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THE POINT OF COINCIDENCE AND COMMON FIXED POINT RESULTS IN CONE METRIC SPACE

BINAYAK S. CHOUDHURY[1], LOKESH KUMAR[2], T. SOM[2*] AND N. METIYA[3]

[1]Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah - 711103, West Bengal, India

[2]Department of Applied Mathematics, Indian Institute of Technology (BHU), Varanasi- 221005, India

[3]Department of Mathematics, Bengal Institute of Technology, Kolkata - 700150, West Bengal, India

Abstract. In this paper we have shown that under certain assumptions an arbitrary family of mappings will have a unique point of coincidence and a unique common fixed point with another function satisfying (ψ, η, ϕ)-weak contractive condition in cone metric space. The theorems have several corollaries and a supporting example.

Keywords: Cone metric space, Control function, Point of coincidence, Common fixed point, Weak contraction.

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1. Introduction

The Banach contraction mapping principle is widely recognized as the source of metric fixed point theory. This contraction principle has further several generalizations in metric spaces as well as in cone metric spaces. Huang and Zhang [12] introduced the concept of cone metric space, where every pair of elements is assigned to an element of a Banach space and defined a partial order on the Banach space with the help of a subset of the Banach space called cone which satisfy certain properties. Fixed point studies were initiated in such spaces in the same work. After that, fixed point theory has experience the rapid growth in cone metric spaces. A review of this development is given in [14]. References [3, 16, 17, and 22] are some more recent examples of this work.

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*Corresponding author

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Weak contraction principle is a generalization of Banach's contraction principle which was first given by Alber *et al.* in Hilbert spaces [2]. It was subsequently extended to metric spaces by Rhoades [19]. In weak contraction results the contractive inequality involves a control function. The concept of control function in metric fixed point theory was introduced by Khan *et al.* [21] as Altering distance function. This function and its generalizations have been used in fixed and coincidence point problems in a large number of works; some of these works are in [4, 5, 20]. In particular, Choudhury *et al.*[6, 7, 8] established some fixed point results in cone metric spaces with the help of control functions.

In this paper we establish some point of coincidence theorems for an arbitrary family of self mappings with another self mapping in cone metric spaces with the help of three different control functions ψ, η and ϕ. The existence of the common fixed point is ensured by imposing, amongst other conditions, the condition of weak compatibility. It may be mentioned that some fixed point results for weakly compatible maps in cone metric spaces have been deduced by Abbas and Jungck [1]. An illustrative example is given to support our main results.

Before coming to our main result we give some preliminaries of cone metric space which was firstly introduced by Huang and Zhang [12].

2. Mathematical preliminaries

Definition 2.1 [12] Let E be a real Banach space and θ is the zero of the Banach space E. Let P be a subset of E. P is called a cone if

(i) P is closed, non-empty and $P \neq \{\theta\}$

(ii) $ax + by \in P$ for all $x, y \in P$ and non negative real numbers a, b

(iii) $P \cap (-P) = \{\theta\}$

For a given cone P we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. Here $x < y$ will stand for $x \leq y$ and $x \neq y$,while $x \ll y$ will stand for $y - x \in \mathrm{int}P$, where int$P$ denotes the interior of P. $x \leq y$ is same as $y \geq x$ and $x \ll y$ is same as $y \gg x$.
A cone P is called normal if there is a real number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies} \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of cone P. The cone P is called regular if every increasing and bounded above sequence $\{x_n\}$ in E is convergent. Equivalently, the cone P is regular if and only if every decreasing and bounded below sequence is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose that E is a real Banach space withcone P in E with int$P \neq \emptyset$ and $\leq$ is the partial ordering with respect to P.

Definition 2.2 [12] Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(i) $\theta \leq d(x, y)$, for all $x, y \in X$and $d(x, y) = \theta$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all$x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3 [12] Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \ll c$.We denote this by

$$\lim_n x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0, d(x_n, x_m) \ll c$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that if P is a normal cone, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if and only if

$$d(x_n, x_m) \to \theta \text{ as } n, m \to \infty.$$[12]

Definition 2.4 Letψ : int$P \cup \{\theta\} \to$ int$P \cup \{\theta\}$ be a function.

(i) We say ψ is strongly monotone increasing if for $x, y \in$ int$P \cup \{\theta\}$,

$$x \leq y \iff \psi(x) \leq \psi(y).$$

(ii) ψ is said to be continuous at $x_0 \in$ int$P \cup \{\theta\}$ if for any sequence $\{x_n\}$ in int$P \cup \{\theta\}$,

$$x_n \to x_0 \Rightarrow \psi(x_n) \leq \psi(x_0).$$

The following is the definition of Altering distance function in cone metric space.

Definition 2.5 A function ψ : int$P \cup \{\theta\} \to$ int$P \cup \{\theta\}$ is called an Altering distance function if the following properties are satisfied:

(i) ψ is strongly monotone increasing and continuous,

(ii) $\psi(t) = \theta$ if and only if $t = \theta$.

Lemma 2.1 Let E be a real Banach space with cone P in E. Then

(i) if $a \leq b$ and $b \ll c$, then $a \ll c$ [13],

(ii) if $a \ll b$ and $b \ll c$, then $a \ll c$ [13],

(iii) if $0 \leq x \leq y$ and $a \geq 0$, where a is real number, then $0 \leq ax \leq ay$ [13],

(iv) if $0 \leq x_n \leq y_n$, for $n \in \mathbb{N}$ and $\lim_n x_n = x$, $\lim_n y_n = y$, then $0 \leq ax \leq ay$ [13],

(v) P is normal iff $x_n \leq y_n \leq z_n$ and $\lim_n x_n = \lim_n z_n = x$ imply $\lim_n y_n = x$ [9].

Lemma 2.2 [7] Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in \operatorname{int}P$, for $x, y \in X$ with $x \neq y$. Let $\phi : \operatorname{int}P \cup \{\theta\} \to \operatorname{int}P \cup \{\theta\}$ be a function with the following properties:

(i) $\phi(t) = \theta$ if and only if $t = \theta$,

(ii) $\phi(t) \ll t$, for $t \in \operatorname{int}P$ and

(iii) either $\phi(t) \leq d(x, y)$ or $d(x, y) \leq \phi(t)$, for $t \in \operatorname{int}P \cup \{\theta\}$ and $x, y \in X$.

Let $\{x_n\}$ be a sequence in X for which $\{d(x_n, x_{n+1})\}$ is monotonic decreasing. Then $\{d(x_n, x_{n+1})\}$ is convergent to either $r = \theta$ or $r \in \operatorname{int}P$.

Lemma 2.3 [8] Let(X, d) be a cone metric space. Let $\phi : \operatorname{int}P \cup \{\theta\} \to \operatorname{int}P \cup \{\theta\}$ be a function such that

(i) $\phi(t) = \theta$ if and only if $t = \theta$,

(ii) $\phi(t) \ll t$, for $t \in \operatorname{int}P$

Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for every $c \in E$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll \phi(c)$, for all $n, m > n_0$.

Definition 2.6 [15] Let g and f be two self-maps of a set X. If $w = gx = fx$ for some $x \in X$, then x is called a coincidence point of g and f, and w is called a point of coincidence of g and f. Self-maps g and f are said to be weakly compatible if they commute at their coincidence point; that is, if

$$gx = fx \text{ for some } x \in X, \text{ then } gfx = fgx.$$

Lemma 2.4 [1] Let g and f be weakly compatible self maps of a set X. If g and f have a unique point of coincidence $w = gx = fx$, then w is the unique common fixed point of g and f.

3. Main Results

Theorem 3.1. Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in \mathrm{int}P$, for $x, y \in X$ with $x \neq y$. Let $g : X \to X$ be a mapping such that $g(X)$ is a complete subspace of X. Let $\{f_\alpha : X \to X : \alpha \in \Lambda\}$ be a family of mappings. Suppose that there exists $\alpha_0 \in \Lambda$ such that $f_{\alpha_0}(X) \subseteq g(X)$ and for $x, y \in X$,

$$\psi\left(d\left(f_{\alpha_0}x, f_\alpha y\right)\right) \leq \eta\left(\frac{1}{2}\left[\, d\left(gx, f_{\alpha_0}x\right) + d(gy, f_\alpha y)\right]\right)$$
$$-\phi\left(\tfrac{1}{2}\left[d\left(gx, f_{\alpha_0}x\right) + d(gy, f_\alpha y)\right]\right) \qquad (1)$$

where $\alpha \in \Lambda$ and ψ, η, ϕ: $\mathrm{int}P \cup \{\theta\} \to \mathrm{int}P \cup \{\theta\}$ are such that ψ and η are continuous, ϕ is lower semicontinuous and also

(i) ψ is strongly monotonic increasing,

(ii) $\psi(t) = \eta(t) = \phi(t) = \theta$ if and only if $t = \theta$,

(iii) $\psi(t) - \eta(t) + \phi(t) > \theta$ for all $t \in \mathrm{int}P$,

(iv) $\phi(t) \ll t$ for $t \in \mathrm{int}P$ and

(v) either $\phi(t) \leq d(x, y)$ or $d(x, y) \ll \phi(t)$, for $t \in \mathrm{int}P \cup \{\theta\}$ and $x, y \in X$.

Then g and $\{f_\alpha : \alpha \in \Lambda\}$ have a unique point of coincidence in X. Moreover, if g and f_{α_0} are weakly compatible, then g and $\{f_\alpha : \alpha \in \Lambda\}$ have a unique common fixed point in X.

Proof: First we establish that any point of coincidence of g and $f_{\alpha_0}$is a point of coincidence of g and $\{f_\alpha : \alpha \in \Lambda\}$ and conversely. Suppose that $p \in X$ be a point of coincidence of g and f_{α_0}. Then there exists a $z \in X$ such that $p = gz = f_{\alpha_0}z$. From (1) and using the monotone property of ψ, we have

$$\psi\left(\frac{1}{2}d(gz, f_\alpha z)\right) \leq \psi\big(d(gz, f_\alpha z)\big) = \psi\left(d\left(f_{\alpha_0}z, f_\alpha z\right)\right)$$
$$\leq \eta\left(\frac{1}{2}\left[\, d\left(gz, f_{\alpha_0}z\right) + d(gz, f_\alpha z)\right]\right) - \phi\left(\frac{1}{2}\left[d\left(gz, f_{\alpha_0}z\right) + d(gz, f_\alpha z)\right]\right)$$
$$= \eta\left(\frac{1}{2}\left[\, d(gz, f_\alpha z)\right]\right) - \phi\left(\frac{1}{2}\left[d(gz, f_\alpha z)\right]\right)$$

That is,

$$\psi\left(\tfrac{1}{2}d(gz, f_\alpha z)\right) - \eta\left(\tfrac{1}{2}\left[\, d(gz, f_\alpha z)\right]\right) + \phi\left(\tfrac{1}{2}\left[d(gz, f_\alpha z)\right]\right) \leq \theta,$$

which by (ii) and (iii) implies that $d(gz, f_\alpha z) = \theta$, that is, $gz = f_\alpha z$, for all $\alpha \in \Lambda$. Hence we have $p = gz = f_\alpha z$, for all $\alpha \in \Lambda$, that is, p is a point of coincidence of g and $\{f_\alpha : \alpha \in \Lambda\}$. The converse part is trivial.

Now, it is sufficient to prove that g and f_{α_0} have a unique point of coincidence. Let $x_0 \in X$. Since $f_{\alpha_0}(X) \subseteq g(X)$, we can construct a sequence $\{x_n\}$ in X such that $f_{\alpha_0} x_n = g x_{n+1}$, for all $n \geq 0$. If there exists an integer $N \geq 0$ such that $f_{\alpha_0} x_N = f_{\alpha_0} x_{N+1}$, then $g x_{N+1} = f_{\alpha_0} x_{N+1}$, which means that g and f_{α_0} have a point of coincidence. Hence we will assume that

$$f_{\alpha_0} x_n \neq f_{\alpha_0} x_{n+1}, \text{ for all } n \geq 0.$$

For $\alpha = \alpha_0, x = x_{n+1}$ and $y = x_{n+2}$, from (1), we have

$$\begin{aligned}\psi\left(d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right) \leq \eta\left(\tfrac{1}{2}\left[d\left(g x_{n+1}, f_{\alpha_0} x_{n+1}\right) + d\left(g x_{n+2}, f_{\alpha_0} x_{n+2}\right)\right]\right)\\ -\phi\left(\tfrac{1}{2}\left[d\left(g x_{n+1}, f_{\alpha_0} x_{n+1}\right) + d\left(g x_{n+2}, f_{\alpha_0} x_{n+2}\right)\right]\right)\end{aligned}$$

that is,

$$\begin{aligned}\psi\left(d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right) \leq \eta\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) + d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right]\right)\\ -\phi\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) + d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right]\right) \qquad (2)\end{aligned}$$

For all $n \geq 0$, we have

$$\begin{aligned}&\psi\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) + d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right]\right) - \psi\left(d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right)\\ &\geq \psi\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) + d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right]\right) - \eta\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) + d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right]\right) +\\ &\phi\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) + d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right]\right) \geq \theta \quad \text{(by (ii) and (iii))}.\end{aligned}$$

This implies that

$$\begin{aligned}&\psi\left(d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right) \leq \psi\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) + d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right]\right)\\ &\Rightarrow d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right) \leq \tfrac{1}{2}\left[d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) + d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right)\right] \text{(by (i))}\\ &\Rightarrow d\left(f_{\alpha_0} x_{n+1}, f_{\alpha_0} x_{n+2}\right) \leq d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) \text{ for all } n \geq 0.\end{aligned}$$

This implies that the sequence $\{d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right)\}$ is monotone decreasing and bounded below by θ. Then by Lemma 2.2, there exists $r \in \operatorname{int} P \cup \{\theta\}$ such that $d\left(f_{\alpha_0} x_n, f_{\alpha_0} x_{n+1}\right) \to r$ as $n \to \infty$.

Letting $n \to \infty$. in equation (2) and using the lower semi continuity of ϕ and continuities of ψ and η, we have

$$\psi(r) \leq \eta(r) - \phi(r)$$
$$\Rightarrow \psi(r) - \eta(r) + \phi(r) \leq \theta \Rightarrow r = \theta \qquad \text{(by (ii) and (iii)).}$$

Then, $\lim_{n\to\infty} d\big(f_{\alpha_0}x_n, f_{\alpha_0}x_{n+1}\big) = \theta$ (3)

Next we show that $\{f_{\alpha_0}x_n\}$ is a Cauchy sequence. If $\{f_{\alpha_0}x_n\}$ is not a Cauchy sequence, then by Lemma 2.3, there exists a $c \in E$ with $\theta \ll c$ such that $\forall\, n_0 \in \mathbb{N}, \exists\, n, m \in \mathbb{N}$ with $n > m > n_0$ such that $d\big(f_{\alpha_0}x_n, f_{\alpha_0}x_{n+1}\big) <\not\leq \phi(c)$. Hence by a property of ϕ in (v) of the theorem,

$\phi(r) \leq d\big(f_{\alpha_0}x_n, f_{\alpha_0}x_{n+1}\big)$.

Therefore, there exist sequences $\{n(k)\}$ and $\{m(k)\}$ in $\mathbb{N}$ such that for all positive integers k,

$n(k) > m(k) > k$ and $d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\big) \geq \phi(c)$.

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\big) \geq \phi(c)$$

and

$$d\big(f_{\alpha_0}x_{n(k)-1}, f_{\alpha_0}x_{m(k)}\big) \ll \phi(c)$$

Now,

$$\phi(c) \leq d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\big) \leq d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{n(k)-1}\big) + d\big(f_{\alpha_0}x_{n(k)-1}, f_{\alpha_0}x_{m(k)}\big)$$

that is,

$$\phi(c) \leq d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\big) \leq d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{n(k)-1}\big) + \phi(c).$$

Letting $k \to \infty$in the above inequality, using (3) and property (v) of Lemma 2.1, we have

$$\lim_{k\to\infty} d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\big) = \phi(c) \qquad (4)$$

Again,

$$d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\big) \leq d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{n(k)+1}\big) + d\big(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}\big)$$
$$+ d\big(f_{\alpha_0}x_{m(k)+1}, f_{\alpha_0}x_{m(k)}\big)$$

and

$$d\big(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}\big) \leq d\big(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{n(k)}\big) + d\big(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\big)$$
$$+ d\big(f_{\alpha_0}x_{m(k)}, f_{\alpha_0}x_{m(k)+1}\big).$$

Letting $k \to \infty$ in above inequalities, using (3) and (4), we have

$$\lim_{k\to\infty} d\big(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}\big) = \phi(c) \qquad (5)$$

For $\alpha = \alpha_0, x = x_{n(k)+1}$ and $y = x_{m(k)+1}$, from (1), we have

$$\psi\left(d\left(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}\right)\right)$$
$$\leq \eta\left(\frac{1}{2}\left[d\left(gx_{n(k)+1}, f_{\alpha_0}x_{n(k)+1}\right) + d\left(gx_{m(k)+1}, f_{\alpha_0}x_{m(k)+1}\right)\right]\right)$$
$$- \phi\left(\frac{1}{2}\left[d\left(gx_{n(k)+1}, f_{\alpha_0}x_{n(k)+1}\right) + d\left(gx_{m(k)+1}, f_{\alpha_0}x_{m(k)+1}\right)\right]\right)$$

That is,

$$\psi\left(d\left(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}\right)\right)$$
$$\leq \eta\left(\frac{1}{2}\left[d\left(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{n(k)+1}\right) + d\left(f_{\alpha_0}x_{m(k)}, f_{\alpha_0}x_{m(k)+1}\right)\right]\right)$$
$$- \phi\left(\frac{1}{2}\left[d\left(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{n(k)+1}\right) + d\left(f_{\alpha_0}x_{m(k)}, f_{\alpha_0}x_{m(k)+1}\right)\right]\right)$$

Letting $k \to \infty$ in the above inequality and using (3) and (5) with the properties of ψ, η and ϕ we obtain

$$\psi(\phi(c)) \leq \theta \Rightarrow \phi(c) = \theta$$

which is a contradiction. Hence $\{f_{\alpha_0}x_n\}$ is a Cauchy sequence in $g(X)$.

From the completeness of $g(X)$, there exists $z \in g(X)$ such that

$$f_{\alpha_0}x_n \to z \text{ as } n \to \infty \tag{6}$$

Since $z \in g(X)$, we can find $p \in X$ such that $z = gp$.

For $\alpha = \alpha_0,\ x = x_{n+1}$ and $y = p$, and using the monotone property of ψ, from (1), we have

$$\psi\left(\tfrac{1}{2}d\left(f_{\alpha_0}x_{n+1}, f_{\alpha_0}p\right)\right) \leq \psi\left(d\left(f_{\alpha_0}x_{n+1}, f_{\alpha_0}p\right)\right)$$
$$\leq \eta\left(\tfrac{1}{2}\left[d\left(gx_{n+1}, f_{\alpha_0}x_{n+1}\right) + d\left(gp, f_{\alpha_0}p\right)\right]\right) - \phi\left(\tfrac{1}{2}\left[d\left(gx_{n+1}, f_{\alpha_0}x_{n+1}\right) + d\left(gp, f_{\alpha_0}p\right)\right]\right)$$
$$\Rightarrow \psi\left(\tfrac{1}{2}d\left(f_{\alpha_0}x_{n+1}, f_{\alpha_0}p\right)\right) \leq \eta\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0}x_n, f_{\alpha_0}x_{n+1}\right) + d\left(z, f_{\alpha_0}p\right)\right]\right)$$
$$-\phi\left(\tfrac{1}{2}\left[d\left(f_{\alpha_0}x_n, f_{\alpha_0}x_{n+1}\right) + d\left(z, f_{\alpha_0}p\right)\right]\right)$$

Letting $n \to \infty$ in the above inequality, using (6) and the properties of ψ, η and ϕ, we have

$$\psi\left(\frac{1}{2}d\left(z, f_{\alpha_0}p\right)\right) \leq \eta\left(\frac{1}{2}\left[d\left(z, f_{\alpha_0}p\right)\right]\right) - \phi\left(\frac{1}{2}\left[d\left(z, f_{\alpha_0}p\right)\right]\right)$$

that is,

$$\psi\left(\frac{1}{2}d\left(z, f_{\alpha_0}p\right)\right) \leq \eta\left(\frac{1}{2}\left[d\left(z, f_{\alpha_0}p\right)\right]\right) - \phi\left(\frac{1}{2}\left[d\left(z, f_{\alpha_0}p\right)\right]\right)$$

$$\Rightarrow d(z, f_{\alpha_0}p) = \theta \qquad \text{(by (ii) and (iii))}$$

$$\Rightarrow z = f_{\alpha_0}p$$

Therefore, we have

$$z = gp = f_{\alpha_0}p \tag{7}$$

Hence p is a coincidence point and z is a point of coincidence of g and f_{α_0}.

For uniqueness, suppose that there exists another point $q \in X$ such that

$z_1 = gq = f_{\alpha_0}q$ and $z \neq z_1$. For $\alpha = \alpha_0$, $x = p$ and $y = q$, from (1), we have

$$\psi\left(d(f_{\alpha_0}p, f_{\alpha_0}q)\right) \leq \eta\left(\frac{1}{2}\left[\, d(gp, f_{\alpha_0}p) + d(gp, f_{\alpha_0}q)\right]\right)$$

$$-\phi\left(\tfrac{1}{2}\left[d(gp, f_{\alpha_0}p) + d(gp, f_{\alpha_0}q)\right]\right)$$

$$\Rightarrow \psi\left(d(z, f_{\alpha_0}p)\right) \leq \theta \Rightarrow d(z, z_1) = \theta$$

$$\Rightarrow z = z_1$$

Therefore, z is the unique point of coincidence of g and f_{α_0} . By what we have already proved z is the unique point of coincidence of g and $\{f_\alpha : \alpha \in \Lambda\}$.

Now, we establish that any common fixed point of g and f_{α_0} is a common fixed point of g and $\{f_\alpha : \alpha \in \Lambda\}$ and conversely. Suppose that $p \in X$ be a common fixed point of g and f_{α_0}. Then $p = gp = f_{\alpha_0}p$. From (1) and using the monotone property of ψ, we have

$$\psi\left(\tfrac{1}{2}d(p, f_\alpha p)\right) \leq \psi\left(d(p, f_\alpha p)\right) = \psi\left(d(f_{\alpha_0}p, f_\alpha p)\right)$$

$$\leq \eta\left(\frac{1}{2}\left[\, d(gp, f_{\alpha_0}p) + d(gp, f_\alpha p)\right]\right) - \phi\left(\frac{1}{2}\left[d(gp, f_{\alpha_0}p) + d(gp, f_\alpha p)\right]\right)$$

$$= \eta\left(\frac{1}{2}\left[\, d(p, f_\alpha p)\right]\right) - \phi\left(\frac{1}{2}\left[d(p, f_\alpha p)\right]\right)$$

that is,

$$\psi\left(\frac{1}{2}d(p, f_\alpha p)\right) - \eta\left(\frac{1}{2}\left[\, d(p, f_\alpha p)\right]\right) + \phi\left(\frac{1}{2}\left[d(p, f_\alpha p)\right]\right) \leq \theta$$

which by (ii) and (iii) implies that $d(p, f_\alpha p) = \theta$, that is, $p = f_\alpha p$, for all $\alpha \in \Lambda$. Hence we have $p = gp = f_\alpha p$, for all $\alpha \in \Lambda$, that is, p is a common fixed point of g and $\{f_\alpha : \alpha \in \Lambda\}$. The converse part is trivial.

We have proved that z is the unique point of coincidence of g and f_{α_0}. Now, if g and f_{α_0} are weakly compatible, then by Lemma 2.4, z is the unique common fixed point of g and f_{α_0}. By what we have proved z is the unique common fixed point of g and $\{f_\alpha : \alpha \in \Lambda\}$.

Theorem 3.2. Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in \text{int}P$, for $x, y \in X$ with $x \neq y$. Let $g : X \to X$ be a mapping such that $g(X)$ is a complete subspace of X. Let $\{f_\alpha : X \to X : \alpha \in \Lambda\}$ be a family of mappings. Suppose that there exists $\alpha_0 \in \Lambda$ such that $f_{\alpha_0}(X) \subseteq g(X)$ and for $x, y \in X$,

$$\psi\left(d\left(f_{\alpha_0}x, f_\alpha y\right)\right) \leq \eta\left(\frac{1}{2}\left[\, d(gx, f_\alpha y) + d\left(gy, f_{\alpha_0}x\right)\right]\right) - \phi\left(\tfrac{1}{2}\left[d(gx, f_\alpha y) + d\left(gy, f_{\alpha_0}x\right)\right]\right) \qquad (8)$$

where $\alpha \in \Lambda$ and the conditions upon (ψ, η, ϕ) are the same as in Theorem 3.1. Then g and $\{f_\alpha : \alpha \in \Lambda\}$ have a unique common fixed point in X.

Proof: Arguing like in the proof of Theorem 3.1, we establish that any point of coincidence of g and f_{α_0} is a point of coincidence of g and $\{f_\alpha : \alpha \in \Lambda\}$ and conversely. We take the same sequence $\{x_n\}$ as in the proof of Theorem 3.1. Arguing like in the proof of Theorem 3.1, by the condition (8) we prove that $\{d\left(f_{\alpha_0}x_n, f_{\alpha_0}x_{n+1}\right)\}$ is monotone decreasing and

$$\lim_{n\to\infty} d\left(f_{\alpha_0}x_n, f_{\alpha_0}x_{n+1}\right) = \theta \qquad (9)$$

Next we show that $\{f_{\alpha_0}x_n\}$is a Cauchy sequence. If $\left\{f_{\alpha_0}x_n\right\}$ is not a Cauchy sequence, then using an argument similar to that given in Theorem 3.1, we can find two sequences of positive integers $\{n(k)\}$ and $\{m(k)\}$ for which

$$\lim_{k\to\infty} d\left(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\right) = \phi(c) \qquad (10)$$

$$\lim_{k\to\infty} d\left(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}\right) = \phi(c) \qquad (11)$$

Again,

$$d\left(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\right) \leq d\left(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)+1}\right) + d\left(f_{\alpha_0}x_{m(k)+1}, f_{\alpha_0}x_{m(k)}\right)$$

and

$$d\left(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)+1}\right) \leq d\left(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\right) + d\left(f_{\alpha_0}x_{m(k)}, f_{\alpha_0}x_{m(k)+1}\right)$$

Further,

$$d\left(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)}\right) \leq d\left(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{n(k)+1}\right) + d\left(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)}\right)$$

and

$$d(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)}) \leq d(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}) + d(f_{\alpha_0}x_{m(k)+1}, f_{\alpha_0}x_{m(k)})$$

Letting $k \to \infty$ in above four inequalities, using (9), (10) and (11), we have

$$\lim_{k\to\infty} d(f_{\alpha_0}x_{n(k)}, f_{\alpha_0}x_{m(k)+1}) = \phi(c) \tag{12}$$

and

$$\lim_{k\to\infty} d(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)}) = \phi(c) \tag{13}$$

For $\alpha = \alpha_0, x = x_{n(k)+1}$ and $y = x_{m(k)+1}$, from (8), we have

$$\psi\left(d(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1})\right) \leq \eta\left(\frac{1}{2}\left[d(gx_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}) + d(gx_{m(k)+1}, f_{\alpha_0}x_{n(k)+1})\right]\right)$$
$$-\phi\left(\tfrac{1}{2}\left[d(gx_{n(k)+1}, f_{\alpha_0}x_{m(k)+1}) + d(gx_{m(k)+1}, f_{\alpha_0}x_{n(k)+1})\right]\right),$$

that is,

$$\psi\left(d(f_{\alpha_0}x_{n(k)+1}, f_{\alpha_0}x_{m(k)+1})\right) \leq \eta\left(\frac{1}{2}\left[d(gx_{n(k)}, f_{\alpha_0}x_{m(k)+1}) + d(gx_{m(k)}, f_{\alpha_0}x_{n(k)+1})\right]\right)$$
$$-\phi\left(\tfrac{1}{2}\left[d(gx_{n(k)}, f_{\alpha_0}x_{m(k)+1}) + d(gx_{m(k)}, f_{\alpha_0}x_{n(k)+1})\right]\right).$$

Letting $k \to \infty$ in the above inequality, using (11), (12) and (13) and the properties of ψ, η and ϕ we have

$$\psi(\phi(c)) \leq \eta(\phi(c)) - \phi(\phi(c)),$$

which is a contradiction by virtue of a property of ϕ. Therefore, $\{f_{\alpha_0}x_n\}$is a Cauchy sequence in $g(X)$. From the completeness of $g(X)$ there exists $z \in g(X)$ such that

$$f_{\alpha_0}x_n \to z \text{ as } n \to \infty. \tag{14}$$

Since $z \in g(X)$, we can find $p \in X$ such that $z = gp$.

For $\alpha = \alpha_0$, $x = x_{n+1}$and $y = q$ and using the monotone property of ψ, from (8), we have

$$\psi\left(\frac{1}{2}d(f_{\alpha_0}x_{n+1}, f_{\alpha_0}p)\right) \leq \psi\left(d(f_{\alpha_0}x_{n+1}, f_{\alpha_0}p)\right) \leq \eta\left(\frac{1}{2}\left[d(gx_{n+1}, f_{\alpha_0}p) + d(gp, f_{\alpha_0}x_{n+1})\right]\right)$$
$$-\phi\left(\tfrac{1}{2}\left[d(gx_{n+1}, f_{\alpha_0}p) + d(gp, f_{\alpha_0}x_{n+1})\right]\right)$$

that is,

$$\psi\left(\frac{1}{2}d(f_{\alpha_0}x_{n+1}, f_{\alpha_0}p)\right) \leq \eta\left(\frac{1}{2}\left[d(f_{\alpha_0}x_n, f_{\alpha_0}p) + d(z, f_{\alpha_0}x_{n+1})\right]\right)$$
$$-\phi\left(\tfrac{1}{2}\left[d(f_{\alpha_0}x_n, f_{\alpha_0}p) + d(z, f_{\alpha_0}x_{n+1})\right]\right).$$

Letting $n \to \infty$ in the above inequality, using (14) and the properties of ψ, η and ϕ we have

$$\psi\left(\frac{1}{2}d\left(z,f_{\alpha_0}p\right)\right) \le \eta\left(\frac{1}{2}\left[\,d\left(z,f_{\alpha_0}p\right)\right]\right) - \phi\left(\frac{1}{2}\left[d\left(z,f_{\alpha_0}p\right)\right]\right)$$

$$\Rightarrow \psi\left(\frac{1}{2}d\left(z,f_{\alpha_0}p\right)\right) - \eta\left(\frac{1}{2}\left[\,d\left(z,f_{\alpha_0}p\right)\right]\right) + \phi\left(\frac{1}{2}\left[d\left(z,f_{\alpha_0}p\right)\right]\right) \le \theta$$

$$\Rightarrow d\left(z,f_{\alpha_0}p\right) = \theta \Rightarrow f_{\alpha_0}p = z(\text{ by (ii) and (iii) }).$$

Therefore, we have

$$z = gp \; = f_{\alpha_0}p. \tag{15}$$

Hence p is a coincidence point and z is a point of coincidence of g and f_{α_0}. Like in the proof of Theorem 3.1, by the condition (8), we prove that z is the unique point of coincidence of g and f_{α_0}. Then, by what we have already proved z is the unique point of coincidence of g and $\{f_\alpha : \alpha \in \Lambda\}$.

Arguing in the same manner as in the proof of Theorem 3.1, we establish that any common fixed point of g and f_{α_0} is a common fixed point of g and $\{f_\alpha : \alpha \in \Lambda\}$ and conversely. Now, if g and f_{α_0} are weakly compatible, then by Lemma 2.4, z is the unique common fixed point of g and f_{α_0} and hence z is the unique common fixed point of g and $\{f_\alpha : \alpha \in \Lambda\}$.

Considering $\{f_\alpha : \alpha \in \Lambda\} = f$ in Theorem 3.1, we have the following corollary.

Corollary 3.1 Let (X,d) be a cone metric space with regular cone P such that $d(x,y) \in \mathrm{int}P$, for $x,y \in X$ with $x \neq y$. Let $g : X \to X$ be a mapping such that $g(X)$ is a complete subspace of X. Let $f : X \to$ X be a mapping such that $f(X) \subseteq g(X)$ and for $x,y \in X$,

$$\psi\left(\frac{1}{2}d(fx,fy)\right) \le \eta\left(\frac{1}{2}\left[\,d(gx,fx) + d(gy,fy)\right]\right)$$
$$-\phi\left(\tfrac{1}{2}\left[d(gx,fx) + d(gy,fy)\right]\right) \tag{16}$$

where the conditions upon $(\psi,\eta,\phi\,)$ are the same as in Theorem 3.1. Then g and f have a unique point of coincidence in X. Moreover, if g and f are weakly compatible, then g and f have a unique common fixed point in X.

Considering $\{f_\alpha : \alpha \in \Lambda\} = f$ in Theorem 3.2, we have the following corollary.

Corollary 3.2 Let (X,d) be a cone metric space with regular cone P such that $d(x,y) \in \mathrm{int}P$, for $x,y \in X$ with $x \; \neq y$. Let $g : X \to$ X be a mapping such that $g(X)$ is a complete subspace of X. Let $f : X \to X$ be a mapping such that $f(X) \; \subseteq g(X)$ and for $x,y \in X$,

$$\psi\big(d(fx,fy)\big) \leq \eta\left(\frac{1}{2}[\,d(gx,fy)+d(gy,fx)]\right)$$
$$-\phi\left(\tfrac{1}{2}[d(gx,fy)+d(gy,fx)]\right) \qquad (17)$$

where the conditions upon $(\psi,\eta,\phi\,)$ are the same as in Theorem 3.1. Then g and f have a unique point of coincidence in X. Moreover, if g and f are weakly compatible, then g and f have a unique common fixed point in X.

Considering $\{f_\alpha : \alpha \in \Lambda\} = f$ and η to be the function ψ in Theorem 3.1, we have the following corollary.

Corollary 3.3 Let (X,d) be a cone metric space with regular cone P such that $d(x,y) \in \mathrm{int}P$, for $x,y \in X$ with $x \neq y$. Let $g : X \to X$ be a mapping such that $g(X)$ is a complete subspace of X. Let $f : X \to$ X be a mapping such that $f(X) \subseteq g(X)$ and for $x,y \in X$,

$$\psi\big(d(fx,fy)\big) \leq \psi\left(\frac{1}{2}[\,d(gx,fx)+d(gy,fy)]\right)$$
$$-\phi\left(\tfrac{1}{2}[d(gx,fx)+d(gy,fy)]\right) \qquad (18)$$

where $\psi,\phi : \mathrm{int}P \cup \{\theta\} \to \mathrm{int}P \cup \{\theta\}$ are such that ψ is continuous, ϕ lower semi-continuous and also

(i) ψ is strongly monotonic increasing,

(ii) $\psi(t) = \phi(t) = \theta$ if and only if $t = \theta$,

(iii) $\phi(t) \ll t$, for $t \in \mathrm{int}P$ and

(iv) either $\phi(t) \leq d(x,y)$ or $d(x,y) \ll \phi(t)$, for $t \in \mathrm{int}P \cup \{\theta\}$ and $x,y \in X$.

Then g and f have a unique point of coincidence in X. Moreover, if g and f are weakly compatible, then g and f have a unique common fixed point in X.

Considering $\{f_\alpha : \alpha \in \Lambda\} = f$ and η to be the function ψ in Theorem 3.2, we have the following corollary.

Corollary 3.4 Let$(X\ d)$ be a cone metric space with regular cone P such that $d(x\ y) \in \mathrm{int}P$, for $x,y \in X$ with $x \neq y$. Let $g: X \to X$ be a mapping such that $g(X)$ *is* a complete subspace of X. Let $f : X \to X$ be a mapping such that $f(X) \subseteq g(X)$ and for $x,y \in X$,

$$\psi\big(d(fx,fy)\big) \leq \psi\left(\tfrac{1}{2}[\,d(gx,fy)+d(gy,fx)]\right) - \phi\left(\tfrac{1}{2}[d(gx,fy)+d(gy,fx)]\right) \qquad (19)$$

where the conditions upon (ψ,ϕ) are the same as in corollary 3.5. Then g and f have a unique point of coincidence in X. Moreover, if g and f are weakly compatible, then g and f have a unique common fixed point in X.

Example 3.1 Let $X = [0, 220), E = \mathbb{R}^2, \theta = (0, 0)$ with usual norm, be a real Banach space. We define $P = \{(x, y) \in E : x, y \geq 0\}$. The partial ordering $\leq$ with respect to the cone P be the partial ordering in E. It is obvious that P is a regular cone.

Let $d : X \times X \to E$ be given as $d(x, y) = (|x - y|, |x - y|)$, for $x, y \in X$.

Then (X, d) is a cone metric space with the required properties of Theorems 3.1 and 3.2.

$$gx = \begin{cases} \frac{x}{2}, & \text{if } 0 \leq x \leq 1 \\ & 200, \text{if } x > 1 \end{cases}$$

Then g has the properties mentioned in Theorems 3.1 and 3.2.

Let $\Lambda = \{1,2,3, \ldots\}$. Let the family of mappings $\{f_\alpha : X \to X: \alpha \in \Lambda\}$ be defined as follows:

$f_1 x = 0$, for $x \in X$, and for $\alpha \geq 2$

$$f_\alpha x = \begin{cases} 0, & \text{if } 0 \leq x \leq 1 \\ \frac{2\alpha}{\alpha + 1}, & \text{if } x > 1 \end{cases}$$

Then $f_1(X) \subseteq g(X)$ and the pair (g, f_1) is weakly compatible.

Let $\psi, \eta, \phi : \text{int}P \cup \{\theta\} \to \text{int}P \cup \{\theta\}$ be defined respectively as follows:

for $t = (x, y) \in \text{int}P \cup \{\theta\}$,

$$\psi(x) = \begin{cases} 0, & \text{if x } = 0 \text{ and y} = 0 \\ (\text{x}, \text{y}), & \text{if } 0 < x \leq 1 \text{ and } 0 < \text{y} \leq 1 \\ (\text{x}^2, \text{y}), & \text{if x } > 1 \text{ and } 0 < y \leq 1 \\ (\text{x}, \text{y}^2), & \text{if } 0 < \text{x} \leq 1 \text{ and y } > 1 \\ (\text{x}^2, \text{y}^2), & \text{if x } > 1 \text{ and } y > 1 \end{cases}$$

and for $t = (x, y) \in \text{int}P \cup \{\theta\}$ with $v = \min\{x, y\}$,

$\eta(t) = (v^2, v^2)$ and $\phi(t) = \left(\frac{v^2}{10}, \frac{v^2}{10}\right)$.

Then ψ, η and ϕ have the properties mentioned in Theorems 3.1 and 3.2, conditions (1) and (8) are satisfied for all $x, y \in X$. Hence the conditions of Theorems 3.1 and 3.2 are satisfied. Here it is found that 0 is the unique point of coincidence and also the unique common fixed point of g and $\{f_\alpha : \alpha \in \Lambda\}$.

Remark 3.1 In the above example the family of mapping $\{f_\alpha : \alpha \in \Lambda\}$ contains infinitely many mappings. So it is applicable to corollaries 3.1, 3.2, 3.3 and 3.4. Hence Theorem 3.1 properly contains corollaries 3.1 and 3.3 and Theorem 3.2 properly contains corollaries 3.2 and 3.4.

Remark 3.2 In some recent works [10, 11, 14, 18] it has been pointed out that a cone metric generates a metric in a natural way and several fixed point problems on the cone metric space are reducible to the corresponding problems in the associated metric space. This is particularly true with the contraction mapping principle. But this cannot be claimed in general. Particularly, weak contraction is not transferable to a corresponding weak contraction in the generated metric space and, therefore, cannot be claimed to have been derived from the results of weak contractions in metric spaces. This is the reason why the fixed point problems of weak contractions and their generalizations conceived an cone metric space are relevant. Also, in our case a weak contraction in a cone metric space is not a weak contraction in the corresponding metric space. In fact there is even no assurance that a cone metric space inequality will generate an inequality condition in metric spaces, although, as pointed out in [10, 11, 14, 18], it does in several important cases. Our problem in this paper is outside this category.

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