frac{t_i(t_i - t_{i+1})}{t_i - Q(t_i)} \\ &\leq \sum_{i=n}^{m-1} \int_{t_{i+1}}^{t_i} \frac{t}{t - Q(t)} dt \\ &= \int_{t_m}^{t_n} p(t) dt. \end{aligned}$$

From the convergence of the sequence $\{t_n\}$ and the condition (iii) we assure that

$$\lim_{n,m\to\infty} \int_{t_m}^{t_n} p(t)dt = 0.$$

Thus, $\{y_n\}$ is a G-Cauchy sequence in X, since (X, G) is a complete G-metric space, there exists $u \in X$ such that $\lim_{n\to\infty} y_n = u$. Since (T, f) are R-weakly commuting mappings, thus we have

$$\begin{aligned} G(Ty_{n-1}, fy_n, fy_n) &= G(Tfx_n, fTx_n, fTx_n) \\ &\leq RG(Tx_n, fx_n, fx_n) \\ &= RG(y_n, y_{n-1}, y_{n-1}). \end{aligned}$$

On taking $n \to \infty$ at both sides, noting that T and f are continuous mappings, we have $G(Tu, fu, fu) \leq RG(u, u, u) = 0$, which gives that $Tu = fu$. Setting $z = Tu = fu$. Since (T, f) are R-weakly commuting mappings, we have

$$G(Tz, fz, fz) = G(Tfu, fTu, fTu) \leq RG(Tu, fu, fu) = RG(z, z, z) = 0.$$

Which gives that $Tz = fz$.

Now we prove that $Tz = fz = z$. If not, we have $G(Tz, z, z) > 0$. By the property (i) of Q that $0 < Q(G(Tz, z, z)) < G(Tz, z, z)$. Again, it follows from the condition (v) that

$$\begin{aligned} G(Tz,z,z) &= G(Tz,Tu,Tu) \\ &\leq Q\left(\max\left\{\begin{array}{c} G(fz,fu,fu), G(fz,Tz,Tz), \\ G(fu,Tu,Tu), G(fu,Tu,Tu) \end{array}\right\}\right) \\ &= Q\left(\max\left\{\begin{array}{c} G(Tz,z,z), G(Tz,Tz,Tz), \\ G(Tu,Tu,Tu), G(Tu,Tu,Tu) \end{array}\right\}\right) \\ &= Q(G(Tz,z,z) < G(Tz,z,z). \end{aligned}$$

It is a contradiction, this implies that $Tz = fz = z$, so z is a common fixed point of T and f.

In the following part, we will show the common fixed point of T and f is unique. In fact, assume v is another common fixed point of T and f. From the condition (v) and the property (i) of Q, we have

$$\begin{aligned} G(z,v,v) &= G(Tz,Tv,Tv) \\ &\leq Q\left(\max\{G(fz,fv,fv), G(fz,Tz,Tz), G(fv,Tv,Tv), G(fv,Tv,Tv)\}\right) \\ &= Q\left(G(z,v,v)\right) \\ &< G(z,v,v). \end{aligned}$$

This is a contradiction, that is T and f have a unique common fixed point in X. This completes the proof of Theorem 2.1.

Now we give an example to support Theorem 2.1.

Example 2.1 Let $X = [0, \infty)$, $G(x, y, z) = |x - y| + |y - z| + |z - x|, \forall x, y, z \in X$. Let $T, f : X \to X$ be defined by $Tx = \frac{x}{8}, fx = \frac{x}{2}$. Clearly, we can get $T(X) \subseteq f(X)$. Through calculation, we have

$$\begin{aligned} G(Tx,Ty,Tz) &= G\left(\frac{x}{8},\frac{y}{8},\frac{z}{8}\right) \\ &= \left|\frac{x}{8}-\frac{y}{8}\right| + \left|\frac{y}{8}-\frac{z}{8}\right| + \left|\frac{x}{8}-\frac{z}{8}\right| \\ &= \frac{1}{8}(|x-y|+|y-z|+|z-y|), \end{aligned}$$

$$\begin{aligned} G(fx,fy,fz) &= G\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right) \\ &= \left|\frac{x}{2}-\frac{y}{2}\right| + \left|\frac{y}{2}-\frac{z}{2}\right| + \left|\frac{z}{2}-\frac{x}{2}\right| \\ &= \frac{1}{2}(|x-y|+|y-z|+|z-y|), \end{aligned}$$

Thus we have

$$\begin{aligned} G(Tx,Ty,Tz) &= \frac{1}{4}G(fx,fy,fz) \\ &\leq \frac{1}{4}\max\{G(fx,fy,fz),G(fx,Tx,Tx),G(fy,Ty,Ty),G(fz,Tz,Tz)\} \\ &= Q\left(\max\{G(fx,fy,fz),G(fx,Tx,Tx),G(fy,Ty,Ty),G(fz,Tz,Tz)\}\right). \end{aligned}$$

Where $Q(t)=\frac{t}{4}$. Since $p(t)=\frac{t}{t-\frac{t}{4}}=\frac{4}{3}$ is a constant, thus conditions (i)-(iii) are satisfied.

On the other hand, we have

$$G(Tfx,fTx,fTx)=G\left(\frac{x}{16},\frac{x}{16},\frac{x}{16}\right)\leq\frac{1}{3}G(Tx,fx,fx)$$

for all $x\in X$. Which means that (T,f) is a pairs of continuous R-weakly commuting mappings in X. So that all the conditions of theorem 2.1 are satisfied. Moreover, 0 is the unique common fixed point of the mappings T and f.

If we taken $f=I$, where I stands for the identity mapping, in Theorem 2.1, then we have

Corollary 2.1. *Let T be a continuou mapping in complete G-metric spaces (X,G). If there exists an increasing function $Q:[0,+\infty)\to[0,+\infty)$ satisfying the condition (i) $\sim$ (iii) and (iv): $G(Tx,Ty,Tz)\leq Q\left(\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\}\right)$, for all $x,y,z\in X$. Then the T have a unique fixed point in X.*

If $Q(t)=kt$ with $0<k<1$ in Theorem 2.1, then we have the following.

Corollary 2.2. *Let (T, f) be a pairs of continuous self mappings in complete G-metric spaces (X, G), and satisfying the following conditions:*

(i) $T(X) \subseteq f(X)$,

(ii) $G(Tx, Ty, Tz) \leq k \max\{G(fx, fy, fz), G(fx, Tx, Tx), G(fy, Ty, Ty), G(fz, Tz, Tz)\}$

for all $x, y, z \in X$, where $0 < k < 1$. Then the pairs (T, f) has a coincidence point in X. Furthermore, if the pairs (T, f) be R-weakly commuting, then T and f have a unique common fixed point in X.

Corollary 2.3. *Let T be a continuous mappings in complete G-metric spaces (X, G) such that $G(Tx, Ty, Tz) \leq k \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}$, for all $x, y, z \in X$, where $0 < k < 1$. Then T have a unique fixed point in X.*

References

[1] M. Altman, A fixed point theorem in compact metric spaces, Amer. Math. Monthly, 82 (1975), 827-829.

[2] Z. Liu, On common fixed points of Altman type mappings, J. Liaoning Normal Univ. 16 (1993), 1-4.

[3] S. Zhang, Common fixed point theorem of Altman type mappings, J. Yantai Normal Univ. 16 (2000), 95-97.

[4] Z. Mustafa, B. Sims, A new approach to a generalized metric space, J. Nonlinear Convex Anal. 7 (2006), 289-297.

[5] M. Abbas, B. E. Rhoades, Common fixed point results for non-commuting mappings without continuity in generalized metric spaces, Appl. Math. Comput. 215 (2009), 262-269.

[6] S. Manro, S. S. Bhatia, S. Kumar, Expansion mapping theorems in G-metric spaces, Int. J. Contemp. Math. Sciences, 5 (2010), 2529-2535.

[7] R. K. Vats, S. Kumar, V. Sihag, Some common fixed point theorems for compatible mappings of type(A) in complete G-metric space, J. Adv. Fuzzy Math. 6 (2011), 27-38.

[8] M. Abbas, T. Nazir, R. Saadati, Common fixed point results for three maps in generalized metric space, Adv. Differ. Equ. 49 (2011), 1-20.

[9] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory Appl. 2009 (2009), Article ID 917175.

[10] Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorems for mappings on complete G-metric space, Fixed Point Theory Appl. 2008 (2008), Article ID 189870.

[11] A. P. Chouhan, A. S. Ranadive, Absorbing maps and common fixed point theorem in Menger space, Adv. Fixed Point Thoery, 2 (2012) 108-119.

[12] M. Abbas, T. Nazir, P. Vetro, Common fixed point results for three maps in G-metric spaces, Filomat, 25 (2011), 1-17.

[13] S. Manro, S. Kumar, S. S. Bhatia, Common fixed point theorems in intuitionistic fuzzy metric spaces using occasionally weakly compatible maps, J. Math. Comput. Sci. 2 (2012) 73-81.

[14] W. Shatanawi, Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces, Fixed Point Theory Appl. 2010 (2010), Article ID 181650.

[15] L. Gajić, Z. Lozanov-Crvenković, A fixed point results for mappings with contractive iterate at a point in G-metric spaces, Filomat, 25 (2011), 53-58.

[16] H. Aydi, A fixed point result involving a generalized weakly contractive condition in G-metric spaces, Bull. Math. Anal. Appl. 3 (2011), 180-188.

[17] M. Abbas, S. H. Khan, T. Nazir, Common fixed points of R-weakly commuting maps in generalized metric spaces, Fixed Point Theory Appl. 2011 (2011) Article ID 784595.

[18] R. Chugh, T. Kadian, A. Rani, B. E. Rhoades, Property P in G-metric spaces, Fixed Point Theory Appl. 2010 (2010), Article ID 401684.

[19] L. Ciric, M. O. Olatinwo, D. Gopal, G. Akinbo, Coupled fixed point theorems for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Adv. Fixed Point Theory 2 (2012), 1-8.

[20] Z. Mustafa, M. Khandagji, W. Shatanawi, Fixed point results on complete G-metric spaces, Stud. Sci. Math. Hung. 48 (2011), 304-319.

[21] H. Aydi, W. Shatanawi, C. Vetro, On generalized weakly G-contraction mapping in G-metric spaces, Comput. Math. Appl. 62 (2011), 4222-4229.

[22] Z. Kadelburg, S. Radenovic, Coupled fixed point results under tvs-cone metric and w-cone-distance, Adv. Fixed Point Theory, 2 (2012), 29-46.

[23] H. Obiedat, Z. Mustafa, Fixed point results on a nonsymmetric G-metric spaces, Jordan J. Math. Stat. 3 (2010), 65-79.

[24] S. Manro, S. Kumar, S. S. Bhatia, R-weakly commuting maps in G-metric spaces, Fasciculi Math. 47 (2010), 11-18.

[25] R. K. Vats, S. Kumar, V. Sihag, Fixed point theorems in complete G-metric spaces, Fasciculi Math. 47 (2011), 127-139.