Available online at http://scik.org
Advances in Fixed Point Theory, 2 (2012), No. 4, 398-416
ISSN: 1927-6303

CONSTRUCTIVE PROOF OF THE EXISTENCE OF NASH EQUILIBRIUM IN A STRATEGIC GAME WITH SEQUENTIALLY LOCALLY NON-CONSTANT PAYOFF FUNCTIONS

YASUHITO TANAKA*

Faculty of Economics, Doshisha University, Kamigyo-ku, Kyoto, 602-8580, Japan

Abstract. In this paper we constructively prove the existence of Nash equilibrium in a finite strategic game with sequentially locally non-constant payoff functions by a constructive version of Kakutani's fixed point theorem for sequentially locally non-constant multi-functions (multi-valued functions or correspondences). We also examine the existence of Nash equilibrium in a game with continuous strategies and quasi-concave payoff functions which has sequentially locally at most one maximum. We follow the Bishop style constructive mathematics.

Keywords: constructive mathematics; sequentially locally non-constant payoff functions; Nash equilibrium.

2000 AMS Subject Classification: 26E40; 91A10

1. Introduction

*Corresponding author

This research was partially supported by the Ministry of Education, Science, Sports and Culture of Japan, Grant-in-Aid for Scientific Research (C), 20530165.

Received September 9, 2012

It is well known that Brouwer's fixed point theorem can not be constructively proved[1]. Thus, Kakutani's fixed point theorem for multi-functions (multi-valued functions or correspondences) and the existence of Nash equilibrium in a strategic game also can not be constructively proved. On the other hand, Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's fixed point theorem using Sperner's lemma (See [6] and [11]). Also Dalen in [6] states a conjecture that a function f from a simplex to itself, with property that each open set contains a point x which is not equal to $f(x)$ ($x \neq f(x)$) and on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. Recently Berger and Ishihara[2] showed that the following theorem is equivalent to Brouwer's fan theorem, and so it is non-constructive.

> Each uniformly continuous function from a compact metric space into itself with at most one fixed point has a fixed point.

By reference to the notion of *sequentially at most one maximum* in Berger, Bridges and Schuster[1] we require a more general and somewhat stronger condition of *sequential local non-constancy* for functions, and in [8] we have shown the following result.

> If each uniformly continuous function from a compact metric space into itself is *sequentially locally non-constant*, then it has a fixed point,

without the fan theorem. It is a partial answer to Dalen's conjecture.

In [9] we have proved the existence of Nash equilibrium in a finite strategic game with sequentially locally non-constant payoff functions by a constructive version of Brouwer's fixed point theorem. A proof by Kakutani's fixed point theorem is more smart than a proof by Brouwer's fixed point theorem. In [10] we have proved the mini-max theorem of zero-sum games by a constructive version of Kakutani's fixed point theorem for multi-functions with sequentially at most one fixed point. The condition of sequentially at most

[1] [7] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics à la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive (See [4] or [6]).

one fixed point is stronger than the condition of sequential local non-constancy in this paper. Thus, a theorem in this paper is more general than a theorem in [10].

In this paper we extend the sequential local non-constancy for functions to multi-functions, and prove the existence of Nash equilibrium in a finite strategic game with sequentially locally non-constant payoff functions by a constructive version of Kakutani's fixed point theorem for sequentially locally non-constant multi-functions. We also examine the existence of Nash equilibrium in a game with continuous strategies and quasi-concave payoff functions which has sequentially locally at most one maximum. We follow the Bishop style constructive mathematics according to [3], [4] and [5].

In the next section we will prove our Kakutani's fixed theorem. In Section 3 we will prove the existence of Nash equilibrium in a finite strategic game. In Section 4 we study a game with continuous strategies and quasi-concave payoff functions.

2. Kakutani's fixed point theorem for sequentially locally non-constant multi-functions

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S.

> Note that in order to show that S is inhabited, we cannot just prove that it is impossible for S to be empty: we must actually construct an element of S (see page 12 of [5]).

Also in constructive mathematics compactness of a set means *total boundedness with completeness*. A set S is *finitely enumerable* if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S. An ε-approximation to S is a subset of S such that for each $\mathbf{p} \in S$ there exists $\mathbf{q}$ in that ε-approximation with $|\mathbf{p}-\mathbf{q}| < \varepsilon$($|\mathbf{p}-\mathbf{q}|$ is the distance between $\mathbf{p}$ and $\mathbf{q}$). S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S. Completeness of a set, of course, means that every Cauchy sequence in the set converges.

Let $\mathbf{p}$ be a point in a compact metric space X, and f be a uniformly continuous function from X into itself. According to [6] and [11] f has an approximate fixed point (or an ε-approximate fixed point). It means

$$\text{For each } \varepsilon > 0 \text{ there exists } \mathbf{p} \in X \text{ such that } |\mathbf{p} - f(\mathbf{p})| < \varepsilon.$$

Now consider an n-dimensional simplex Δ as a compact metric space. According to Corollary 2.2.12 of [5], we have the following result.

Lemma 2.1. For each $\varepsilon > 0$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^n H_i$.

The notion that a function f has at most one fixed point by [2] is defined as follows;

Definition 2.1. (At most one fixed point) For all $\mathbf{p}, \mathbf{q} \in \Delta$, if $\mathbf{p} \neq \mathbf{q}$, then $f(\mathbf{p}) \neq \mathbf{p}$ or $f(\mathbf{q}) \neq \mathbf{q}$.

By reference to the notion of *sequentially at most one maximum* in [1], we define the property of *sequential local non-constancy* as follows;

Definition 2.2. (Sequential local non-constancy of functions) There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(\mathbf{p}_n)_{n\geq 1}$, $(\mathbf{q}_n)_{n\geq 1}$ in each H_i, $|f(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ and $|f(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$, then $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$.

Let F be a compact and convex valued multi-function from Δ to the collection of its inhabited subsets. Since Δ and $F(\mathbf{p})$ for $\mathbf{p} \in \Delta$ are compact, $F(\mathbf{p})$ is located (see Proposition 2.2.9 in [5]), that is, $|F(\mathbf{p}) - \mathbf{q}| = \inf_{\mathbf{r}\in F(\mathbf{p})} |\mathbf{r} - \mathbf{q}|$ for $\mathbf{q} \in \Delta$ exists.

The definition of sequential local non-constancy for multi-functions is as follows;

Definition 2.3. (Sequential local non-constancy of multi-functions) There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(\mathbf{p}_n)_{n\geq 1}$, $(\mathbf{q}_n)_{n\geq 1}$ in each H_i, $|F(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ and $|F(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$, then $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$.

A graph of a multi-function F from Δ to the collection of its inhabited subsets is

$$G(F) = \cup_{\mathbf{p}\in\Delta}\{\mathbf{p}\} \times F(\mathbf{p}).$$

If $G(F)$ is a closed set, we say that F has a closed graph. It implies the following fact.

For sequences $(\mathbf{p}_n)_{n\geq 1}$ and $(\mathbf{q}_n)_{n\geq 1}$ such that $\mathbf{q}_n \in F(\mathbf{p}_n)$, if $\mathbf{p}_n \longrightarrow \mathbf{p}$, then for some $\mathbf{q} \in F(\mathbf{p})$ we have $\mathbf{q}_n \longrightarrow \mathbf{q}$.

On the other hand, if the following condition is satisfied, we say that F has a uniformly closed graph.

For sequences $(\mathbf{p}_n)_{n\geq 1}$, $(\mathbf{q}_n)_{n\geq 1}$, $(\mathbf{p}'_n)_{n\geq 1}$, $(\mathbf{q}'_n)_{n\geq 1}$ such that $\mathbf{q}_n \in F(\mathbf{p}_n)$, $\mathbf{q}'_n \in F(\mathbf{p}'_n)$, if $|\mathbf{p}_n - \mathbf{p}'_n| \longrightarrow 0$, then for any $\mathbf{q}_n$ and some $\mathbf{q}'_n$, we have $|\mathbf{q}_n - \mathbf{q}'_n| \longrightarrow 0$, and for any $\mathbf{q}'_n$ and some $\mathbf{q}_n$, we have $|\mathbf{q}_n - \mathbf{q}'_n| \longrightarrow 0$.

Let $\mathbf{q} \in F(\mathbf{p})$, $(\mathbf{p}'_n)_{n\geq 1} = \{\mathbf{p}, \mathbf{p}, \dots\}$ and $(\mathbf{q}'_n)_{n\geq 1} = \{\mathbf{q}, \mathbf{q}, \dots\}$ be sequences with a constant points $\mathbf{p}$ and $\mathbf{q}$. If $|\mathbf{p}_n - \mathbf{p}'_n| = |\mathbf{p}_n - \mathbf{p}| \longrightarrow 0$, then $|\mathbf{q}_n - \mathbf{q}'_n| = |\mathbf{q}_n - \mathbf{q}| \longrightarrow 0$, that is, if $\mathbf{p}_n \longrightarrow \mathbf{p}$, then $\mathbf{q}_n \longrightarrow \mathbf{q}$, and so uniformly closed graph property implies closed graph property.

In this definition

$|\mathbf{p}_n - \mathbf{p}'_n| \longrightarrow 0$ means that for any $\delta > 0$ there exists n_0 such that when $n \geq n_0$ we have $|\mathbf{p}_n - \mathbf{p}'_n| < \delta$, and $|\mathbf{q}_n - \mathbf{q}'_n| \longrightarrow 0$ means that for any $\varepsilon > 0$ there exists n'_0 such that when $n \geq n'_0$, we have $|\mathbf{q}_n - \mathbf{q}'_n| < \varepsilon$.

Now we show the following lemma.

Lemma 2.2. Let F be a convex and compact valued multi-function with uniformly closed graph from Δ to the collection of its inhabited subsets. Assume $\inf_{\mathbf{p}\in H_i} |F(\mathbf{p}) - \mathbf{p}| = 0$ in some H_i such that $\Delta = \cup_{i=1}^m H_i$. If the following property holds:

For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{p}, \mathbf{q} \in H_i$, $|F(\mathbf{p}) - \mathbf{p}| < \delta$ and $|F(\mathbf{q}) - \mathbf{q}| < \delta$, then $|\mathbf{p} - \mathbf{q}| \leq \varepsilon$.

Then, there exists a point $\mathbf{r} \in H_i$ such that $\mathbf{r} \in F(\mathbf{r})$.

Proof. Choose a sequence $(\mathbf{p}_n)_{n\geq 1}$ in H_i such that $|F(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$. Compute N such that $|F(\mathbf{p}_n) - \mathbf{p}_n| < \delta$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|\mathbf{p}_m - \mathbf{p}_n| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $(\mathbf{p}_n)_{n\geq 1}$ is a Cauchy sequence in H_i, and converges to a limit $\mathbf{r} \in H_i$. The uniformly closed graph property of F yields $\mathbf{r} \in F(\mathbf{r})$.

This completes the proof.

A fixed point of a multi-function is defined as follows;

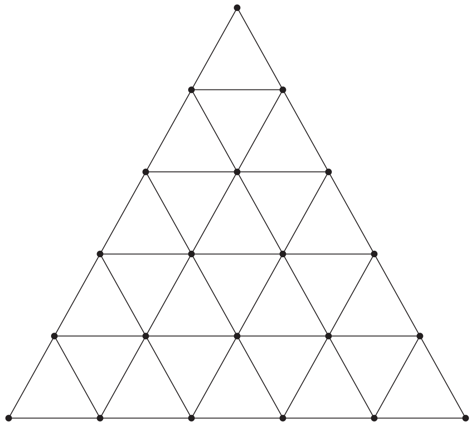

FIGURE 1. Subdivision of 2-dimensional simplex

Definition 2.4. $\mathbf{p}$ is a fixed point of a multi-function F if $\mathbf{p} \in F(\mathbf{p})$.

We define an approximate fixed point of a multi-function F as follows;

Definition 2.5. For each $\varepsilon > 0$ $\mathbf{p}$ is an approximate fixed point (or an ε-approximate fixed point) of a multi-function F if $|\mathbf{p} - F(\mathbf{p})| < \varepsilon$.

We will constructively show that if the value of a sequentially locally non-constant multi-function F from Δ to the collection of inhabited subsets of Δ with uniformly closed graph is compact and convex, it has a fixed point. If a set X is homeomorphic to Δ (so X is also compact), we can show the same result for a multi-function from X to the collection of inhabited subsets of X.

Our Kakutani's fixed point theorem is as follows;

Theorem 2.1. If F is compact and convex valued sequentially locally non-constant multi-function with uniformly closed graph from an n-dimensional simplex Δ to the collection of its inhabited subsets, then it has a fixed point.

Proof.

(1) Let Δ be an n-dimensional simplex, and consider m-th subdivision of Δ. Subdivision in a case of 2-dimensional simplex is illustrated in Figure 1. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider subdivision of Δ inductively for cases of higher dimension.

Let us partition Δ sufficiently fine, and define a uniformly continuous function $f^m : \Delta \longrightarrow \Delta$ as follows. If $\mathbf{p}$ is a vertex of a simplex constructed by m-th subdivision of Δ, let $f^m(\mathbf{p}) = \mathbf{q}$ for some $\mathbf{q} \in F(\mathbf{p})$. For other $\mathbf{p} \in \Delta$ we define $f^m(\mathbf{p})$ by a convex combination of the values of F at vertices of a simplex $\mathbf{p}_0^m$, $\mathbf{p}_1^m$, $\dots$, $\mathbf{p}_n^m$. Let $\sum_{i=0}^n \lambda_i = 1$, $\lambda_i \geq 0$,

$$f^m(\mathbf{p}) = \sum_{i=0}^n \lambda_i f^m(\mathbf{p}_i^m) \text{ with } \mathbf{p} = \sum_{i=0}^n \lambda_i \mathbf{p}_i^m.$$

Since f^m is clearly uniformly continuous, it has an approximate fixed point according to [6] and [11]. Let $\mathbf{p}^*$ be an approximate fixed point of f^m, then for each $\frac{\varepsilon}{2} > 0$ there exists $\mathbf{p}^* \in \Delta$ which satisfies

$$|\mathbf{p}^* - f^m(\mathbf{p}^*)| < \frac{\varepsilon}{2}.$$

Consider a sequence, $(\Delta_m)_{m\geq 1}$, of partition of Δ and a sequence of the distance between vertices of simplices constructed by partition $(|\mathbf{p}_i^m - \mathbf{p}_j^m|)_{m\geq 1}$, $i \neq j$. Suppose $|\mathbf{p}_i^m - \mathbf{p}_j^m| \longrightarrow 0$. Since F has a uniformly closed graph, for any $\mathbf{q}_i^m \in F(\mathbf{p}_i^m)$ and some $\mathbf{q}_j^m \in F(\mathbf{p}_j^m)$, $|\mathbf{q}_i^m - \mathbf{q}_j^m| \longrightarrow 0$, and for any $\mathbf{q}_j^m \in F(\mathbf{p}_j^m)$ and some $\mathbf{q}_i^m \in F(\mathbf{p}_i^m)$, $|\mathbf{q}_i^m - \mathbf{q}_j^m| \longrightarrow 0$. $\mathbf{p}^*$ is represented by $\mathbf{p}^* = \sum_{i=0}^n \lambda_i \mathbf{p}_i^m$. If $|\mathbf{p}_i^m - \mathbf{p}_j^m| \longrightarrow 0$ for each pair of i and j $(j \neq i)$, $|\mathbf{p}_i^m - \mathbf{p}^*| \longrightarrow 0$. Thus, for any $\mathbf{q}_i^m \in F(\mathbf{p}_i^m)$ and some $\mathbf{q}_i^* \in F(\mathbf{p}^*)$, we have $|\mathbf{q}_i^m - \mathbf{q}_i^*| < \frac{\varepsilon}{2}$. For different i, that is, different $\mathbf{p}_i^m$, $\mathbf{q}_i^*$ may be different. But, the convexity of $F(\mathbf{p}^*)$ implies

$$\mathbf{q}^* = \sum_{i=0}^n \lambda_i \mathbf{q}_i^* \in F(\mathbf{p}^*).$$

Since, for sufficiently large m we have $|\mathbf{q}_i^m - \mathbf{q}_i^*| < \frac{\varepsilon}{2}$ for each i, and

$$f^m(\mathbf{p}^*) = \sum_{i=0}^n \lambda_i f^m(\mathbf{p}_i^m) = \sum_{i=0}^n \lambda_i \mathbf{q}_i^m,$$

we obtain $|f^m(\mathbf{p}^*) - \mathbf{q}^*| < \frac{\varepsilon}{2}$. From $|\mathbf{p}^* - f^m(\mathbf{p}^*)| < \frac{\varepsilon}{2}$

$$|\mathbf{p}^* - \mathbf{q}^*| < \varepsilon. \tag{1}$$

Since $\mathbf{q}^* \in F(\mathbf{p}^*)$, $\mathbf{p}^*$ is an approximate fixed point of F. ε is arbitrary, and so

$$\inf_{\mathbf{p}^* \in \Delta} |\mathbf{p}^* - F(\mathbf{p}^*)| = 0.$$

This means

$$\inf_{\mathbf{p}^* \in H_i} |\mathbf{p}^* - F(\mathbf{p}^*)| = 0$$

in some H_i such that $\Delta = \cup_{i=1}^{m} H_i$.

(2) Choose a sequence $(\mathbf{r}_n)_{n\geq 1}$ in Δ such that $|\mathbf{r}_n - F(\mathbf{r}_n)| \longrightarrow 0$. In view of Lemma 2.2 it is enough to prove that the following condition holds.

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $\mathbf{p}, \mathbf{q} \in \Delta$, $|F(\mathbf{p}) - \mathbf{p}| < \varepsilon$ and $|F(\mathbf{q}) - \mathbf{q}| < \varepsilon$, then $|\mathbf{p} - \mathbf{q}| \leq \delta$.

Assume that the set

$$K = \{(\mathbf{p}, \mathbf{q}) \in \Delta \times \Delta : \ |\mathbf{p} - \mathbf{q}| \geq \delta\}$$

is nonempty and compact (see Theorem 2.2.13 of [5].). Since the mapping $(\mathbf{p}, \mathbf{q}) \longrightarrow \max(|F(\mathbf{p}) - \mathbf{p}|, |F(\mathbf{q}) - \mathbf{q}|)$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n\geq 1}$ such that

$$\lambda_n = 0 \Rightarrow \inf_{(\mathbf{p},\mathbf{q})\in K} \max(|F(\mathbf{p}) - \mathbf{p}|, |F(\mathbf{q}) - \mathbf{q}|) < 2^{-n},$$

$$\lambda_n = 1 \Rightarrow \inf_{(\mathbf{p},\mathbf{q})\in K} \max(|F(\mathbf{p}) - \mathbf{p}|, |F(\mathbf{q}) - \mathbf{q}|) > 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $|F(\mathbf{p}) - \mathbf{p}| < 2^{-n-1}$, $|F(\mathbf{q}) - \mathbf{q}| < 2^{-n-1}$, we have $(\mathbf{p}, \mathbf{q}) \notin K$ and $|\mathbf{p} - \mathbf{q}| \leq \delta$. Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(\mathbf{p}_n, \mathbf{q}_n) \in K$ such that $\max(|F(\mathbf{p}_n) - \mathbf{p}_n|, |F(\mathbf{q}_n) - \mathbf{q}_n|) < 2^{-n}$, and if $\lambda_n = 1$, set $\mathbf{p}_n = \mathbf{q}_n = \mathbf{r}_n$. Then, $|F(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ and $|F(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$, so $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$. Computing N such that $|\mathbf{p}_N - \mathbf{q}_N| < \delta$, we must have $\lambda_N = 1$.

This completes the proof.

3. Nash equilibrium of a finite strategic game

Consider a strategic game such that there are n players with m alternative pure strategies for each player. n and m are finite positive integers which are not smaller than 2. Denote the set of strategies of player i by S_i, and denote his each pure strategy by s_{ij}. His mixed strategy is defined to be a probability distribution over the set of his pure strategies, and is denoted by p_i. Denote the set of all p_i by P_i, which is totally bounded.

Call a combination of mixed strategies of players a *profile*, and denote it by $\mathbf{p}$. $\mathbf{p}$ is a vector with $n \times m$ components, but only $n(m-1)$ components are independent. The set of all $\mathbf{p}$ is n-times product of $m-1$-dimensional simplices, and so it is compact and convex. It is homeomorphic to an $n(m-1)$-dimensional simplex. Denote the expected payoff of player i at a profile $\mathbf{p}$ by $\pi_i(\mathbf{p})$, and his expected payoff when he chooses a pure strategy s_{ij} at that profile by $\pi_i(s_{ij}, \mathbf{p}_{-i})$. Then, $\pi_i(\mathbf{p})$ is written as follows;

$$\pi_i(\mathbf{p}) = \sum_{\{j: p_{ij}>0\}} p_{ij}\pi_i(s_{ij}, \mathbf{p}_{-i}).$$

$\mathbf{p}_{-i}$ denotes a combination of strategies of players other than i at $\mathbf{p}$. We assume that payoff of each player is finite. Then, since the expected payoff of each player is linear with respect to probability distributions over the sets of pure strategies of players, it is a uniformly continuous function. We define the best response pure strategies of player i to $\mathbf{p}_{-i}$, $s^*_{ij}(\mathbf{p}_{-i})$, as follows;

$$\pi_i(s^*_{ij}, \mathbf{p}_{-i}) \geq \pi_i(s'_{ij}, \mathbf{p}_{-i}) \text{ for all } s'_{ij} \in S_i.$$

Since S_i is finite, we can find s_{ij} which realizes $\max_{s_{ij}\in S_i} \pi_i(s_{ij}, \mathbf{p}_{-i})$. Linearity of the expected payoff functions implies that if there are multiple best response pure strategies for player i to $\mathbf{p}_{-i}$, convex combinations of these pure strategies are also best responses to $\mathbf{p}_{-i}$. Call them the best response mixed strategies for player i to $\mathbf{p}_{-i}$. Let denote the set of the best response pure and mixed strategies to $\mathbf{p}_{-i}$ by $BR_i(\mathbf{p}_{-i})$.

Each player chooses one of his best response pure or mixed strategies given a combination of strategies of other players. A Nash equilibrium is a state where all players choose their best responses each other. The set of best responses of all players is represented as follows;

$$\begin{aligned}&\mathbf{BR}(\mathbf{p})\\&=(BR_1(\mathbf{p}_{-1}), BR_2(\mathbf{p}_{-2}), \dots, BR_i(\mathbf{p}_{-i}), \dots, BR_n(\mathbf{p}_{-n})).\end{aligned}$$

This is a multi-function from the set of players' mixed strategies, which is denoted by $\mathbf{P}$, to the collection of its inhabited subsets.

We assume the following condition.

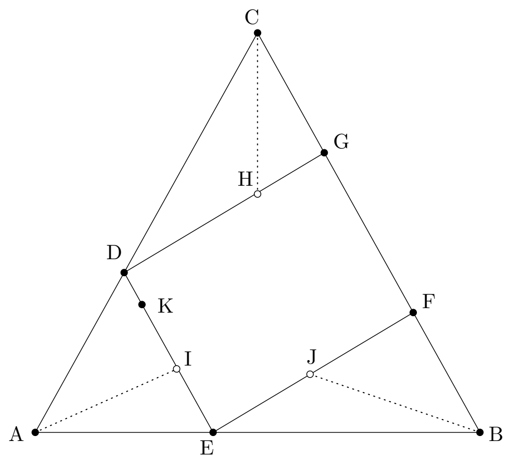

FIGURE 2. Homeomorphism between simplex and combination of strategies

Definition 3.1. (Sequential local non-constancy of payoff functions) There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $\mathbf{P} = \cup_{i=1}^{m} H_i$, and if for all sequences $(\mathbf{p}_n, \mathbf{q}_n)_{n\geq 1}$ we have $\max(\pi_i(s_{ij}, (\mathbf{p}_n)_{-i}) - \pi_i(\mathbf{p}_n), 0) \longrightarrow 0$, $\max(\pi_i(s_{ij}, (\mathbf{q}_n)_{-i}) - \pi_i(\mathbf{q}_n), 0) \longrightarrow 0$ for all $s_{ij} \in S_i$ and all i, then $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$.

Let us consider a homeomorphism between an $n(m-1)$-dimensional simplex and the set of players' mixed strategies $\mathbf{P}$ which is illustrated by a rectangle $DEFG$ in Figure 2. This figure depicts an example of a case of two players with two pure strategies for each player. Vertices D, E, F and G represent states where two players choose pure strategies, and points on edges DE, EF, FG and GD represent states where one player chooses a pure strategy. In such a homeomorphism vertices of the simplex do not correspond to any vertex of $\mathbf{P}$. But points on faces (simplices whose dimension is lower than $n(m-1)$) of the simplex as well as the vertices of the simplex correspond to the points on edges or faces, not inside, of $\mathbf{P}$. For example, in Figure 2, A, B and C correspond, respectively, to I, J and H. On the other hand, each of vertices of $\mathbf{P}$, D, E, F and G correspond, respectively, to itself on the faces of the simplex.

Let us check some properties of $\mathbf{BR}$.

(1) The set of all $\mathbf{p}$ is clearly compact and convex.

(2) $\mathbf{BR}$ is a multi-function from the set of all $\mathbf{p}$ to the set of all inhabited subsets.

(3) We show convexity of $\mathbf{BR}(\mathbf{p})$. It is sufficient to show the convexity of $BR_i(\mathbf{p}_{-i})$ for each i. Suppose $p_i \in BR_i(\mathbf{p}_{-i})$ and $p_i'' \in BR_i(\mathbf{p}_{-i})$. Then,

$$\pi_i(p_i, \mathbf{p}_{-i}) \geq \pi_i(s_{ij}, \mathbf{p}_{-i}) \text{ for all } s_{ij} \in S_i,$$

and

$$\pi_i(p_i'', \mathbf{p}_{-i}) \geq \pi_i(s_{ij}, \mathbf{p}_{-i}) \text{ for all } s_{ij} \in S_i.$$

Let $0 \leq \lambda \leq 1$. We have

$$\lambda\pi_i(p_i, \mathbf{p}_{-i}) + (1-\lambda)\pi_i(p_i'', \mathbf{p}_{-i}) = \pi_i(\lambda p_i + (1-\lambda)p_i'', \mathbf{p}_{-i}) \geq \pi_i(s_{ij}, \mathbf{p}_{-i}) \text{ for all } s_{ij} \in S_i.$$

Therefore, $\lambda p_i + (1-\lambda)p_i'' \in BR_i(\mathbf{p}_{-i})$, and $BR_i(\mathbf{p}_{-i})$ is convex.

(4) Next we show that $\mathbf{BR}$ has a uniformly closed graph. Consider $BR_i(\mathbf{p}_{-i})$. Let $(\mathbf{p}_n)_{n\geq 1} = ((p_i, \mathbf{p}_{-i})_n)_{n\geq 1}$, $(\mathbf{p}'_n)_{n\geq 1} = ((p'_i, \mathbf{p}'_{-i})_n)_{n\geq 1}$ be sequences such that $(p_i)_n \in BR_i((\mathbf{p}_{-i})_n)$ and $(p'_i)_n \in BR_i((\mathbf{p}'_{-i})_n)$ for all n. By the uniform continuity of $\pi_i(p_i, \mathbf{p}_{-i})$, when $|(\mathbf{p}_{-i})_n - (\mathbf{p}'_{-i})_n| \longrightarrow 0$, we have $|\pi_i((p_i)_n, (\mathbf{p}_{-i})_n) - \pi_i((p_i)_n, (\mathbf{p}'_{-i})_n)| \longrightarrow 0$ and $|\pi_i((p'_i)_n, (\mathbf{p}_{-i})_n) - \pi_i((p'_i)_n, (\mathbf{p}'_{-i})_n)| \longrightarrow 0$. Let $(\varepsilon_n)_{n\geq 1}$ be a sequence such that $\varepsilon_1 > \varepsilon_2 > \dots$ and $\varepsilon_n \longrightarrow 0$. Then, there exists N such that if $n \geq N$, we have

$$\pi_i((p_i)_n, (\mathbf{p}'_{-i})_n) > \pi_i((p_i)_n, (\mathbf{p}_{-i})_n) - \varepsilon_n,$$

and

$$\pi_i((p'_i)_n, (\mathbf{p}_{-i})_n) > \pi_i((p'_i)_n, (\mathbf{p}'_{-i})_n) - \varepsilon_n.$$

On the other hand, by the definition of $BR_i(\mathbf{p}_{-i})$

$$\pi_i((p'_i)_n, (\mathbf{p}'_{-i})_n) \geq \pi_i((p_i)_n, (\mathbf{p}'_{-i})_n).$$

Thus, we obtain

$$\pi_i((p'_i)_n, (\mathbf{p}_{-i})_n) > \pi_i((p_i)_n, (\mathbf{p}_{-i})_n) - 2\varepsilon_n, \text{ for } n \geq N.$$

Since if $\varepsilon_n \longrightarrow 0$, $|\pi_i((p'_i)_n, (\mathbf{p}_{-i})_n) - \pi_i((p_i)_n, (\mathbf{p}_{-i})_n)| \longrightarrow 0$, for some $(\tilde{p}_i)_n \in BR_i((\mathbf{p}_{-i})_n)$ we have $|(p'_i)_n - (\tilde{p}_i)_n| \longrightarrow 0$, and so BR_i has a uniformly closed graph. This relation holds for all i. Therefore, $\mathbf{BR}$ has a uniformly closed graph.

(5) For all sequences $(\mathbf{p}_n)_{n\geq 1}$ and $(\mathbf{q}_n)_{n\geq 1}$, if $|\mathbf{BR}(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ and $|\mathbf{BR}(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$, then $\max(\pi_i(s_{ij}, (\mathbf{p}_n)_{-i}) - \pi_i(\mathbf{p}_n), 0) \longrightarrow 0$, $\max(\pi_i(s_{ij}, (\mathbf{q}_n)_{-i}) - \pi_i(\mathbf{q}_n), 0) \longrightarrow 0$ for all $s_{ij} \in S_i$ for all i. The sequential local non-constancy of payoff functions implies $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$. Thus, **BR** satisfies the sequential local non-constancy for multi-functions.

Then, a multi-function **BR** satisfies all of the conditions for Kakutani's fixed point theorem for sequentially locally non-constant multi-functions proved in the previous section, and so it has a fixed point. Let $\mathbf{p}^*$ be a fixed point of **BR**, that is, $\mathbf{BR}(\mathbf{p}^*) = \mathbf{p}^*$. At $\mathbf{p}^*$ the strategy of each player is a best response each other. Therefore, $\mathbf{p}^*$ is a Nash equilibrium.

We have proved the following theorem.

Theorem 3.1. Any finite strategic game with sequentially locally non-constant payoff functions has a Nash equilibrium.

Consider an example.

		Player 2	
		X	Y
Player	X	2, 2	0, 3
1	Y	3, 0	1, 1

Table 1. Example of game 1

Example 1. See a game in Table 1. It is an example of the so-called Prisoners' dilemma game. Pure strategies of Player 1 and 2 are X and Y. The left side number in each cell represents the payoff of Player 1 and the right side number represents the payoff of Player 2. Let p_X and $1 - p_X$ denote the probabilities that Player 1 chooses, respectively, X and Y, and q_X and $1 - q_X$ denote the probabilities for Player 2. Denote the expected payoffs of Player 1 and 2 by $\pi_1(p_X, q_X)$ and $\pi_2(p_X, q_X)$. Then,

$$\begin{aligned}\pi_1(p_X, q_X) &= 2p_Xq_X + 3(1 - p_X)q_X + (1 - p_X)(1 - q_X) \\ &= 1 - p_X + 2q_X,\end{aligned}$$

and

$$\pi_2(p_X, q_X) = 2p_X q_X + 3p_X(1 - q_X) + (1 - p_X)(1 - q_X)$$
$$= 1 - q_X + 2p_X.$$

We have

$$\pi_1(Y, q_X) > \pi_1(X, q_X), \text{ and } \pi_2(p_X, Y) > \pi_2(p_X, X).$$

Let $(p_X(n))_{n\geq 1}$, $(p'_X(n))_{n\geq 1}$, $(q_X(n))_{n\geq 1}$ and $(q'_X(n))_{n\geq 1}$ be sequences.

(1) If $\max(\pi_1(Y, q_X) - \pi_1(p_X(n), q_X), 0) \longrightarrow 0$ and $\max(\pi_1(Y, q_X) - \pi_1(p'_X(n), q_X), 0) \longrightarrow 0$, then $p_X(n) \longrightarrow 0$, $p'_X(n) \longrightarrow 0$ and $|p_X(n) - p'_X(n)| \longrightarrow 0$.

(2) If $\max(\pi_2(p_X, Y) - \pi_2(p_X, q_X(n)), 0) \longrightarrow 0$ and $\max(\pi_2(p_X, Y) - \pi_2(p_X, q'_X(n)), 0) \longrightarrow 0$, then $q_X(n) \longrightarrow 0$, $q'_X(n) \longrightarrow 0$ and $|q_X(n) - q'_X(n)| \longrightarrow 0$.

Therefore, the payoff functions are sequentially locally non-constant.

Let us consider another example.

		Player 2	
		X	Y
Player 1	X	2, 1	0, 0
	Y	0, 0	1, 2

TABLE 2. Example of game 2

Example 2. See a game in Table 2. It is an example of the so-called Battle of the Sexes Game. Notations are the same as those in the previous example. The expected payoffs of players are as follows;

$$\pi_1(p_X, q_X) = 2p_X q_X + (1 - p_X)(1 - q_X) = 1 + p_X(3q_X - 1) - q_X,$$

and

$$\pi_2(p_X, q_X) = p_X q_X + 2(1 - p_X)(1 - q_X) = 2 + q_X(3p_X - 2) - 2p_X,$$

Then

(1) When $q_X > \frac{1}{3}$, $\pi_1(p_X, q_X)$ is strictly increasing in p_X. Every strategy of Player 1 with $p_X < 1$ is not a best response to any strategy of Player 2 with $q_X > \frac{1}{3}$.

(2) When $q_X < \frac{1}{3}$, $\pi_1(p_X, q_X)$ is strictly decreasing in p_X. Every strategy of Player 1 with $p_X > 0$ is not a best response to any strategy of Player 2 with $q_X < \frac{1}{3}$.

(3) When $p_X > \frac{2}{3}$, $\pi_2(p_X, q_X)$ is strictly increasing in q_X. Every strategy of Player 2 with $q_X < 1$ is not a best response to any strategy of Player 1 with $p_X > \frac{2}{3}$.

(4) When $p_X < \frac{2}{3}$, $\pi_2(p_X, q_X)$ is strictly decreasing in q_X. Every strategy of Player 2 with $q_X > 0$ is not a best response to any strategy of Player 1 with $p_X < \frac{2}{3}$.

Let $(p_X(n))_{n\geq 1}$, $(p'_X(n))_{n\geq 1}$, $(q_X(n))_{n\geq 1}$ and $(q'_X(n))_{n\geq 1}$ be sequences.

(1) When $p_X > \frac{2}{3}$, $q_X > \frac{1}{3}$, if $\max(\pi_1(X, q_X)-\pi_1(p_X(n), q_X), 0) \longrightarrow 0$ and $\max(\pi_1(X, q_X)-\pi_1(p'_X(n), q_X), 0) \longrightarrow 0$, then $p_X(n) \longrightarrow 1$, $p'_X(n) \longrightarrow 1$ and $|p_X(n)-p'_X(n)| \longrightarrow 0$.

If $\max(\pi_2(p_X, X)-\pi_2(p_X, q_X(n)), 0) \longrightarrow 0$ and $\max(\pi_2(p_X, X)-\pi_2(p_X, q'_X(n)), 0) \longrightarrow 0$, then $q_X(n) \longrightarrow 1$, $q'_X(n) \longrightarrow 1$ and $|q_X(n) - q'_X(n)| \longrightarrow 0$.

(2) When $p_X < \frac{2}{3}$, $q_X < \frac{1}{3}$, if $\max(\pi_1(Y, q_X)-\pi_1(p_X(n), q_X), 0) \longrightarrow 0$ and $\max(\pi_1(Y, q_X)-\pi_1(p'_X(n), q_X), 0) \longrightarrow 0$, then $p_X(n) \longrightarrow 0$, $p'_X(n) \longrightarrow 0$ and $|p_X(n)-p'_X(n)| \longrightarrow 0$.

If $\max(\pi_2(p_X, Y)-\pi_2(p_X, q_X(n)), 0) \longrightarrow 0$ and $\max(\pi_2(p_X, Y)-\pi_2(p_X, q'_X(n)), 0) \longrightarrow 0$, then $q_X(n) \longrightarrow 0$, $q'_X(n) \longrightarrow 0$ and $|q_X(n) - q'_X(n)| \longrightarrow 0$.

(3) When $p_X < \frac{2}{3}$, $q_X > \frac{1}{3}$, there exists no pair of sequences $(p_X(n))_{n\geq 1}$ and $(q_X(n))_{n\geq 1}$ such that $\max(\pi_1(X, q_X)-\pi_1(p_X(n), q_X), 0) \longrightarrow 0$ and $\max(\pi_2(p_X, Y)-\pi_2(p_X, q_X(n)), 0) \longrightarrow 0$.

(4) When $p_X > \frac{2}{3}$, $q_X < \frac{1}{3}$, there exists no pair of sequences $(p_X(n))_{n\geq 1}$ and $(q_X(n))_{n\geq 1}$ such that $\max(\pi_1(Y, q_X)-\pi_1(p_X(n), q_X), 0) \longrightarrow 0$ and $\max(\pi_2(p_X, X)-\pi_2(p_X, q_X(n)), 0) \longrightarrow 0$.

(5) When $\frac{2}{3} - \varepsilon < p_X < \frac{2}{3} + \varepsilon$, $\frac{1}{3} - \varepsilon < q_X < \frac{1}{3} + \varepsilon$ with $0 < \varepsilon < \frac{1}{3}$, if $\max(\pi_1(X, q_X) - \pi_1(p_X(n), q_X), 0) \longrightarrow 0$, $\max(\pi_1(Y, q_X)-\pi_1(p_X(n), q_X), 0) \longrightarrow 0$, $\max(\pi_2(p_X, X)-\pi_2(p_X, q_X(n)), 0) \longrightarrow 0$ and $\max(\pi_2(p_X, Y) - \pi_2(p_X, q_X(n)), 0) \longrightarrow 0$, then $(p_X(n), q_X(n)) \longrightarrow (\frac{2}{3}, \frac{1}{3})$ for all sequences $(p_X(n))_{n\geq 1}$ and $(q_X(n))_{n\geq 1}$.

The payoff functions are sequentially locally non-constant.

4. Nash equilibrium in a game with continuous strategies and quasi-concave payoff functions

In this section we will prove the existence of Nash equilibrium in a game with continuous strategies and quasi-concave payoff functions which has sequentially locally at most one maximum

Let us consider a strategic game such that there are m players with an infinite number of strategies for each player. The set of strategies of player i is denoted by $S_i, i = 1, 2, \dots, m$. S_i is a compact (totally bounded and complete) and convex subset of Euclidean space $\mathbf{R}^N$ where N is finite. Let $\mathbf{S} = \Pi_{i=1}^m S_i$ be the set of profiles of strategies of all players. Denote a strategy of player i by s_i, a profile of strategies of all players by $\mathbf{s} = (s_1, s_2, \dots, s_m)$, and a profile of strategies of players other than i by $\mathbf{s}_{-i}$. The payoff function of player i is denoted by $\pi_i(s_i, \mathbf{s}_{-i})$. It is uniformly continuous. Since S_i is compact, it has the supremum in S_i.

Quasi-concavity of payoff functions with respect to s_i is defined as follows;

Definition 4.1. $\pi_i(s_i, \mathbf{s}_{-i})$ is quasi-concave if for any $s_i, s_i' \in s_i$ and $\delta > 0$ we have

$$\pi_i(\lambda s_i + (1-\lambda)s_i', \mathbf{s}_{-i}) > \min(\pi_i(s_i, \mathbf{s}_{-i}), \pi_i(s_i', \mathbf{s}_{-i})) - \delta.$$

We assume that the payoff function for each i satisfies the following condition.

Definition 4.2. (Sequentially locally at most one maximum) Let $M = \sup \pi_i(s_i, \mathbf{s}_{-i})$ in S_i. There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $S_i = \cup_{j=1}^m H_j$, and if for all sequences $(s_i^n)_{n\geq 1}$, $({s_i}'^n)_{n\geq 1}$ in each H_j, $|\pi_i(s_i^n, \mathbf{s}_{-i}) - M| \longrightarrow 0$ and $|\pi_i({s_i}'^n, \mathbf{s}_{-i}) - M| \longrightarrow 0$, then $|s_i^n - {s_i}'^n| \longrightarrow 0$.

We show the following lemma.

Lemma 4.1. Assume $\sup_{s_i \in H_j} \pi_i(s_i, \mathbf{s}_{-i}) = M$ for some $H_j \subset S_i$ defined above. If the following property holds:

> For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $s_i, s_i' \in H_j$, $\pi_i(s_i, \mathbf{s}_{-i}) \geq M - \varepsilon$ and $\pi_i(s_i', \mathbf{s}_{-i}) \geq M - \varepsilon$, then $|s_i - s_i'| \leq \delta$.

Then, there exists $s_i^* \in H_j$ such that $\pi_i(s_i^*, \mathbf{s}_{-i}) = M$.

Proof.

Choose a sequence $(s_i^n)_{n\geq 1}$ in H_j such that $\pi_i(s_i^n, \mathbf{s}_{-i}) \longrightarrow M$. Compute N such that $\pi_i(s_i^n, \mathbf{s}_{-i}) \geq M - \varepsilon$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|s_i^m - s_i^n| \leq \delta$. Since $\delta > 0$ is arbitrary, $(s_i^n)_{n\geq 1}$ is a Cauchy sequence in H_j, and converges to a limit $s_i^* \in H_j$. The continuity of $\pi_i(s_i, \mathbf{s}_{-i})$ yields $\pi_i(s_i^*, \mathbf{s}_{-i}) = M$.

This completes the proof.

Next we show the following lemma.

Lemma 4.2. Under above assumptions $\pi_i(s_i, \mathbf{s}_{-i})$ has the maximum.

Proof. Choose a sequence $(\bar{s}_i^n)_{n\geq 1}$ in H_j defined above such that $\pi_i(\bar{s}_i^n, \mathbf{s}_{-i}) \longrightarrow M$. In view of Lemma 4.1 it is enough to prove that the following condition holds.

> For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $s_i, s_i' \in H_j$, $\pi_i(s_i, \mathbf{s}_{-i}) \geq M - \varepsilon$ and $\pi_i(s_i', \mathbf{s}_{-i}) \geq M - \varepsilon$, then $|s_i - s_i'| \leq \delta$.

Assume that the set

$$K = \{(s_i, s_i') \in H_j \times H_j : \ |s_i - s_i'| \geq \delta\}$$

is inhabited and compact (see Theorem 2.2.13 of [5]). Since the mapping $(s_i, s_i') \longrightarrow \min(\pi_i(s_i, \mathbf{s}_{-i}), \pi_i(s_i', \mathbf{s}_{-i}))$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n\geq 1}$ such that

$$\lambda_n = 0 \Rightarrow \sup_{(s_i, s_i') \in K} \min(\pi_i(s_i, \mathbf{s}_{-i}), \pi_i(s_i', \mathbf{s}_{-i})) > M - 2^{-n},$$

$$\lambda_n = 1 \Rightarrow \sup_{(s_i, s_i') \in K} \min(\pi_i(s_i, \mathbf{s}_{-i}), \pi_i(s_i', \mathbf{s}_{-i})) < M - 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $\pi_i(s_i, \mathbf{s}_{-i}) > M - 2^{-n-1}$, $\pi_i(s_i', \mathbf{s}_{-i}) > M - 2^{-n-1}$, we have $(s_i, s_i') \notin K$ and $|s_i - s_i'| \leq \delta$. Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(s_i^n, s_i'^n) \in K$ such that $\min(\pi_i(s_i, \mathbf{s}_{-i}), \pi_i(s_i', \mathbf{s}_{-i})) > M - 2^{-n}$, and if $\lambda_n = 1$, set $s_i^n = s_i'^n = \bar{s}_i^n$. Then, $\pi_i(s_i^n, \mathbf{s}_{-i}) \longrightarrow M$ and $\pi_i(s_i'^n, \mathbf{s}_{-i}) \longrightarrow M$, so $|s_i^n - s_i'^n| \longrightarrow 0$. Computing N such that $|s_i^N - s_i'^N| < \delta$, we must have $\lambda_N = 1$.

This completes the proof.

This lemma means that $\pi_i(s_i, \mathbf{s}_{-i})$ has the maximum in S_i, that is, $\max_{s_i \in S_i} \pi_i(s_i, \mathbf{s}_{-i})$ exists.

Each player chooses one of strategies s_i satisfying

$$\pi_i(s_i, \mathbf{s}_{-i}) \geq \pi_i(s_i', \mathbf{s}_{-i}) \text{ for all } s_i' \in S_i,$$

or

$$s_i = \operatorname*{argmax}_{s_i' \in S_i} \pi_i(s_i', \mathbf{s}_{-i})$$

s_i is a best response of player i to $\mathbf{s}_{-i}$, and denote the set of best responses of player i by $BR_i(\mathbf{s}_{-i})$.

A set of best responses of all players at a profile $\mathbf{s}$ is a multi-function from $\mathbf{S} = (S_1, S_2, \dots. S_m)$ to the collection of its inhabited subsets, and it is denoted by

$$\mathbf{BR}(\mathbf{s}) = (BR_1(\mathbf{s}_{-1}), BR_2(\mathbf{s}_{-2}), \dots, BR_m(\mathbf{s}_{-m})).$$

A Nash equilibrium is a state where all players choose their best responses each other. Now we show that $\mathbf{BR}(\mathbf{s})$ satisfies the conditions of Kakutani's fixed point theorem for sequentially locally non-constant multi-functions.

(1) $\mathbf{BR}(\mathbf{s})$ is convex.

Let $\mathbf{s}, \mathbf{s}' \in \mathbf{BR}(\mathbf{s})$. Denote $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{s}' = (s_1', s_2', \dots, s_n')$. By the quasi-concavity of payoff functions we have, for each player i

$$\pi_i(\lambda s_i + (1-\lambda)s_i', \mathbf{s}_{-i}) > \pi_i(s_i, \mathbf{s}_{-i}) - \delta,$$

or

$$\pi_i(\lambda s_i + (1-\lambda)s_i', \mathbf{s}_{-i}) > \pi_i(s_i', \mathbf{s}_{-i}) - \delta.$$

Since $s_i,\ s_i' \in BR_i(\mathbf{s}_{-i})$, we have

$$\pi_i(\lambda s_i + (1-\lambda)s_i', \mathbf{s}_{-i}) > \pi_i(s_i'', \mathbf{s}_{-i}) - \delta \text{ for all } s_i'' \in S_i.$$

Since δ is arbitrary,

$$\pi_i(\lambda s_i + (1-\lambda)s_i', \mathbf{s}_{-i}) \geq \pi_i(s_i'', \mathbf{s}_{-i}) \text{ for all } s_i'' \in S_i.$$

By Lemma 4.2 π_i has a maximum. It is attained at $\lambda s_i + (1-\lambda)s_i'$. Thus, $\lambda s_i + (1-\lambda)s_i'$ is a best response of player i to $\mathbf{s}_{-i}$, and $\mathbf{BR}(\mathbf{s})$ is a convex set.

(2) Next we show that $\mathbf{BR}$ has a uniformly closed graph. Consider $BR_i(\mathbf{s}_{-i})$. Let $(\mathbf{s}_n)_{n\geq 1} = ((s_i, \mathbf{s}_{-i})_n)_{n\geq 1}$, $(\mathbf{s}_n')_{n\geq 1} = ((s_i', \mathbf{s}_{-i}')_n)_{n\geq 1}$ be sequences such that $(s_i)_n \in BR_i((\mathbf{s}_{-i})_n)$ and $(s_i')_n \in BR_i((\mathbf{s}_{-i}')_n)$ for each n. By the uniform continuity of $\pi_i(s_i, \mathbf{s}_{-i})$, when $|(\mathbf{s}_{-i})_n - (\mathbf{s}_{-i}')_n| \longrightarrow 0$, we have $|\pi_i((s_i)_n, (\mathbf{s}_{-i})_n) - \pi_i((s_i)_n, (\mathbf{s}_{-i}')_n)| \longrightarrow 0$ and $|\pi_i((s_i')_n, (\mathbf{s}_{-i})_n) - \pi_i((s_i')_n, (\mathbf{s}_{-i}')_n)| \longrightarrow 0$. Let $(\varepsilon_n)_{n\geq 1}$ be a sequence such that $\varepsilon_1 > \varepsilon_2 > \ldots$ and $\varepsilon_n \longrightarrow 0$. Then, there exists N such that if $n \geq N$, we have

$$\pi_i((s_i)_n, (\mathbf{s}_{-i}')_n) > \pi_i((s_i)_n, (\mathbf{s}_{-i})_n) - \varepsilon_n,$$

and

$$\pi_i((s_i')_n, (\mathbf{s}_{-i})_n) > \pi_i((s_i')_n, (\mathbf{s}_{-i}')_n) - \varepsilon_n.$$

On the other hand, by the definition of $BR_i(\mathbf{s}_{-i})$

$$\pi_i((s_i')_n, (\mathbf{s}_{-i}')_n) \geq \pi_i((s_i)_n, (\mathbf{s}_{-i}')_n).$$

Thus, we obtain

$$\pi_i((s_i')_n, (\mathbf{s}_{-i})_n) > \pi_i((s_i)_n, (\mathbf{s}_{-i})_n) - 2\varepsilon_n, \text{ for } n \geq N.$$

If $\varepsilon_n \longrightarrow 0$, $|\pi_i((s_i')_n, (\mathbf{s}_{-i})_n) - \pi_i((s_i)_n, (\mathbf{s}_{-i})_n)| \longrightarrow 0$, and so for some $\tilde{s}_i \in BR_i(\mathbf{s}_{-i})$ we have $|(s_i')_n - \tilde{s}_i| \longrightarrow 0$. Thus, $BR_i(\mathbf{s}_{-i})$ has a uniformly closed graph. This relation holds for all i. Therefore, $\mathbf{BR}$ has a uniformly closed graph.

(3) For all sequences $(\mathbf{s}_n)_{n\geq 1}$ and $(\mathbf{s}_n')_{n\geq 1}$, if $|\mathbf{BR}(\mathbf{s}_n) - \mathbf{s}_n| \longrightarrow 0$ and $|\mathbf{BR}(\mathbf{s}_n') - \mathbf{s}_n'| \longrightarrow 0$, then $\max(\pi_i(s_{ij}, (\mathbf{s}_n)_{-i}) - \pi_i(\mathbf{s}_n), 0) \longrightarrow 0$, $\max(\pi_i(s_{ij}, (\mathbf{s}_n')_{-i}) - \pi_i(\mathbf{s}_n'), 0) \longrightarrow 0$ for all $s_{ij} \in S_i$ for all i. Since the payoff functions has sequential locally at most one maximum, $|\mathbf{s}_n - \mathbf{s}_n'| \longrightarrow 0$. Thus, $\mathbf{BR}$ satisfies the sequential local non-constancy for multi-functions.

Then, $\mathbf{BR}$ satisfies the conditions of Kakutani's fixed point theorem for sequentially locally non-constant multi-functions, and there exists a point $\mathbf{s}^*$ such that $\mathbf{s}^* \in \mathbf{BR}(\mathbf{s}^*)$. Since s_i^* is a best response of player i to $\mathbf{s}_{-i}^*$ for all players, $\mathbf{s}^*$ is a Nash equilibrium.

References

[1] J. Berger, D. Bridges, and P. Schuster, The fan theorem and unique existence of maxima, Journal of Symbolic Logic, vol. 71, pp. 713–720, 2006.

[2] J. Berger and H. Ishihara, Brouwer's fan theorem and unique existence in constructive analysis, Mathematical Logic Quarterly, vol. 51, no. 4, pp. 360–364, 2005.

[3] E. Bishop and D. Bridges, Constructive Analysis. Springer, 1985.

[4] D. Bridges and F. Richman, Varieties of Constructive Mathematics. Cambridge University Press, 1987.

[5] D. Bridges and L. Vîţă, Techniques of Constructive Mathematics. Springer, 2006.

[6] D. van Dalen, Brouwer's ε-fixed point from Sperner's lemma, Theoretical Computer Science, vol. 412, no. 28, pp. 3140–3144, June 2011.

[7] R. B. Kellogg, T. Y. Li, and J. Yorke, A constructive proof of Brouwer fixed-point theorem and computational results, SIAM Journal on Numerical Analysis, vol. 13, pp. 473–483, 1976.

[8] Y. Tanaka, Constructive proof of Brouwer's fixed point theorem for sequentially locally non-constant functions, http://arxiv.org/abs/1103.1776, 2011.

[9] Y. Tanaka, Constructive proof of the existence of nash equilibrium in a finite strategic game with sequentially locally non-constant payoff functions, ISRN Computational Mathematics (Hindawi Publishing Corporation), vol. 2012, Article ID 459459, pp. 1–9, 2012.

[10] Y. Tanaka, Kakutani's fixed point theorem for multi-functions with sequentially at most one fixed point and the minimax theorem for two-person zero-sum games: A constructive analysis, Advances in Fixed Point Theory (Science & Knowledge Publishing Corporation Limited), vol. 2, no. 2, pp. 120–134, 2012.

[11] W. Veldman, Brouwer's approximate fixed point theorem is equivalent to Brouwer's fan theorem, in Logicism, Intuitionism and Formalism, S. Lindström, E. Palmgren, K. Segerberg, and V. Stoltenberg-Hansen, Eds. Springer, 2009.