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EXISTENCE AND UNIQUENESS OF BANACH AND KANNAN FIXED POINT THEOREMS FOR OPERATOR ON HILBERT C^*-MODULES

R. A. RASHWAN[1], SALEH OMRAN[2], ASMAA FANGARY[2,*]

[1]Department of Mathematics, Faculty of Science, Assuit University, Assuit, Egypt

[2]Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt

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Abstract. In this paper we introduce some Banach fixed point theorems in operators of Hilbert C^*-modules, based on a definition of valued operator Hilbert C^*-modules normed space. Also We give some examples to clear our definitions. Finally we discuss the existence and uniqueness of the solution of system of operators on Hilbert C^*-modules.

Keywords: fixed point theorems; C^*-algebra; operators on Hilbert C^*-modules.

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1. INTRODUCTION

In 1922, the Polish Mathematician Banach introduced the most well known fixed point theorem so-called Banach contraction principle [1]. This theorem states that a contraction mapping on a complete metric space into itself has a unique fixed point. This theorem is a very useful, simple and classical tool in moderen analysis. It is consider an important tool for solving existance problems in many branches of mathmatics and physics.

Hilbert C^*-modules were first introduced in 1953 by Kaplansky [9]. Later, the theory was developed independently by Paschke [13] and Rieffel [15] where the research on HilbertC^*-modules

*Corresponding author

E-mail address: asmaa.fangary44@yahoo.com

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began in the 70's in the work of the induced representations of C^*-algebras by M. A. Rieffel [15] also Kaplansky[9] used this object to prove that derivations of type IAW^*-algebras are inner where he was to generalise Hilbert space by allowing the inner product to take values in a (commutative, unital) C^*-algebra rather than in the field of complex numbers, Kasparov [10] introduced the definition of KK-theory by using HilbertC^*-modules.

Ma and et al [17], introduced the concept of C^*-algebra-valued metric spaces. The main idea consists in using the set of all positive elements of a unital C^*-algebra instead of the set of real numbers. They presented some fixed point results for mapping under contractive or expansive conditions in these spaces.

An element $x \in \mathbb{A}$ is a positive element, denote it by $x \succeq 0$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset [0, +\infty[$, where $\sigma(x)$ is the spectrum of x and $\mathbb{A}_h = \{x \in \mathbb{A} : x^* = x\}$. Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows: $x \preceq y$ if and only if $y - x \succeq 0$. From now on, by $\mathbb{A}_+$ we denote the set $\{x \in \mathbb{A} : x \succeq 0\}$ and $|x| = (x^*x)^{\frac{1}{2}}$.

2. Preliminaries

In this section, we begin with some basic notations and definition C^*-algebra and fixed point theory that will be very important and useful in the sequal.

Definition 2.1 [18] A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|ab\| \leq \|a\|\|b\|$ (for all $a, b \in \mathbb{A}$). A C^*algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ (for all $a \in \mathbb{A}$).

Definition 2.2 [18] An element $a \in \mathbb{A}$ is positive element, if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}^+$, where $\sigma(a)$ is the spectrum of a, we denote $\mathbb{A}_+$ the set of all positive element in $\mathbb{A}$.

Definition 2.3 [12, 22] A pre-Hilbert C^*-module $\mathscr{E}$ over a C^*-algebra $\mathbb{A}$, is a right $\mathbb{A}$-module together with an $\mathbb{A}$-valued inner product $< .,. >: \mathscr{E} \times \mathscr{E} \longrightarrow \mathbb{A}$ satisfying the conditions:

(1) $< x,x > \succeq 0$ for all $x \in \mathscr{E}$;

(2) $< x,x >= 0$ if and only if $x = 0$;

(3) $< x, \alpha y + \beta z >= \alpha < x,y > + \beta < x,z >$ for all $x,y,z \in \mathscr{E}, \alpha, \beta \in \mathbb{C}$;

(4) $< x, ya >=< x,y > a$ for all $x,y \in \mathscr{E}, a \in \mathbb{A}$;

(5) $< x,y >^* =< y,x >$ for all $x,y \in \mathscr{E}$.

Definition 2.4 [12] The norm of an element $e \in \mathscr{E}$ is defined as

$$\|x\|_{\mathscr{E}} := \sqrt{\|\langle x,x\rangle\|_{\mathbb{R}}}, \text{ where } \|.\|_{\mathbb{R}} \text{ is the } \mathbb{R}\text{-valued norm.}$$

If a pre-Hilbert $\mathbb{A}$ -module is complete with respect to its norm,it is said to be a Hilbert $\mathbb{A}$ -module.

Example 2.1

(i) Every C^*-algebra $\mathbb{A}$ is a Hilbert $\mathbb{A}$-module over itself when equipped with the $\mathbb{A}$-valued inner product given simply by

$$\langle a,b\rangle = a^*b,\ (a,b \in \mathbb{A}).$$

(ii) Let $\{\mathscr{E}_i\}_{1\leq i\leq n}$ be a finite family of Hilbert $\mathbb{A}$-modules. Then the direct sum $\oplus\mathscr{E}_i$ is a Hilbert $\mathbb{A}$-modules with the module action and inner product defined by

$$(x_1,x_2,\cdots,x_n)a = (x_1a,x_2a,\cdots,x_na)$$
$$<(x_1,x_2,\cdots,x_n),(y_1,y_2,\cdots,y_n)>=\sum_{i=1}^{n}<x_i,y_i>_{\mathscr{E}},\ x_i,y_i \in \mathscr{E}_i.$$

Definition 2.5 [22] Let $\mathscr{E}$ be a Hilbert $\mathbb{A}$-module. A map $T:\mathscr{E}\longrightarrow\mathscr{E}$ is said to be adjointable if there exists a map $T^*:\mathscr{E}\longrightarrow\mathscr{E}$ satisfying

$$<x,Ty>=<T^*x,y>$$

for all $x,y\in\mathscr{E}$.

Definition 2.6 [7] An element $T\in l(\mathscr{E})$ is positive if for every $x\in\mathscr{E}$ we have $<Tx,x>_{\mathbb{A}}\succeq 0$ and we write it by $T\succeq 0$ and we denote the set $l(\mathscr{E})_+ = \{T\in\mathscr{E}\quad;\quad T\succeq 0\}$, we define a partial ordering relation on $l(\mathscr{E})_+$ as

$$\text{if } T_1,T_2\in l(\mathscr{E}),T_1\preceq_{l(\mathscr{E})} T_2 \text{ if and only if } T_2-T_1\in l(\mathscr{E})_+$$

Definition 2.7 [7] $l(\mathscr{E})$ = $\{T:\mathscr{E}\longrightarrow\mathscr{E}\}$ is the set of all adjiontable linear operators with $\|T\|$ = $sup\{\|Tx\|_{\mathscr{E}};\|x\|_{\mathscr{E}}\leq 1\}$ is a C^*-algebra.

3. Main Results

Definition 3.1 [3] Let $l(\mathscr{E})_+$ be a subset of $l(\mathscr{E})$. $l(\mathscr{E})_+$ is called Cone of $l(\mathscr{E})$ if and only if:

(1) $l(\mathscr{E})_+\cap(-l(\mathscr{E})_+)=\{0_{l(\mathscr{E})}\}$, ($0_{l(\mathscr{E})}$ is the zero vector);

(2) $l(\mathscr{E})_+$ is closed in $l(\mathscr{E})$;

(3) $Ta+Sb\in l(\mathscr{E})_+$; $aT+bS\in l(\mathscr{E})_+$ $a,b\in A$, $T\lambda+S\beta\in l(\mathscr{E})_+ : \lambda,\beta\in\mathbb{C}$;

(4) $l(\mathscr{E})_+ \cdot l(\mathscr{E})_+ \subseteq l(\mathscr{E})_+$.

Definition 3.2 [3] An $l(\mathscr{E})$-valued metric on a set X is a function $d_{l(\mathscr{E})} : X \times X \longrightarrow l(\mathscr{E})$ such that for all x, y and z in X the following conditions are hold:

(1) $d_{l(\mathscr{E})}(x,y) \succeq 0$;
(2) $d_{l(\mathscr{E})}(x,y) = 0$ if and only if $x = y$;
(3) $d_{l(\mathscr{E})}(x,y) = d_{l(\mathscr{E})}(y,x)$;
(4) $d_{l(\mathscr{E})}(x,y) \preceq d_{l(\mathscr{E})}(x,z) + d_{l(\mathscr{E})}(z,y)$.

Then the triple $(X, l(\mathscr{E}), d_{l(\mathscr{E})})$ is called an $l(\mathscr{E})$-valued metric space.

Definition 3.3[17] Let X be a nonempty set. Suppose the mapping $d : X \times X \longrightarrow \mathbb{A}$ satisfies:

(1) $0_{\mathbb{A}} \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0_{\mathbb{A}}$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$.
(3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space.

Definition 3.4 [3] Let $(X, l(\mathscr{E}), d_{l(\mathscr{E})})$ be an $l(\mathscr{E})$- valued metric spacs. Suppose that $x_n \subset X$ and $x \in X$ If for any $\varepsilon_{l(\mathscr{E})} \succ 0_{l(\mathscr{E})}$ (where $0_{l(\mathscr{E})}$ is the zero element in $l(\mathscr{E})$) there exists $N \in \mathbb{N}$ such that for all $n > N$, $d_{l(\mathscr{E})}(x_n, x) \preceq \varepsilon_{l(\mathscr{E})}$, then $\{x_n\}$ is said to be converge with respect to $l(\mathscr{E})$, and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote it by $\lim_{n \longrightarrow +\infty} \{x_n\} = x$.
If for any $\varepsilon_{l(\mathscr{E})} \succ 0_{l(\mathscr{E})}$ there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $d(x_n, x_m) \preceq \varepsilon_{l(\mathscr{E})}$, then $\{x_n\}$ is said to be a Cauchy with respect to $l(\mathscr{E})$.
We say $(X, l(\mathscr{E}), d_{l(\mathscr{E})})$ is a complete $l(\mathscr{E})$- valued metric spacs if every Cauchy sequence with respect to $l(\mathscr{E})$ is convergent.

Lemma 3.1 [3] A sequence $x_n \subset X$ is convergence if $\|x_n\| \longrightarrow 0 \quad \forall n > N$ such that $N \in \mathbb{N}$.

Example 3.1 [3] Let $X = \mathbb{A}^{\oplus n}, \mathscr{E} = \mathbb{A}^{\oplus n}$ and $L(\mathscr{E}) = \{T : \mathbb{A}^{\oplus n} \longrightarrow \mathbb{A}^{\oplus n} \ : T(a_1, a_2, ..., a_n) = (Ta_1, Ta_2, ..., Ta_n)\}$. Define

$$d((a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n)) = (\|Ta_1 - Tb_1\|_{\mathbb{R}}, \|Ta_2 - Tb_2\|_{\mathbb{R}}, ..., \|Ta_n - Tb_n\|_{\mathbb{R}}) I_{\mathbb{A}},$$

where $(a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n) \in \mathbb{A}^{\oplus n}$ and $I_{\mathbb{A}}$ is the identity element of $\mathbb{A}$. It is easy to verify that $d_{l(\mathscr{E})}$ is an $l(\mathscr{E})$ -valued metric space and $(X, \mathbb{A}^{\oplus n}, d_{l(\mathscr{E})})$ is a complete $l(\mathscr{E})$ -valued metric space, since $\mathbb{A}$ is complete.

Example 3.2 Let $X=\mathbb{A}^{\oplus n},\mathscr{E}=\mathbb{A}$ and $l(\mathscr{E})=\{T:\mathbb{A}\longrightarrow\mathbb{A}\}$. Define

$$d((a_1,a_2,...,a_n),(b_1,b_2,...,b_n))=\sum_{i=1}^{n}\|Ta_i-Tb_i\|_{\mathbb{R}}I_{\mathbb{A}},$$

where $(a_1,a_2,...,a_n),(b_1,b_2,...,b_n)\in\mathbb{A}^{\oplus n}$ and $I_{\mathbb{A}}$ is the identity element of $\mathbb{A}$. It easy to verify that $(X,\mathbb{A},d_{l(\mathscr{E})})$ is a complete $l(\mathscr{E})$ metric space.

Definition 3.5 [3] let $(X,l(\mathscr{E}))$ is an $l(\mathscr{E})$-metric space, we define the open ball on X

$$B_{l(\mathscr{E})}(a,\varepsilon_{l(\mathscr{E})})=\{x\in X;\|x-a\|\prec\varepsilon_{l(\mathscr{E})}\}$$

Definition 3.6 [3] Suppose that $(X,d_{l(\mathscr{E})})$ is $l(\mathscr{E})$-metric space, let $x\in X$ then a neighhborhood of x is any set containing $B_{l(\mathscr{E})}(x,\varepsilon_{l(\mathscr{E})})$ for some $\varepsilon_{l(\mathscr{E})}\succ 0_{l(\mathscr{E})}$.

Definition 3.7 [3] Suppose that $(X,d_{l(\mathscr{E})})$ is $l(\mathscr{E})$-metric space, a subset $U\subset X$ is open if for every $x\in U$ there exist an open ball $B_{l(\mathscr{E})}(a,\varepsilon_{l(\mathscr{E})})$ such that $x\in B_{l(\mathscr{E})}(x,\varepsilon_{l(\mathscr{E})})\subset U$.

Definition 3.8 The union of open set define a topology on X related to $l(\mathscr{E})$.

Motivaied by the idea in [11],[16],[18], we give the following definations.

Definition 3.9 Let X be vector space, if the function $\|.\|_{l(\mathscr{E})}:X\longrightarrow l(\mathscr{E})$ has the following properties:

(1) $\|x\|_{l(\mathscr{E})}\succeq 0$ i.e $\|x\|_{l(\mathscr{E})}$ is a positive operator, $\|x\|_{l(\mathscr{E})}=0\Leftrightarrow x=0$;
(2) $\|\lambda x\|_{l(\mathscr{E})}=|\lambda|\|x\|_{l(\mathscr{E})}$; $\lambda\in\mathbb{C}$;
(3) $\|x+y\|_{l(\mathscr{E})}\preceq\|x\|_{l(\mathscr{E})}+\|y\|_{l(\mathscr{E})}$.

Then $\|.\|$ is said to be $l(\mathscr{E})$-valued norm defined on X, and $(X,\|.\|)$ is said to be $l(\mathscr{E})$-valued normed $l(\mathscr{E})$ space.

Also we will set the relation between $l(\mathscr{E})$-valued metric space and $l(\mathscr{E})$-valued normed space as follow $\quad d_{l(\mathscr{E})}(x,y)=\|x-y\|_{l(\mathscr{E})}$.

Definition 3.10 Let X be a vector space over a field $(F=\mathbb{C},\mathbb{R})$ we say that X is a right $l(\mathscr{E})$-vector space if satisfy:

(1) $(x+y)T=xT+yT$;
(3) $x(T_1+T_2)=xT_1+xT_2$;
(3) $(xS)T=x(ST)$.

Where $x,y \in X$ and $S,T \in l(\mathscr{E})$.

Lemmae 3.2 Let X be a right $l(\mathscr{E})$-vector space then,

$$\|xT\|_{l(\mathscr{E})} \preceq \|x\|\|T\|_{l(\mathscr{E})}.$$

Definition 3.11 Let $\mathbb{A}$ be C^*-algebra,and $l(\mathscr{E})$ be an $l(\mathscr{E})$-normed spac. We say that $l(\mathscr{E})$ is right $\mathbb{A}$-module if the mapping is right module multiplication $(a,T) \longmapsto xa$ of $\mathbb{A} \times l(\mathscr{E}) \longrightarrow l(\mathscr{E})$ such that the following axioms are satisfied:

(1) For each fixed $a \in \mathbb{A}$ the map $(a,T) \longrightarrow Ta$ is linear on $l(\mathscr{E})$: $T \in l(\mathscr{E})$;

(2) For each fixed $T \in l(\mathscr{E})$ the map $(a,T) \longrightarrow Ta$ is linear on $\mathbb{A}$;

(3) For all $a_1,a_2 \in \mathbb{A}$ and all $T \in l(\mathscr{E})$ we have that $(Ta_1)a_2 = T(a_1a_2)$.

Example 3.3 If we define the norm $\|x\|_{l(\mathscr{E})}$ = $\|x\|I_{l(\mathscr{E})}$ (where $I_{l(\mathscr{E})}$ is the identity operator of $l(\mathscr{E})$) then we have that $l(\mathscr{E})$ with this norm is $l(\mathscr{E})$-norm.

Example 3.4 Let $X = \mathbb{A}^{\oplus n}$ and $l(\mathscr{E}) = \mathbb{A}$. Define

$$\|(a_1,a_2,...,a_n)\| = \sum_{i=1}^{n} \|a_i\|I_{\mathbb{A}},$$

where $(a_1,a_2,...,a_n) \in \mathbb{A}^{\oplus n}$ and $I_{\mathbb{A}}$ is the identity element of $\mathbb{A}$. It is easy to verify that X is $l(\mathscr{E})$ -valued normed space.

Lemma 3.3 If S is positive operator then for any operator T implies T^*ST is positive operator.

Proof. Since $S \succeq 0$, we can write $S = R^*R$, for any $R \in (l_{\mathscr{E}})$ implies $T^*(R^*R)T = (T^*R^*)(RT) = (RT)^*(RT) \succeq 0$ □

Definition 3.12 A sequence $\{x_n\}$ in X is said to be convergent if for every $\varepsilon > 0$, there is a natural number N such that for $n > N$ we have

$$\|x_n - x\| \preceq_{l(\mathscr{E})} \varepsilon I_{l(\mathscr{E})} \text{ (where } I_{l(\mathscr{E})} \text{ the identity operator of } l(\mathscr{E}) \text{).}$$

Definition 3.13 A sequence $\{x_n\}$ in X is said to be a Cuachy sequence if for every $\varepsilon > 0$, there is a natural number N such that for $n,m > N$ we have

$$\|x_n - x_m\| \preceq_{l(\mathscr{E})} \varepsilon I_{l(\mathscr{E})}.$$

Lemma 3.3 A sequence $\{x_n\}$ in X is convergence in X if $\|x_n\|_{\mathbb{R}} \longrightarrow 0$ at $n \longrightarrow +\infty$.

Lemma 3.4 [5, 18] Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I:

(1) for any $x \in \mathbb{A}_+$ we have $x \preceq I$ if and only if $\|x\| \leq 1$;

(2) If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I-a$ is invertable and $\|a(I-a)^{-1}\| < 1$;

(3) suppose that $a,b \in \mathbb{A}$ with $a,b \succeq 0$ and $ab = ba$,then $ab \succeq 0$.

(4) by $\acute{\mathbb{A}}$ we denote the set $\{a \in \mathbb{A} : ab = ba \text{ forall } b \in \mathbb{A}\}$ Let $a \in \acute{\mathbb{A}}$, if $b,c \in \mathbb{A}$ with $b \succeq c \succeq 0$

$(I-a)^{-1}b \succeq (I-a)^{-1}c$.

Definition 3.14 Let $(X, l(\mathscr{E}), \|.\|_{l(\mathscr{E})})$ be an $l(\mathscr{E})$ normed space. We call a mapping $T : X \longrightarrow X$ is $l(\mathscr{E})$ contractive mapping on X if there exists an $M \in l(\mathscr{E})$ with $\|M\|_{l(\mathscr{E})} \leq 1$ such that

$$\|Tx - Ty\|_{l(\mathscr{E})} \preceq M^* \|x-y\|_{l(\mathscr{E})} M \text{ for all } x,y \in X.$$

Definition 3.15 An $l(\mathscr{E})$- Banach space is a complete $l(\mathscr{E})$-normed space $(X, \|.\|_{l(\mathscr{E})})$.

Many results on fixed point theorems have been extended from metric spaces to C^*-algebra valued metric spaces with different contraction conditions (see for example [17],[18],[19],[20],[21])

Theorem 3.1 Let $(X, l(\mathscr{E}), \|.\|_{l(\mathscr{E})})$ be $l(\mathscr{E})$ complete normed space and $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition

$$\|Tx - Ty\|_{l(\mathscr{E})} \preceq M^* \|x-y\|_{l(\mathscr{E})} M,$$

where $M \in (\grave{l(\mathscr{E})})_+$ with $\|M\|_{l(\mathscr{E})} < 1$, Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{+\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, \ldots.., x_{n+1} = Tx_n$

$$\begin{aligned}
\|x_{n+1} - x_n\|_{l(\mathscr{E})} &= \|Tx_n - Tx_{n-1}\|_{l(\mathscr{E})} \\
&\preceq M^* \|x_n - x_{n-1}\|_{l(\mathscr{E})} M \\
&= M^* \|Tx_{n-1} - Tx_{n-2}\|_{l(\mathscr{E})} M \\
&\preceq (M^*)^2 \|x_{n-1} - x_{n-2}\|_{l(\mathscr{E})} (M)^2 \\
&\vdots \\
&\preceq (M^*)^n \|x_1 - x_0\|_{l(\mathscr{E})} (M)^n.
\end{aligned}$$

Let $B = \|x_1 - x_0\|_{l(\mathscr{E})}$. Then $\|x_{n+1} - x_n\|_{l(\mathscr{E})} \preceq (M^*)^n B (M)^n$.

For any $n,m \in N$ such that $n \geq m$ the triangle inequality tells that

$$\begin{aligned}
\|x_n - x_m\|_{l(\mathscr{E})} &\preceq \|x_n - x_{n-1}\|_{l(\mathscr{E})} + \|x_{n-1} - x_{n-2}\|_{l(\mathscr{E})} + \ldots + \|x_{m+1} - x_m\|_{l(\mathscr{E})} \\
&\preceq (M^*)^{n-1} B (M)^{n-1} + (M^*)^{n-2} B (M)^{n-2} + \ldots + (M^*)^m B (M)^m
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=m}^{n-1}(M^*)^k B(M)^k \\
&= \sum_{k=m}^{n-1}((M^*)^k B^{1/2})(B^{1/2}(M)^k) \\
&= \sum_{k=m}^{n-1}(B^{1/2}M^k)^*(B^{1/2}M^k) \\
&= \sum_{k=m}^{n-1}|B^{1/2}M^k|^2 \\
&\preceq \sum_{k=m}^{n-1}\||B^{1/2}M^k|^2\|_{l(\mathscr{E})} I_{l(\mathscr{E})} \\
&\preceq \sum_{k=m}^{n-1}\|B^{1/2}\|_{l(\mathscr{E})}^2 \|M^k\|_{l(\mathscr{E})}^2 I_{l(\mathscr{E})} \\
&\preceq \|B\|_{l(\mathscr{E})} \sum_{k=m}^{n-1}\|M\|_{l(\mathscr{E})}^{2k} I_{l(\mathscr{E})} \\
&\preceq \|B\|_{l(\mathscr{E})} \frac{\|M\|_{l(\mathscr{E})}^{2m}}{1-\|M\|_{l(\mathscr{E})}} I_{l(\mathscr{E})} \longrightarrow 0_{l(\mathscr{E})} (m \longrightarrow +\infty),
\end{aligned}
$$

where $I_{l(\mathscr{E})}$ the unite element in $l(\mathscr{E})$, Therefore $\{x_n\}$ is a Cauchy sequence with respect to $l(\mathscr{E})$. By the completeness of $(X, l(\mathscr{E}), \|.\|_{l(\mathscr{E})})$, there exists an $x \in X$ such that $\lim_{n\longrightarrow+\infty} x_n = \lim_{n\longrightarrow+\infty} Tx_{n-1} = x$.
Since

$$
\begin{aligned}
0 \leq \|Tx - x\|_{l(\mathscr{E})} &\preceq \|Tx - Tx_n\|_{l(\mathscr{E})} + \|Tx_n - x\|_{l(\mathscr{E})} \\
&\preceq M^*\|x - x_n\|_{l(\mathscr{E})}M + \|Tx_n - x\|_{l(\mathscr{E})} \longrightarrow 0_{l(\mathscr{E})} \text{ at } n \longrightarrow \infty
\end{aligned}
$$

Implies $\|Tx - x\|_{l(\mathscr{E})} = 0 \Rightarrow Tx = x$. Hence T hase a fixed point .

To prove the uniquness suppose that $y(\neq x)$ is another fixed point of T, since

$$
0 \preceq \|x - y\|_{l(\mathscr{E})} = \|Tx - Ty\|_{l(\mathscr{E})} \preceq M^*\|x - y\|_{l(\mathscr{E})}M,
$$

then we have

$$
\begin{aligned}
0 \leq \|\|x - y\|_{l(\mathscr{E})}\| &= \|\|Tx - Ty\|_{l(\mathscr{E})}\| \\
&\leq \|M^*\|\|\|x - y\|_{l(\mathscr{E})}\|\|M\| \\
&\leq \|M^*\|\|\|x - y\|_{l(\mathscr{E})}\|\|M\| \\
&\leq \|M\|^2\|\|x - y\|_{l(\mathscr{E})}\| \\
&< \|\|x - y\|_{l(\mathscr{E})}\|.
\end{aligned}
$$

It is impossible. So $\|x - y\|_{l(\mathscr{E})} = 0$ and $x = y$, which implies that the fixed point is unique. □

Next, we introduce a version of kannan fixed point in the case of operator on Hilbert C^*-modules

Theorem 3.2 (Kannan Type Theorem [8]) Let $(X, l(\mathscr{E}), \|.\|_{l(\mathscr{E})})$ be an $l(\mathscr{E})$ complete normed space and $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition

$$\|Tx - Ty\|_{l(\mathscr{E})} \preceq \frac{M}{2}[\|Tx - x\|_{l(\mathscr{E})} + \|Ty - y\|_{l(\mathscr{E})}],$$

where $M \in (l(\mathscr{E}))\grave{}_{+}$ with $\|M\|_{l(\mathscr{E})} < 1$, Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{+\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, \ldots.., x_{n+1} = Tx_n$

$$\begin{aligned}
\|x_{n+1} - x_n\|_{l(\mathscr{E})} &= \|Tx_n - Tx_{n-1}\|_{l(\mathscr{E})} \\
&\preceq \frac{M}{2}[\quad \|Tx_n - x_n\|_{l(\mathscr{E})} + \|Tx_{n-1} - x_{n-1}\|_{l(\mathscr{E})} \quad] \\
&= \frac{M}{2}[\quad \|x_{n+1} - x_n\|_{l(\mathscr{E})} + \|x_n - x_{n-1}\|_{l(\mathscr{E})} \quad] \\
&\preceq \frac{M}{2}\|x_{n+1} - x_n\|_{l(\mathscr{E})} + \frac{M}{2}\|x_n - x_{n-1}\|_{l(\mathscr{E})}.
\end{aligned}$$

Thus,

$(I_{l(\mathscr{E})} - \frac{M}{2})\|x_{n+1} - x_n\|_{l(\mathscr{E})} \preceq \frac{M}{2}\|x_n - x_{n-1}\|_{l(\mathscr{E})}$.

Since $M \in (l(\mathscr{E}))\grave{}_{+}$ with $\|\frac{M}{2}\|_{l(\mathscr{E})} < \frac{1}{2}$, one have $(I_{l(\mathscr{E})} - \frac{M}{2})^{-1} \in (l(\mathscr{E}))\grave{}_{+}$, and furthermore $\frac{M}{2}(I_{l(\mathscr{E})} - \frac{M}{2})^{-1} \in (l(\mathscr{E}))\grave{}_{+}$ with $\|\frac{M}{2}(I_{l(\mathscr{E})} - \frac{M}{2})^{-1}\|_{l(\mathscr{E})} < 1$. Therefore,

$$\begin{aligned}
\|x_{n+1} - x_n\|_{l(\mathscr{E})} &\preceq (\frac{\frac{M}{2}}{I_{l(\mathscr{E})} - \frac{M}{2}})\|x_n - x_{n-1}\|_{l(\mathscr{E})} \\
&\preceq (\frac{\frac{M}{2}}{I_{l(\mathscr{E})} - \frac{M}{2}})^2\|x_{n-1} - x_{n-2}\|_{l(\mathscr{E})} \\
&\vdots \\
&\preceq (\frac{\frac{M}{2}}{I_{l(\mathscr{E})} - \frac{M}{2}})^n\|x_1 - x_0\|_{l(\mathscr{E})}.
\end{aligned}$$

Let $t = \frac{M}{2}(I_{l(\mathscr{E})} - \frac{M}{2})^{-1}, B = \|x_1 - x_0\|_{l(\mathscr{E})}$.

Implies $\|x_{n+1} - x_n\|_{l(\mathscr{E})} \preceq t^n B$.

For $n + 1 > m$

$$\begin{aligned}
\|x_{n+1} - x_m\|_{l(\mathscr{E})} &\preceq \|x_{n+1} - x_n\|_{l(\mathscr{E})} + \|x_n - x_{n-1}\|_{l(\mathscr{E})} + \cdots + \|x_{m+1} - x_m\|_{l(\mathscr{E})} \\
&\preceq t^n B + t^{n-1}B + \cdots + t^m B \\
&\preceq (t^n + t^{n-1} + \cdots + t^m)B \\
&= \textstyle\sum_{k=m}^{n} t^k B \\
&= \textstyle\sum_{k=m}^{n} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} \\
&= \textstyle\sum_{k=m}^{n} B^{\frac{1}{2}} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}}
\end{aligned}$$

$$= \sum_{k=m}^{n} (t^{\frac{k}{2}} B^{\frac{1}{2}})^{*} (t^{\frac{k}{2}} B^{\frac{1}{2}})$$
$$= \sum_{k=m}^{n} |t^{\frac{k}{2}} B^{\frac{1}{2}}|^{2}$$
$$\preceq \| \sum_{k=m}^{n} |t^{\frac{k}{2}} B^{\frac{1}{2}}|^{2} \|_{l(\mathscr{E})} I_{l(\mathscr{E})}$$
$$\preceq \sum_{k=m}^{n} \|B^{\frac{1}{2}}\|_{l(\mathscr{E})}^{2} \|t^{\frac{k}{2}}\|_{l(\mathscr{E})}^{2} I_{l(\mathscr{E})}$$
$$= \|B\|_{l(\mathscr{E})} \sum_{k=m}^{n} \|t\|_{l(\mathscr{E})}^{k} I_{l(\mathscr{E})}$$
$$\preceq \|B\|_{l(\mathscr{E})} \frac{\|t\|_{l(\mathscr{E})}^{m}}{1-\|t\|_{l(\mathscr{E})}^{m}} I_{l(\mathscr{E})} \longrightarrow 0_{l(\mathscr{E})} (m \longrightarrow +\infty),$$

where $I_{l(\mathscr{E})}$ the unite element in $l(\mathscr{E})$, Therefore $\{x_n\}$ is a Cauchy sequence with respect to $l(\mathscr{E})$. By the completeness of $(X, l(\mathscr{E}), \|.\|_{l(\mathscr{E})})$, there exists an $x \in X$ such that $\lim_{n\longrightarrow+\infty} x_n = \lim_{n\longrightarrow+\infty} Tx_{n-1} = x$.
Since

$$\|Tx - x\|_{l(\mathscr{E})} \preceq \|Tx - Tx_n\|_{l(\mathscr{E})} + \|Tx_n - x\|_{l(\mathscr{E})}$$
$$\preceq \frac{M}{2}(\|Tx - x\|_{l(\mathscr{E})} + \|Tx_n - x_n\|_{l(\mathscr{E})}) + \|Tx_n - x\|_{l(\mathscr{E})}$$
$$= \frac{M}{2}\|Tx - x\|_{l(\mathscr{E})} + \frac{M}{2}\|Tx_n - x_n\|_{l(\mathscr{E})} + \|Tx_n - x\|_{l(\mathscr{E})}.$$

Implies $\|Tx - x\|_{l(\mathscr{E})} \preceq \frac{\frac{M}{2}}{I_{l(\mathscr{E})} - \frac{M}{2}} \|Tx_n - x_n\|_{l(\mathscr{E})} + \frac{1}{I_{l(\mathscr{E})} - \frac{M}{2}} \|Tx_n - x\|_{l(\mathscr{E})}$
$\|Tx - x\|_{l(\mathscr{E})} \preceq \frac{\frac{M}{2}}{I_{l(\mathscr{E})} - \frac{M}{2}} \|x_{n+1} - x_n\|_{l(\mathscr{E})} + \frac{1}{I_{l(\mathscr{E})} - \frac{M}{2}} \|x_{n+1} - x\|_{l(\mathscr{E})} \longrightarrow 0 (n \longrightarrow +\infty),$
Implies $\|Tx - x\|_{l(\mathscr{E})} = 0 \Rightarrow Tx = x$.

To prove the uniquness suppose that $y(\neq x)$ is another fixed point of T, then

$$0 \preceq \|x - y\|_{l(\mathscr{E})} = \|Tx - Ty\|_{l(\mathscr{E})}$$
$$\preceq \frac{M}{2}(\|Tx - x\|_{l(\mathscr{E})} + \|Ty - y\|_{l(\mathscr{E})})$$
$$\preceq 0$$

This means that

$\|x - y\|_{l(\mathscr{E})} = 0$ implies $x = y$.

Therefore the fixed point is unique. □

Theorem 3.3 (Extension of Kannan Type Theorem) Let $(X, l(\mathscr{E}), \|.\|_{l(\mathscr{E})})$ be an $l(\mathscr{E})$ complete normed space and $T : X \longrightarrow X$ be a self mapping satisfy the following contraction condition

$$\|Tx - Ty\|_{l(\mathscr{E})} \preceq M[\frac{\|x-y\|_{l(\mathscr{E})}}{2} + \frac{\|Tx-x\|_{l(\mathscr{E})} + \|Ty-y\|_{l(\mathscr{E})}}{2}],$$

where $M \in (l(\mathscr{E}))\grave{}_+$ with $\|M\|_{l(\mathscr{E})} < \frac{1}{2}$, Then T has a unique fixed point.

Proof. Le $x_0 \in X$ be arbitrary point and construct a sequence $\{x_n\}_{n=0}^{+\infty} \subseteq X$ by the way: $x_1 = Tx_0, x_2 = Tx_1, \ldots.., x_{n+1} = Tx_n$.

$$\begin{aligned}
\|x_{n+1} - x_n\|_{l(\mathscr{E})} &= \|Tx_n - Tx_{n-1}\|_{l(\mathscr{E})} \\
&\preceq M[\quad \frac{\|x_n - x_{n-1}\|_{l(\mathscr{E})}}{2} + \frac{\|Tx_n - x_n\|_{l(\mathscr{E})} + \|Tx_{n-1} - x_{n-1}\|_{l(\mathscr{E})}}{2}] \\
&= M[\quad \frac{\|x_n - x_{n-1}\|_{l(\mathscr{E})}}{2} + \frac{\|x_{n+1} - x_n\|_{l(\mathscr{E})} + \|x_n - x_{n-1}\|_{l(\mathscr{E})}}{2} \quad] \\
&= M[\quad \|x_n - x_{n-1}\|_{l(\mathscr{E})} + \frac{\|x_{n+1} - x_n\|_{l(\mathscr{E})}}{2} \quad] \\
&= M \quad \|x_n - x_{n-1}\|_{l(\mathscr{E})} + \frac{M}{2}\|x_{n+1} - x_n\|_{l(\mathscr{E})} \quad].
\end{aligned}$$

Thus,

$(I_{l(\mathscr{E})} - \frac{M}{2})\|x_{n+1} - x_n\|_{l(\mathscr{E})} \preceq M\|x_n - x_{n-1}\|_{l(\mathscr{E})}$.

Since $M \in (l(\mathscr{E}))\grave{}_+$ with $\|M\|_{l(\mathscr{E})} \leq \frac{1}{2}$, one have $(I_{l(\mathscr{E})} - M)^{-1} \in (l(\mathscr{E}))\grave{}_+$, and furthermore $M(I-M)^{-1} \in (l(\mathscr{E}))\grave{}_+$ with $\|M(I_{l(\mathscr{E})} - M)^{-1}\|_{l(\mathscr{E})} \leq 1$. Therefore,

$$\begin{aligned}
\|x_{n+1} - x_n\|_{l(\mathscr{E})} &\preceq (\frac{M}{I_{l(\mathscr{E})} - \frac{M}{2}})\|x_n - x_{n-1}\|_{l(\mathscr{E})} = t\|x_n - x_{n-1}\|_{l(\mathscr{E})} \\
&\preceq t^2\|x_{n-1} - x_{n-2}\|_{l(\mathscr{E})} \\
&\vdots \\
&\preceq t^n\|x_1 - x_0\|_{l(\mathscr{E})},
\end{aligned}$$

where $t = M(I_{l(\mathscr{E})} - \frac{M}{2})^{-1}$.

For $n+1 > m$.

$$\begin{aligned}
\|x_{n+1} - x_m\|_{l(\mathscr{E})} &\preceq \|x_{n+1} - x_n\|_{l(\mathscr{E})} + \|x_n - x_{n-1}\|_{l(\mathscr{E})} + \cdots + \|x_{m+1} - x_m\|_{l(\mathscr{E})} \\
&\preceq (t^n + t^{n-1} + \cdots + t^m)\|x_1 - x_0\|_{l(\mathscr{E})}.
\end{aligned}$$

Let $B = \|x_1 - x_0\|_{l(\mathscr{E})}$, implies

$$\begin{aligned}
\|x_{n+1} - x_m\|_{l(\mathscr{E})} &= \textstyle\sum_{k=m}^{n} t^k B \\
&= \textstyle\sum_{k=m}^{n} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} \\
&= \textstyle\sum_{k=m}^{n} B^{\frac{1}{2}} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}} \\
&= \textstyle\sum_{k=m}^{n} (t^{\frac{k}{2}} B^{\frac{1}{2}})^* (t^{\frac{k}{2}} B^{\frac{1}{2}}) \\
&= \textstyle\sum_{k=m}^{n} |t^{\frac{k}{2}} B^{\frac{1}{2}}|^2 \\
&\preceq \|\textstyle\sum_{k=m}^{n} |t^{\frac{k}{2}} B^{\frac{1}{2}}|^2\|_{l(\mathscr{E})} I_{l(\mathscr{E})}
\end{aligned}$$

$$\begin{aligned}&\preceq \sum_{k=m}^{n}\|B^{\frac{1}{2}}\|_{l(\mathscr{E})}^{2}\|t^{\frac{k}{2}}\|_{l(\mathscr{E})}^{2}I_{l(\mathscr{E})}\\ &=\|B\|_{l(\mathscr{E})}\sum_{k=m}^{n}\|t\|_{l(\mathscr{E})}^{k}I_{l(\mathscr{E})}\\ &\preceq \|B\|_{l(\mathscr{E})}\frac{\|t\|_{l(\mathscr{E})}^{m}}{1-\|t\|_{l(\mathscr{E})}^{m}}I_{l(\mathscr{E})}\longrightarrow 0_{l(\mathscr{E})}(m\longrightarrow +\infty),\end{aligned}$$

where $I_{l(\mathscr{E})}$ the unite element in $l(\mathscr{E})$, Therefore $\{x_n\}$ is a Cauchy sequence with respect to $l(\mathscr{E})$. By the completeness of$(X,l(\mathscr{E}),\|.\|_{l(\mathscr{E})})$, there exists an $x\in X$ such that $\lim_{n\longrightarrow+\infty} x_n = \lim_{n\longrightarrow+\infty} Tx_{n-1}=x$.

Since

$$\begin{aligned}\|Tx-x\|_{l(\mathscr{E})} &\preceq \|Tx-Tx_n\|_{l(\mathscr{E})}+\|Tx_n-x\|_{l(\mathscr{E})}\\ &\preceq M\Big(\frac{\|x-x_n\|_{l(\mathscr{E})}}{2}+\frac{\|Tx-x\|_{l(\mathscr{E})}+\|Tx_n-x_n\|_{l(\mathscr{E})}}{2}\Big)+\|Tx_n-x\|_{l(\mathscr{E})}\\ &\preceq M\Big(\frac{\|x-x_n\|_{l(\mathscr{E})}}{2}+\frac{\|Tx-x\|_{l(\mathscr{E})}}{2}+\frac{\|x_{n+1}-x_n\|_{l(\mathscr{E})}}{2}\Big)+\|Tx_n-x\|_{l(\mathscr{E})}.\end{aligned}$$

Implies $\|Tx-x\|_{l(\mathscr{E})}\preceq \frac{M}{I_{l(\mathscr{E})}-\frac{M}{2}}\Big(\frac{\|x-x_{n-1}\|_{l(\mathscr{E})}}{2}+\frac{\|x_{n+1}-x_n\|_{l(\mathscr{E})}}{2}\Big)+\frac{1}{I_{l(\mathscr{E})}-\frac{M}{2}}\|x_{n+1}-x\|_{l(\mathscr{E})}\longrightarrow 0(at \quad n\longrightarrow +\infty)$.

Then This implies that $Tx=x$ i.e., x is fixed point of T .

To prove the uniquencess suppose that $y(\neq x)$ is another fixed point of T, then

$$\begin{aligned}0\leq \|x-y\|_{l(\mathscr{E})} &= \|Tx-Ty\|_{l(\mathscr{E})}\\ &\preceq M\Big(\frac{\|x-y\|_{l(\mathscr{E})}}{2}+\frac{\|Tx-x\|_{l(\mathscr{E})}+\|Ty-y\|_{l(\mathscr{E})}}{2}\Big)\\ &\preceq \frac{M}{2}\|x-y\|_{l(\mathscr{E})},\end{aligned}$$

This is contradiction, implies $x=y$.

Therefore the fixed point is unique. □

4. APPLICATION

The soluation of operator on Hilbert C^*-module is important and studied by many authers see ([6], [4]). Hence we give the existance and uniquness of such solution of operator equations by using fixed point theorem.

example Suppose that $\mathscr{E}$ is a Hilbert space, $l(\mathscr{E})$ is the set of linear bounded operatoes on $\mathscr{E}$. Let $T_1,T_2,\cdots\in l(\mathscr{E})$, which satisfy $\sum_{n=1}^{+\infty}\|T_n\|^2<1$ and $S\in l(\mathscr{E}), R\in l(\mathscr{E})_+$.

Then the operator equation

$S-\sum_{n=1}^{\infty} T_n^* S T_n = R$

has a unique solution in $l(\mathscr{E})$.

Proof. Set $B=\sum_{n=1}^{+\infty}\|T_n\|^2 I_{l(\mathscr{E})}$. Clear if $\alpha=0$, then $T_n=\theta(n\in\mathbb{N})$, and the equation has a unique solution in $l(\mathscr{E})$. Without loss of generality, one can suppose that $B>0$.

For $S,Q\in l(\mathscr{E})$, set

$\|S-Q\|_{l(\mathscr{E})}=\|S-Q\|I_{l(\mathscr{E})}$.

It is easy to verify tha $\|S-Q\|_{l(\mathscr{E})}$ is an $l(\mathscr{E})$-valued metric space and $(l(\mathscr{E}),\|.\|)$ is complete since $l(\mathscr{E})$ is a Banach space.

Consider the map $F: l(\mathscr{E})\longrightarrow l(\mathscr{E})$ defined by

$$F(S)=\sum_{n=1}^{\infty} T_n^* S T_n + R. \tag{1}$$

Then

$\|F(S)-F(Q)\| = \|F(S)-F(Q)\|I_{l(\mathscr{E})} = \|\sum_{n=1}^{+\infty} T_n^*(S-Q)T_n\|I_{l(\mathscr{E})}$

$\leq \sum_{n=1}^{\infty}\|T_n\|^2\|S-Q\|I_{l(\mathscr{E})}$

$= B\|S-Q\|$

$= (B^{\frac{1}{2}}I_{l(\mathscr{E})})^*\|S-Q\|(B^{\frac{1}{2}}I_{l(\mathscr{E})})$.

Using Theorem 3.1 there exists a unique fixed point $S\in l(\mathscr{E})$. Furthermore, since $\sum_{n=1}^{+\infty} T_n^* S T_n + R$ is positive operator, then the operator equation (1) has a unique solution.

□

5. Conclusions

In this paper, we introduced the notions of metric space valued-operator of Hilbert C^*-module. We define some contraction mapping and prove some Banach fixed point theorems for a self mappings T on the Banach space $l(\mathscr{E})$. Finally we give an application to study the existence and uniqueness soluation of systems of operators on Hilbert C^*-module.

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