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THE SYMMETRY CASE FOR G-METRIC SPACES

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Abstract. In this paper, with D_p distance, we introduce a new notion of convex structure and we present some fixed point results in a complete metric spaces (X, D_p) and in a convex metric spaces (X, D_p, W).

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1. Introduction and Preliminaries

It well known that Banach contraction principle was published in 1922 by S. Banach as follows:

Theorem 1. *Let (X,d) be a complete metric space and a self mapping $T : X \longrightarrow X$. T is said to be contraction if there exist $k \in [0,1)$ such that for all $x,y \in X$, $d(Tx,Ty) \leq kd(x,y)$ then T has a unique fixed point in X.*

The Banach contraction principle has been extensively studied in various spaces and different generalizations were proposed. See for example $[1,3,9,10,13,14,15]$.

In 2006, Mustapha and Sims[4] introduced a new concept of generalized metric space called G-metric space and studied the fixed point result for a self-mapping in G-metric space.

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In 2020 Sabiri and all [12] introduced a new concept of the measure between p-points where $p \geq 2$ and studied convergence and existence results of best proximity points for p-cyclic contraction in (S) convex metric space.

In 2021 Sabiri and all [13] proved the existence and uniqueness for a fixed point for various types of tricyclic contractions.

In 1970 W. Takahashi [2] intoduced the notion of convex structure in meric space as follows :

Definition 2. ([2]) *Let (X,d) be metric space, a mapping $W : X \times X \times I \longrightarrow X$ is to be a convex structure on X provided that*

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,y) \textit{for all } u,x,y \in X \textit{ and } \lambda \in I := [0,1].$$

A metric space (X,d) with a convex structure W is called a convex metric space and is denoted by (X,d,W).

In 2019 Isa Yildirim and al ([8]) gave an analogue to definition of convex structure in G-metric space of Takahashi as follows:

Definition 3. ([8])*Let (X,G) be a G-metric space. A mapping $W : X^2 \times I^2 \longrightarrow X$ is termed as a convex structure on X if*

$G(W(x,y;\lambda,\beta)u,v) \leq \lambda G(x,u,v) + \beta G(y,u,v)$ for real numbers λ and β in $I = [0,1]$ satisfying $\lambda + \beta = 1$ and x,y,u and $v \in X$.

In 2008, 2009 and 2010 Mustapha, Z. and all in $[([5]),([6]),([7])]$ studied existence and uniquness of fixed point of contractive mapping defined on a G-metric space.

Theorem 4. ([5])*Let (X,G) be a G-metric space and let $T : X \longrightarrow X$ be a mapping such that T satisfies the following conditions:*

$$G(Tx,Ty,Tz) \leq aG(x,y,z) + bG(x,Tx,Tx) + cG(y,Ty,Ty) + dG(z,Tz,Tz)$$

or

$$G(Tx,Ty,Tz) \leq aG(x,y,z) + bG(x,x,Tx) + cG(y,y,Ty) + dG(z,z,Tz)$$

for all $x,y,z \in X$ where $0 < a+b+c+d < 1$.

Then T has a unique fixed point (say u,i.e., $Tu = u$), and T is G-continuous at u.

Theorem 5. ([6])*Let (X,G) be a G-metric space and let $T : X \longrightarrow X$ be a mapping such that T satisfies the following conditions:*

(1) $G(Tx,Ty,Tz) \leq aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz)$ for all $x,y,z \in X$ where $0 < a+b+c < 1$,

(2) T is G-continuous at $u \in X$,

(3) there is $x \in X$; $\{T^n x\}$ has a subsequence $\{T^{n_i} x\}$ G-converges to u.

Then u is the unique fixed point of T.

Very recently with d the standard metric, we define a new distance as follows.

Definition 6. ([12])*Let a metric space (X,d), and anteger $p > 2$,*

$$D_p : X^p \longrightarrow \mathbb{R}^+, (x_1,x_2,....,x_p) \longmapsto D_p(x_1,x_2,....,x_p) = \sum_{i<j} d(x_i,x_j) \textit{ for } 1 \leq i,j \leq p.$$

We have:

$$D_p(x_1,x_2,....,x_p) = 0 \Longleftrightarrow x_i = x_{i+1} \text{ for all } x_i \in X \text{ and } 1 \leq i \leq p-1$$

$$0 < D_p(x,x,....x,y) = D_p(y,y,....y,x) = (p-1)d(x,y) \text{ for all } x,y \in X \text{ with } x \neq y$$

$$D_p(x_1,x_2,....,x_p) = D_p(x_p,x_{p-1},....,x_1) = D_p(x_p,x_1,....,x_{p-1}) = ...(\text{symetry in all } p\text{-variables})$$

$$D_p(x_1,x_1,....,x_2) \leq D_p(x_1,x_2,....,x_p) \text{ for all } x_i \in X \text{ and } 1 \leq i \leq p \text{ with } x_i \neq x_2, 3 \leq i \leq p.$$

Then the fonction D_p is called a generelized metric or more specifically, D_p-metric space on X, and the pair (X,D_p) is called a D_p-metric space.

Definition 7. *Let (X,D_p) be a D_p-metric space and let $\{x_n\}$ a sequence in X and let $x \in X$. We say that $\{x_n\}$ converge to x if $\lim_{n \longrightarrow +\infty} D_p(x,x_n,x_{n+i}....,x_{n+i}) = 0$ for all $i \geq 1$.We say that $\{x_n\}$ is D_p-convergent to x.*

Proposition 8. *Let (X,D_p) be a D_p-metric space, then the following are equivalent.*

[i] $\{x_n\}$ is D_p-convergent to x.

[ii] $\lim_{n\longrightarrow+\infty} D_p(x_n,x....,x)=0$

[iii] $\lim_{n\longrightarrow+\infty} D_p(x_n,x_n,....x_n,x)=0$.

In this work, using the famous definition of convexity of Takahashi [2] and inspired by the ideas given in $[4,5,6,7,8]$. we give a new and more pratical definition of convex structure and we generalize some results and give some fixed point results in D_p-metric space, where $p\geq 3$.

2. Main Results

Theorem 9. *Let (X,d) a complete metric space and a map $T:X\longrightarrow X$ such that $D_p(Tx_1,Tx_2,....,Tx_p)\leq a_0D_p(x_1,x_2,....,x_p)+\sum_{i=1}^{p}a_iD_p(x_i,Tx_i,....,Tx_i)$ for all $x_i\in X$ and $0\leq a_i$ and $0\leq\sum_{i=0}^{p} a_i<1$. Then T has a unique fixed point $z\in X: Tz=z$ and T is continuous at z.*

Proof. Let $x,y\in X$, we have:

$$D_p(Tx,Ty,....,Ty)\leq a_0D_p(x,y,....,y)+a_1D_p(x,Tx,....,Tx)+\sum_{i=2}^{p}a_iD_p(y,Ty,....,Ty)$$

and

$$D_p(Ty,Tx,....,Tx)\leq a_0D_p(y,x,....,x)+a_1D_p(y,Ty,....,Ty)+\sum_{i=2}^{p}a_iD_p(x,Tx,....,Tx)$$

implies that

$$2(p-1)d(Tx,Ty)\leq a_0 2(p-1)d(x,y)+\sum_{i=1}^{p}a_iD_p(x,Tx,....,Tx)+\sum_{i=1}^{p}a_iD_p(y,Ty,....,Ty)$$

then

$$d(Tx,Ty)\leq a_0d(x,y)+\frac{1}{2}\sum_{i=1}^{p}a_i(d(x,Tx)+d(y,Ty)) \text{ for all } x,y\in X.$$

Let $x_0\in X$ and put $x_{n+1}=Tx_n$ $n=0,1,2....$

We have

$$d(x_n,x_{n+1})=d(T^nx_0,T^{n+1}x_0)\leq a_0d(T^{n-1}x_0,T^nx_0)+\frac{1}{2}\sum_{i=1}^{p}a_i(d(T^nx_0,T^{n+1}x_0)+d(T^{n-1}x_0,T^nx_0))$$

$$\iff d(T^nx_0,T^{n+1}x_0)(1-\frac{1}{2}\sum_{i=1}^{p}a_i)\leq(a_0+\frac{1}{2}\sum_{i=1}^{p}a_i)d(T^{n-1}x_0,T^nx_0))$$

$$\iff d(T^nx_0,T^{n+1}x_0)(2-\sum_{i=1}^{p}a_i)\leq(2a_0+\sum_{i=1}^{p}a_i)d(T^{n-1}x_0,T^nx_0))$$

$$\iff d(T^nx_0,T^{n+1}x_0)\leq td(T^{n-1}x_0,T^nx_0) \text{ with } t=\frac{2a_0+\sum_{i=1}^{p}a_i}{2-\sum_{i=1}^{p}a_i}<1.$$

Then

$$d(T^nx_0,T^{n+1}x_0)\leq t^nd(x_0,Tx_0)$$

implies that

$$\forall m>n, d(T^mx_0,T^nx_0)\leq\frac{t^n}{1-t}d(x_0,Tx_0).$$

Then $\{x_n\}$ is a Cauchy sequence in a complete metric space (X,d). So there exist $z\in X$ such that $\lim_{n\longrightarrow+\infty}T^nx_0=z$.

Now we will show that $\lim_{n\longrightarrow+\infty}T^{n+1}x_0=Tz$.

We have

$$d(T^{n+1}x_0,Tz)\leq a_0d(T^nx_0,z)+\frac{1}{2}\sum_{i=1}^{p}a_i(d(T^nx_0,T^{n+1}x_0)+d(z,Tz))$$

implies that

$$\lim_{n\longrightarrow+\infty}d(T^{n+1}x_0,Tz)=d(z,Tz)\leq\frac{1}{2}\sum_{i=1}^{p}a_i(d(z,Tz))<d(z,Tz)$$

then $\lim_{n\longrightarrow+\infty}T^{n+1}x_0=Tz$.

Now we prove the uniqueness fixed point. Let z_1,z_2 be two fixed points such that $z_1\neq z_2$.

Then

$$d(z_1,z_2)=d(Tz_1,Tz_2)\leq a_0d(z_1,z_2)+\frac{1}{2}\sum_{i=1}^{p}a_i(d(z_1,Tz_1)+d(z_2,Tz_2))=a_0d(z_1,z_2).$$

As $d(z_1,z_2)>0$ we have $1<a_0$, a contradiction.

To show that T is continuous at z, let $(z_n)\subseteq X$ be a sequence such that $\lim_{n\longrightarrow+\infty}(z_n)=z$.

We have

$$d(z,Tz_n)=d(Tz,Tz_n)\leq a_0d(z,z_n)+\frac{1}{2}\sum_{i=1}^{p}a_i(d(z,Tz)+d(z_n,Tz_n))$$

$\Longleftrightarrow$

$$d(z,Tz_n) \leq a_0 d(z,z_n) + \frac{1}{2}\sum_{i=1}^{p} a_i d(z_n,Tz_n)$$

$$\leq a_0 d(z,z_n) + \frac{1}{2}\sum_{i=1}^{p} a_i (d(z_n,z) + d(z,Tz_n)).$$

Then

$$d(z,Tz_n) \leq \frac{(a_0 + \frac{1}{2}\sum\limits_{i=1}^{p} a_i)}{(1-\frac{1}{2}\sum\limits_{i=1}^{p} a_i)} d(z,z_n).$$

Us $\lim\limits_{n\longrightarrow+\infty}(z_n) = z$, then T is continuous at z. □

Corollary 10. *Let (X,d) a complete metric space and a map $T : X \longrightarrow X$ such that $D_p(Tx_1,Tx_2,....,Tx_p) \leq a_0 D_p(x_1,x_2,....,x_p) + a_1\sum\limits_{i=1}^{p} D_p(x_i,Tx_i,....,Tx_i)$ for all $x_i \in X$ and $0 \leq a_0 + pa_1 < 1$ then T has a unique fixed point $z \in X : Tz = z$.*

Example 11. *For $p = 3$ let $X = \mathbb{R}$ the complete metric space with d the standard metric $d(x,y) = |x-y|$ where $x,y \in X$ and the self map*

$$T : X \longrightarrow X \text{ such that } Tx = \frac{x}{6}$$

we take

$$a_0 = \frac{1}{12},\ a_1 = a_2 = a_3 = a_4 = \frac{1}{5}.$$

We have

$$D_3(Tx_1,Tx_2,Tx_3) = \frac{1}{6}\,(\sum_{1\leq i<j\leq 3} |x_i - x_j|)$$

$$a_0 D_3(x_1,x_2,x_3) = \frac{1}{12}\,(\sum_{1\leq i<j\leq 3} |x_i - x_j|)$$

and

$$a_1\sum_{i=1}^{3} D_3(x_i,Tx_i,Tx_i,) = \frac{1}{3}(\sum_{i=1}^{3}|x_i|)$$

us $|x-y| \leq |x| + |y|$ for all $x,y \in \mathbb{R}$ we have:

$$\frac{1}{6}\,(\sum_{1\leq i<j\leq 3} |x_i - x_j|) \leq \frac{1}{12}\,(\sum_{1\leq i<j\leq 3} |x_i - x_j|) + \frac{1}{3}(\sum_{i=1}^{3}|x_i|)$$

So

$$D_3\left(Tx_1,Tx_2,Tx_3\right) \leq a_0 D_3\left(x_1,x_2,x_3\right)+a_1(\sum_{i=1}^{3} D_3\left(x_i,Tx_i,Tx_i\right)$$

for all $x_i \in X$ and $0 \leq a_0+4a_1<1$.

Then T has a unique fixed point $z=0: Tz=z$.

Corollary 12. *Let (X,d) a complete metric space and a map $T: X \longrightarrow X$ such that*

$d(Tx,Ty) \leq a_0 d(x,y)+a_1(d(x,Tx)+d(y,Ty))$ for all $x,y \in X$ and $0 \leq a_0+2a_1<1$ then T has a unique fixed point $z \in X: Tz=z$.

In the following theorem we will prove the existence and the uniquness of fixed point without completeness property in a metric space (X,d).

Theorem 13. *Let (X,d) a metric space and a map $T: X \longrightarrow X$ such that:*

(1) $D_p\left(Tx_1,Tx_2,....,Tx_p\right) \leq \sum_{i=1}^{p} a_i D_p\left(x_i,Tx_i,....,Tx_i\right)$ for all $x_i \in X$ where $0 \leq a_i$ and $0<\sum_{i=1}^{p} a_i<1$.

(2) There is $x \in X$ such that $\{T^n x\}$ has a subsequence $\{T^{n_k}x\}$ converge to z.

(3) T is continuous at point $z \in X$.

Then T has a unique fixed point $z \in X: Tz=z$.

Proof. We have T is continuous at point z and $\{T^{n_k}x\}$ converge to z wich implies that $\{T^{n_k+1}x\}$ converge to Tz. Assume $Tz \neq z$. For $0<\varepsilon<\frac{1}{3}d(z,Tz)$ there exist $N_0 \in \mathbb{N}$ such that if $k>N_0$ we have

$$d(z,T^{n_k}x)<\varepsilon \text{ and } d(Tz,T^{n_k+1}x)<\varepsilon.$$

Then

$$\varepsilon<\frac{1}{3}d(z,Tz) \leq \frac{1}{3}\left[d(z,T^{n_k}x)+d(T^{n_k}x,T^{n_k+1}x)+d(Tz,T^{n_k+1}x)\right]$$

$$\Longrightarrow 3\varepsilon<2\varepsilon+d(T^{n_k}x,T^{n_k+1}x)$$

$$\varepsilon<d(T^{n_k}x,T^{n_k+1}x) \leq D_p\left(T^{n_k}x,T^{n_k+1}x,....,T^{n_k+1}x\right) \text{ for } k>N_0.$$

(1) implies

$$D_p\left(T^{n_k+1}x,T^{n_k+2}x,...,T^{n_k+p}x\right) \leq a_1 D_p\left(T^{n_k}x,T^{n_k+1}x,...,T^{n_k+1}x\right) + \sum_{i=2}^{p} a_i D_p\left(T^{n_k+i-1}x,T^{n_k+1}x,...,T^{n_k+1}x\right)$$

$$\leq a_1 D_p\left(T^{n_k}x,T^{n_k+1}x,...,T^{n_k+1}x\right) + \sum_{i=2}^{p} a_i D_p\left(T^{n_k+1}x,T^{n_k+2}x,...,T^{n_k+p}x\right).$$

Then

$$D_p\left(T^{n_k+1}x,T^{n_k+2}x,...,T^{n_k+p}x\right) \leq t D_p\left(T^{n_k}x,T^{n_k+1}x,...,T^{n_k+1}x\right)$$

with $0 < t = \frac{a_1}{1-\sum_{i=2}^{p} a_i} < 1$, since $0 < \sum_{i=1}^{p} a_i < 1$.

Then

$$D_p\left(T^{n_k+1}x,T^{n_k+2}x,...,T^{n_k+2}x\right) \leq D_p\left(T^{n_k+1}x,T^{n_k+2}x,...,T^{n_k+p}x\right) \leq t D_p\left(T^{n_k}x,T^{n_k+1}x,...,T^{n_k+1}x\right)$$

Consequently

$$d\left(T^{n_k+1}x,T^{n_k+2}x\right) \leq t d\left(T^{n_k}x,T^{n_k+1}x\right) \text{ for } k > N_0.$$

Similarly we have

$$d\left(T^{n_k+j}x,T^{n_k+j+1}x\right) \leq t d\left(T^{n_k+j-1}x,T^{n_k+j}x\right) \text{ for } k > N_0 \text{ and } j \geq 1.$$

Then For $l > k > N_0$ we have:

$$d\left(T^{n_l}x,T^{n_l+1}x\right) \leq t d\left(T^{n_l-1}x,T^{n_l}x\right) \leq t^2 d\left(T^{n_l-2}x,T^{n_l-3}x\right) \leq .... \leq t^{l-k} d\left(T^{n_k}x,T^{n_k+1}x\right)$$

Then

$$\lim_{l\longrightarrow+\infty} d\left(T^{n_l}x,T^{n_l+1}x\right) = 0 \Longleftrightarrow d(z,Tz) = 0$$

which a contradiction, hence $Tz = z$.

□

Theorem 14. *Let (X,d) a complete metric space and a map $T : X \longrightarrow X$ such that*

$D_p(Tx_1,Tx_2,...,Tx_p) \leq k \max_{1\leq i\leq p} D_p(x_i,Tx_i,...,Tx_i)$ for all $x_i \in X$ and $0 \leq k < 1$, then T has a unique fixed point $z \in X : Tz = z$.

Proof. Let $x,y\in X$ we have:

$$D_p\left(Tx,Ty,...,Ty\right)\leq k\max(D_p\left(x,Tx,...,Tx\right),D_p\left(y,Ty,...,Ty\right))$$

$\Longleftrightarrow$

$$d\left(Tx,Ty\right)\leq k\max(d\left(x,Tx\right),d\left(y,Ty\right)).$$

If $d\left(x,Tx\right)\geq d\left(y,Ty\right)$ let $x_0\in X$ and put $y=T^nx_0$ and $x=T^{n-1}x_0$ for $n\geq 1$. Then $d\left(T^nx_0,T^{n+1}x_0\right)\leq kd(T^{n-1}x_0,T^nx_0)\leq k^nd(x_0,Tx_0)$.

Implies that the sequence $\{T^nx_0\}$ is a Cauchy in a complete metric space (X,d),

If $d\left(y,Ty\right)\geq d\left(x,Tx\right)$,we put $x=T^nx_0$ and $y=T^{n-1}x_0$ and we have olso the sequence $\{T^nx_0\}$ is a Cauchy in a complete metric space $\left(X,d\right)$, then there exist $z\in X$ such that $\lim\limits_{n\longrightarrow+\infty}T^nx_0=z$.

Now we will show that $z=Tz$.

We have

$$d\left(T^{n+1}x_0,Tz\right)\leq k\max(d(T^nx_0,T^{n+1}x_0),d(z,Tz))$$

as T is continuous and $\lim\limits_{n\longrightarrow+\infty}T^nx_0=z$. Then $d(z,Tz)\leq kd(z,Tz)$, this implies that $z=Tz$.

Now we prove the uniqueness fixed point. Let z_1,z_2 be two fixed points.

Then $d(z_1,z_2)=d(Tz_1,Tz_2)\leq k\max(d(z_1,Tz_1),d(z_2,Tz_2))=0\Longrightarrow z_1=z_2$. □

Example 15. *For $p=3$ let $X=[0,1]$ the metric space with the usiel norme $d(x,y)=|x-y|$, and the self map*

$$T:X\longrightarrow X \text{ such that } Tx=\frac{x}{3}$$

we have

$$D_3\left(Tx_1,Tx_2,Tx_3\right)=\frac{1}{3}\,(\sum_{1\leq i<j\leq 3}\left|x_i-x_j\right|)$$

and

$$\max D_3\left(x_i,Tx_i,Tx_i\right)=D_3\left(1,T\left(1\right),T\left(1\right)\right)=\frac{4}{3}\text{ for all } x_i\in X$$

For $k=\frac{3}{4}$, we have

$$D_3\left(Tx_1,Tx_2,Tx_3\right)\leq k\max D_3\left(x_i,Tx_i,Tx_i\right)\forall x_i\in X.$$

Then T has a unique fixed point $z=0:Tz=z$.

Definition 16. *Let a (X,D_p) be D_p-metric space a map*

$$W : X^{p-1} \times I^{p-1} \longrightarrow X$$

is to be a convex structure on X if:

$$D_p\left(W\left(x_1,x_2,....,x_{p-1};\lambda_1,\lambda_2,...,\lambda_{p-1}\right)),u_2,....,u_p\right) \leq \sum_{i=1}^{p-1}\lambda_i D_p(x_i,u_2....,u_p)$$

with $\sum\limits_{i=1}^{p-1}\lambda_i = 1$, $\lambda_i \in I := [0,1]$ and $x_i \in X$ for $i = 1,...p-1$; $u_j \in X$ for $j = 2,...,p$.

(X,D_p,W) is called a convex D_p-metric space. A subset C of a convex D_p-metric space is said to be a convex if $W\left(x_1,x_2,....,x_{p-1};\lambda_1,\lambda_2,...,\lambda_{p-1}\right) \in C$ for all $x_i \in C$, $\lambda_i \in I$, $i = 1,...p-1$.

Definition 17. *Let (X,D_p,W) be convex D_p-metric space and $T : X \longrightarrow X$ be a mapping. Let $\alpha_n^i \in [0,1]$ with $\sum\limits_{i=0}^{p-2}\alpha_n^i = 1$ and $n \in \mathbb{N}$. For $x_0 \in X$, we define the sequence $\{x_n\}$ by :*

$$x_{n+1} = W(x_n,Tx_n,....Tx_n;\alpha_n^0,\alpha_n^1,...,\alpha_n^{p-2}) \tag{(1)}$$

is called Mann iterative process in the convex metric space (X,D_p,W).

Then we have :

$$D_p(x_{n+1},u_2,u_3,....u_p) = D_p(W(x_n,Tx_n,....Tx_n;\alpha_n^0,\alpha_n^1,...\alpha_n^{p-2}),u_2,u_3,....,u_p)$$

$$\leq \alpha_n^0 D_p(x_n,u_2,u_3,....u_p) + \sum_{i=1}^{p-2}\alpha_n^i D_p(Tx_n,u_2,....,u_p)$$

$$= \alpha_n^0 D_p(x_n,u_2,u_3,....,u_p) + \left(1-\alpha_n^0\right) D_p(Tx_n,u_2....,u_p)$$

Theorem 18. *Let (X,D_p,W) be a convex D_p-metric space and $T : X \longrightarrow X$ be a mapping such that $D_p\left(Tx_1,Tx_2,....,Tx_p\right) \leq a_0 D_p\left(x_1,x_2,....,x_p\right) + \sum\limits_{i=1}^{p} a_i D_p\left(x_i,Tx_i,....,Tx_i\right)$ for all $x_i \in X$, $0 \leq a_i$, and $0 \leq \sum\limits_{i=0}^{p} a_i < 1$ and let z a fixed point for T. Let $\{x_n\}$ defined by :(1) with $\sum\limits_{n=0}^{\infty}\alpha_n^i = \infty$ and $\sum\limits_{i=0}^{p-2}\alpha_n^i = 1$ then $\{x_n\}$ converges to fixed point of T.*

Proof. We have:

$$D_p(x_{n+1}, z, z, ..., z) = D_p(W(x_n, Tx_n, ...Tx_n; \alpha_n^0, \alpha_n^1, ..., \alpha_n^{p-2}), z, z, ..., z))$$
$$\leq \alpha_n^0 D_p(x_n, z, z, ...z) + \sum_{i=1}^{p-2} \alpha_n^i D_p(Tx_n, z...., z)$$

We know that

$$D_p(Tx_n, z...., z) = D_p(Tx_n, Tz...., Tz) \leq a_0 D_p(x_n, z, ..., z) + a_1 D_p(x_n, Tx_n, ...Tx_n)$$

$$\leq a_0 D_p(x_n, z, ..., z) + a_1(D_p(x_n, z, ..., z) + D_p(z, z, ..., z, Tx_n))$$

$$\leq a_0 D_p(x_n, z, ..., z) + a_1(D_p(x_n, z, ..., z) + (p-1)D_p(z, z, ..., z, Tx_n))$$

implies that

$$D_p(Tx_n, z...., z) \leq \frac{a_0 + a_1}{1-(p-1)a_1} D_p(x_n, z, ..., z).$$

Then

$$D_p(x_{n+1}, z, z, ...z) \leq \alpha_n^0 D_p(x_n, z, z, ...z) + \sum_{i=1}^{p-2} \alpha_n^i \frac{a_0 + a_1}{1-(p-1)a_1} D_p(x_n, z, ..., z)$$

$$= (\alpha_n^0 + \sum_{i=1}^{p-2} \alpha_n^i \frac{a_0 + a_1}{1-(p-1)a_1}) D_P(x_n, z, ..., z)$$

$$= (1 - \sum_{i=1}^{p-2} \alpha_n^i + \sum_{i=1}^{p-2} \alpha_n^i \frac{a_0 + a_1}{1-(p-1)a_1}) D_P(x_n, z, ..., z)$$

$$= (1 - \sum_{i=1}^{p-2} \alpha_n^i (\mathbf{1} - \frac{a_0 + a_1}{1-(p-1)a_1}) D_P(x_n, z, ..., z)$$

$$= (1 - \sum_{i=1}^{p-2} \alpha_n^i (\mathbf{1} - \delta)) D_p(x_n, z, ..., z) \text{ with } 0 \leq \delta = \frac{a_0 + a_1}{1-(p-1)a_1} < 1.$$

That implies .

$$D_p(x_{n+1}, z, z, ...z) \leq \prod_{k=0}^{n} (1 - \sum_{i=1}^{p-2} \alpha_k^i (\mathbf{1} - \delta)) D_P(x_0, z, ..., z).$$

As

$$\delta < 1, \ \sum_{i=1}^{p-2} \alpha_k^i \in [0, 1] \text{ and} \sum_{n=0}^{\infty} \alpha_n^i = \infty$$

then

$$\lim_{n\longrightarrow+\infty}\prod_{k=0}^{n}(1-\sum_{i=1}^{p-2}\alpha_k^i(\mathbf{1}-\delta)=0$$

which implies that

$$\lim_{n\longrightarrow+\infty}D_p(x_{n+1},z,z,....z)=0.$$

Hence the sequence $\{x_n\}$ converge to z fixed point for T. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] G. Modi. Bhatt, Fixed point results for weakly compatible mapping in convex G-metric space, Int. J. Math. Stat. Invent. 2 (11) (2014), 34-38.

[2] W. Takahashi, A convexity in metric space and nonexpansive mappings. I. Kodai Math. Sem. Rep. 22 (1970), 142-149.

[3] A.A. Eldred, P. Veeramani, Existance and converence of best proximity points, J. Math. Anal. Appl. 323 (2) (2006), 1001-1006.

[4] Z. Mustafa, B. Sims, A new Approach to generalized Metric spaces, J. Nonlinear Convex Anal. 7 (2)2 (2006), 289-297.

[5] Z. Mustafa, H. Obiedat, F. Awawdeh, Some Fixed Point Theorem For mapping on complete G-metric spaces, Fixed Point Theory Appl. 2008 (2008), 189870.

[6] Z. Mustafa, W. Shatanawi, M. Bataineh, Existence of Fixed Point Results in G-metric spaces, Int. J. Math. Math. Sci. 2009 (2009), 28302812.

[7] Z. Mustafa, H. Obiedat, A Fixed Point Theorem of Riech in G-metric spaces, Cubo. 12 (01) (2010), 83-93

[8] I. Yildirim, S.H. Khan, Convexity in G-metric spaces and approximation of fixed points by Mann iterative proces, ArXiv:1911.04867 [Math]. (2019).

[9] S. Karpagam, S. Agrawal, Best proximity points theorems for cyclic Meir-Keeler contraction maps, Nonlinear Anal. 74 (2011), 1040-1046.

[10] W.A. Kirk, P.S. Srinivasan. P. Veeramani, Fixed point fo mappings satisfyaing cyclical contractive conditions. Fixed Point Theory, 4 (200), 79-89.

[11] S. Sadiq Basha, Best proximity points theorems, J. Approx. Theory, 163 (2011), 1772-1781.

[12] M. Sabiri, J. Mouline. A. Bassou, T. Sabar, A new best approximation result in (S) convex metric spaces, Int. J. Math. Math. Sci. 2020 (2020), 4367482.

[13] M. Sabiri, A. Bassou, J. Mouline, T. Sabar, Fixed point results for various types of tricyclic contractions, Axioms, 10 (2021), 72.

[14] L.A. Talman, Fixed point for condensing multifunctions in metric spaces with convex structure, Kodai Math. Sem. Rep. 29 (1977), 62-70.

[15] S. Reich, Some remarks concerning contraction mappings, Canad. Math. Bull. 14 (1971), 121-124.