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PRESIC TYPE HYBRID CONTRACTION AND FIXED POINTS IN CONE METRIC SPACES

RENY GEORGE[1,2]

[1]Department of Mathematics, College of Science, Salmanbin Abdulaziz University, Al-Kharj, Kingdom of Saudi Arabia

[2]Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Chhattisgarh, India

Abstract. A generalised common fixed point theorem of Presic type for a pair of hybrid mappings $f : X \to X$ and $T : X^k \to CB(X)$ in a cone metric space is proved. Our result generalises many well known results.

Keywords: Coincidence and common fixed points; cone metric space; set valued contraction; weakly compatible

2000 AMS Subject Classification: 47H10

1. Introduction

Nadler[18] introuced set valued contractive mappings in metric spaces and proved existence of fixed points for such mappings. Later many authors extended and generalised the work of Nadler in different directions. Huang and Zang [3] generalising the notion of metric space by replacing the set of real numbers by ordered normed spaces, defined a cone metric space and proved some fixed point theorems of contractive mappings defined on these spaces. Rezapour and Hamlbarani [4], omitting the assumption of normality,

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obtained generalisations of results of [3]. In [5], Di Bari and Vetro obtained results on points of coincidence and common fixed points in non-normal cone metric spaces. Further results on fixed point theorems in such spaces were obtained by several authors, see [5-15]. Recently Wardowski[17] introduced set valued contraction of Nadler type in cone metric space and proved a fixed point theorem for this type of mappings.

Considering the convergence of certain sequences, Presic [1] proved the following :

Theorem 1.1. *Let (X,d) be a metric space, k a positive integer, $T : X^k \longrightarrow X$ be a mapping satisfying the following condition :*

$$d(T(x_1,x_2,\ldots,x_k),T(x_2,x_3,\ldots,x_{k+1})) \leq \begin{cases} q_1.d(x_1,x_2)+q_2.d(x_2,x_3) \\ +\cdots+q_k.d(x_k,x_{k+1}) \end{cases} \tag{1.1}$$

where $x_1,x_2,\ldots,x_{k+1}$ are arbitrary elements in X and $q_1,q_2,\ldots,q_k$ are non-negative constants such that $q_1+q_2+\cdots+q_k < 1$. Then, there exists some $x \in X$ such that $x = T(x,x,\ldots,x)$. Moreover if $x_1,x_2,\ldots,x_k$ are arbitrary points in X and for $n \in N$ $x_{n+k} = T(x_n,x_{n+1},\ldots,x_{n+k-1})$, then the sequence $< x_n >$ is convergent and $lim\ x_n = T(lim\ x_n, lim\ x_n,\ldots, lim\ x_n)$.

Note that for $k = 1$ the above theorem reduces to the well-known Banach Contraction Principle. Ciric and Presic [2] generalising the above theorem proved the following:

Theorem 1.2. *Let (X,d) be a metric space, k a positive integer, $T : X^k \longrightarrow X$ be a mapping satisfying the following condition :*

(1.2)

$$d(T(x_1,x_2,\ldots,x_k),T(x_2,x_3,\ldots,x_{k+1})) \leq \lambda.max\{d(x_1,x_2),d(x_2,x_3),\ldots d(x_k,x_{k+1})$$

where $x_1,x_2,\ldots,x_{k+1}$ are arbitrary elements in X and $\lambda \in (0,1)$. Then, there exists some $x \in X$ such that $x = T(x,x,\ldots,x)$. Moreover if $x_1,x_2,\ldots,x_k$ are arbitrary points in X and for $n \in N x_{n+k} = T(x_n,x_{n+1},\ldots,x_{n+k-1})$, then the sequence $< x_n >$ is convergent and $lim\ x_n = T(lim\ x_n, lim\ x_n,\ldots, lim\ x_n)$. If in addition T satisfies

$D(T(u, u, \ldots u), T(v, v, \ldots v)) < d(u, v)$, *for all* $u, v \in X$ *then* x *is the unique point satisfying* $x = T(x, x, \ldots, x)$.

In [16], R. George et al. generlising Theorems (1.1) and (1.2) above proved the existence of common fixed points of two mappings satisfying Presic type contractions in a cone metric space and applied this result in proving the existence of stationary distribution in Markov Process. The purpose of this work is to introduce set valued hybrid contraction of Presic type and prove fixed point theorem for this type of mappings in cone metric space without using normality condition for the cone. Our results provide a proper extension and generalisation of Theorems 3.1 of [17] which in turn will extend and generalise the results of [3, 4].

2. Preliminaries

Let E be a real Banach space and P a subset of E. Then, P is called a *cone* if
(i) P is closed, non-empty, and satisfies $P \neq \{\theta\}$, θ is the zero vector of E.
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$, i.e. $P \cap (-P) = \theta$

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$, and $x \ll y$ if $y - x \in intP$, where $intP$ denote the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\parallel x \parallel \leq K \parallel y \parallel$.

Definition 2.1. [3] *Let* X *be a non empty set. Suppose that the mapping* $d : X \times X \to E$ *satisfies:*

$(d_1)\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$

$(d_2)d(x, y) = d(y, x)$ *for allx,* $y \in X$

$(d_3)d(x, y) \preceq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$

Then, d is called a *cone metric* on X and (X,d) is called a *cone metric space*.

Definition 2.2. [3] *Let (X, d) be a cone metric space . The sequence $\{x_n\}$ in X is said to be:*

(a) A *convergent sequence* if for every $c \in E$ with $\theta \ll c$, there is $n_0 \in N$ such that for all $n \geq n_0, d(x_n, x) \ll c$ for some $x \in X$. We denote this by $lim_{n\to\infty} x_n = x$.

(b) A *Cauchy sequence* if for all $c \in E$ with $\theta \ll c$, there is $n_0 \in N$ such that $d(x_m, x_n) \ll c$, for all $m, n \geq n_0$.

(c) A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent in X.

(d) A self-map T on X is said to be *continuous* if $lim_{n\to\infty} x_n = x$ implies that $lim_{n\to\infty} T(x_n) = T(x)$, for every sequence $\{x_n\}$in X.

A set $A \subset X$ is said to be closed if for any sequence $\{x_n\} \subset A$ convergent to x we have $x \in A$. We denote by $C(X)$ the collection of all non empty closed subsets of X.

In this paper let E be a real Banach space, P be a cone in E with non empty interior and $\preceq$ be a partial ordering with respect to P.

Definition 2.3. [17] *Let (X, d) be a cone metric space and $\mathcal{A}$ be the collection of all non empty subsets of X. Map $H : \mathcal{A} \times \mathcal{A} \to E$ is called a H-cone metric with respect to d if for any $A_1, A_2 \in \mathcal{A}$ the following conditions hold :*

(H1) $H(A_1, A_2) = \theta \Rightarrow A_1 = A_2$;
(H2) $H(A_1, A_2) = H(A_2, A_1)$;
(H3) $\forall \epsilon \in E\ \theta \ll \epsilon$, $\forall x \in A_1\ \exists y \in A_2$ *such that* $d(x, y) \preceq H(A_1, A_2) + \epsilon$;
(H4) One of the following holds :

(i) $\forall \epsilon \in E\ \theta \ll \epsilon$, $\forall y \in A_2\ \exists x \in A_1$ *such that* $H(A_1, A_2) \preceq d(x, y) + \epsilon$
(ii) $\forall \epsilon \in E\ \theta \ll \epsilon$, $\forall y \in A_1\ \exists x A_2 \in$ *such that* $H(A_1, A_2) \preceq d(x, y) + \epsilon$

For examples of H-cone metric see [17].

Lemma 2.4. [17] *Let $(X,\, d)$ be a cone metric space and $\mathcal{A}$ be the collection of all non empty subsets of X. If $H : \mathcal{A} \times \mathcal{A} \to E$ is a H-cone metric with respect to d then the pair $(\mathcal{A}, H)$ is a cone metric space.*

Definition 2.5. *Let (X, d) be a metric space, k a positive integer, $T : X^k \to C(X)$ and $f : X \to X$ be mappings.*

(a) An element $x \in X$ is said to be a *coincidence point* of f and T if and only if $f(x) \in T(x, x, \ldots, x)$. If $x = f(x) \in T(x, x, \ldots, x)$, then we say that x is a *common fixed point* of f and T. If $w = f(x) \in T(x, x, \ldots, x)$, then w is called a point of coincidence of f and T.

(b) Mappings f and T are said to be *commuting* if and only if $f(T(x, x, \ldots x)) \subseteq T(fx, fx, \ldots fx)$ for all $x \in X$.

(c) Mappings f and T are said to be *weakly compatible* if and only if they commute at their coincidence points.

Remark 2.6. *For $k = 1$, the above definitions reduce to the usual definition of commuting and weakly compatible hybrid mappings in a metric space.*

The set of coincidence points and common fixed points of f and T is denoted by $C(f, T)$ and $Fix(f, T)$ respectively.

3. Main results

Consider a function $\phi : E^k \to$ E such that

(a) ϕ is an increasing function, i.e $x_1 \preceq y_1, x_2 \preceq y_2, \ldots, x_k \preceq y_k$ implies $\phi(x_1, x_2, \ldots, x_k) \preceq \phi(y_1, y_2, \ldots, y_k)$.

(b) $\phi(t, t, t, \ldots) \preceq t$, for all $t \in E$

Now, we present our main results as follows :

Theorem 3.1. *Let (X, d) be a cone metric space with solid cone P contained in a real Banach space E, $\mathcal{A}$ be the collection of all non empty subsets of X and $H : \mathcal{A} \times \mathcal{A} \to E$ be a H-cone metric with respect to d . For any positive integer k, let $T : X^k \to C(X)$ and*

$f : X \to X$ be mappings satisfying the following conditions:

$$T(X^k) \subseteq f(X) \tag{3.1}$$

$$\begin{cases} H(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \\ \preceq \lambda\phi(d(fx_1, fx_2), d(fx_2, fx_3), \ldots, (fx_k, fx_{k+1})) \\ \textit{where } x_1, x_2, \ldots, x_{k+1} \textit{are arbitrary elements in } X, \lambda \in (0,1) \end{cases} \tag{3.2}$$

$$f(X) \textit{ is complete} \tag{3.3}$$

$$\begin{cases} \textit{there exist elements } x_1, x_2, \ldots, x_k, x_{k+1} \textit{ in } X \textit{ and } R \textit{ in } E \textit{ such that} \\ fx_{k+1} = T(x_1, x_2, \ldots, x_k), \theta \ll R \textit{ and } R \textit{ is the upper bound} \\ \textit{of the set} \left\{ \frac{d(fx_1,fx_2)}{\alpha}, \frac{d(fx_2,fx_3)}{\alpha^2}, \ldots, \frac{d(fx_k,fx_{k+1})}{\alpha^k} \right\}, \alpha = \lambda^{\frac{1}{k}} \end{cases} \tag{3.4}$$

Then, f and T have a coincidence point, i.e. $C(f,T) \neq \emptyset$.

Proof: Let $\{\epsilon_n\} \subset E$ be a sequence satisfying

$$\theta \ll \epsilon_n \textit{ and } \epsilon_i \preceq R\alpha^{k+i} \; \forall \; i \in \mathcal{N} \tag{3.5}$$

By (3.1), (3.4) and (H3) there exist $y_{k+2} = fx_{k+2} \in T(x_2, \ldots, x_{k+1})$ such that
$d(y_{k+1}, y_{k+2}) = d(fx_{k+1}, fx_{k+2})$
$\preceq H(T(x_1, x_2, \ldots, x_k), T(x_2, \ldots, x_{k+1}) + \epsilon_1$
$\preceq \lambda\phi(d(fx_1, fx_2), d(fx_2, fx_3), \ldots, (fx_k, fx_{k+1})) + \epsilon_1$
$\preceq \lambda\phi(R\alpha, R\alpha^2, \ldots, R.\alpha^k) + \epsilon_1$
$\preceq \lambda R\alpha + \epsilon_1 \preceq R\alpha^{k+1} + \epsilon_1 \preceq 2R\alpha^{k+1}$.
Similarly there exist $y_{k+3} = fx_{k+3} \in T(x_3, \ldots, x_{k+2})$ such that
$d(y_{k+2}, y_{k+3}) = d(fx_{k+2}, fx_{k+3})$
$\preceq H(T(x_2, \ldots, x_{k+1}), T(x_3, \ldots, x_{k+2}) + \epsilon_2$
$\preceq \lambda\phi(d(d(fx_2, fx_3), \ldots, (fx_{k+1}, fx_{k+2})) + \epsilon_2$
$\preceq \lambda\phi(R\alpha^2, \ldots, 2R\alpha^{k+1}) + \epsilon_2$
$\preceq \lambda 2R\alpha^{k+1} + \epsilon_2$

$\preceq 3R\alpha^{k+2}$.

Also from (3.4) we see that $d(y_1, y_2) \preceq R\alpha$, $d(y_2, y_3) \preceq R\alpha^2$ .... $d(y_k, y_{k+1}) \preceq R\alpha^k$. Thus we can define sequence $< y_n >$ in $f(X)$ as $y_n = fx_n$ for $n = 1, 2, \ldots, k$ and $y_{k+n} = f(x_{k+n}) \in T(x_n, x_{n+1}, \ldots, x_{n+k-1}), n = 1, 2, \ldots$ such that

$$d(y_n, y_{n+1}) \preceq (n+1)R\alpha^n \; \forall \; n \tag{3.6}$$

Now for $p, n \in N$, we have

$d(y_n, y_{n+p}) \preceq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p})$,

$\preceq (n+1)R\alpha^n + (n+2)R\alpha^{n+1} + \cdots + (n+p)R\alpha^{n+p-1}$

$= nR\alpha^n \sum_{i=0}^{p-1} \alpha^i + R\alpha^n \sum_{i=1}^{p} i\alpha^{i-1}$

Let $\theta \ll c$ be arbitrary . Choose $\delta > 0$ such that $c - N_\delta(0) \subseteq P$ where $N_\delta(0) = \{y \in E; \| y \| < \delta\}$. Also choose a natural number N_1 such that $nR\alpha^n \sum_{i=0}^{p-1} \alpha^i + R\alpha^n \sum_{i=1}^{p} i\alpha^{i-1} \in N_\delta(0)$, for all $n \geq N_1$. Then, $nR\alpha^n \sum_{i=0}^{p-1} \alpha^i + R\alpha^n \sum_{i=1}^{p} i\alpha^{i-1} \ll c$ for all $n \geq N_1$. Thus, $d(y_n, y_{n+p}) \preceq\ll c$ for all $n \geq N_1$. Hence, sequence $< y_n >$ is a Cauchy sequence in $f(X)$, and since $f(X)$ is complete, there exists $v, u \in X$ such that $lim_{n\to\infty} y_n = v = f(u)$. Choose a natural number N_2 such that $d(y_n, y_{n+1}) \ll \frac{c}{\lambda k}$and $d(fu, y_{n+1}) \ll \frac{c}{\lambda k}$ for all $n \geq N_2$. Then for all $n \geq N_2$

$H(T(x_n, x_{n+1}, \ldots x_{n+k-1}), T(u, u, \ldots u))$

$\preceq H(T(u, u, \ldots u), T(u, u, \ldots x_n)) + H(T(u, u, \ldots x_n), T(u, u, \ldots x_n, x_{n+1}))$

$+ \cdots H(T(u, x_n, \ldots x_{n+k-2}), T(x_n, x_{n+1} \ldots x_{n+k-1})$

$\preceq \lambda\phi\{d(fu, fu), d(fu, fu), \ldots, d(fu, fx_n)\}$

$+\lambda\phi\{d(fu, fu), d(fu, fu), \ldots, d(fu, fx_n), d(fx_n, fx_{n+1})\} + \cdots$

$+\lambda\phi\{d(fu, fx_n), d(fx_n, fx_{n+1}), \ldots d(fx_{n+k-2}, fx_{n+k-1})\}$.

$= \lambda\phi(\theta, \theta, \ldots, d(fu, fx_n))$

$+\lambda\phi(\theta, \theta, \ldots, d(fu, fx_n), d(fx_n, fx_{n+1})) + \cdots$

$+\lambda\phi(d(fu, fx_n), d(fx_n, fx_{n+1}), \ldots d(fx_{n+k-2}, fx_{n+k-1}))$.

$\ll \lambda\phi(\frac{c}{\lambda k}, \frac{c}{\lambda k}, \ldots, \frac{c}{\lambda k}) + \lambda\phi(\frac{c}{\lambda k}, \frac{c}{\lambda k}, \ldots, \frac{c}{\lambda k})$

$+\cdots+\lambda\phi(\frac{c}{\lambda k},\frac{c}{\lambda k},\ldots,\frac{c}{\lambda k})$

$\ll\lambda\frac{c}{\lambda k}\ldots+\lambda\frac{c}{\lambda k}=c.$

Thus, $H(T(x_n,x_{n+1},\ldots x_{n+k-1}),T(u,u,\ldots u))\ll\frac{c}{m}$ for all $m\geq 1$.

So, $\frac{c}{m}-H(T(x_n,x_{n+1},\ldots x_{n+k-1}),T(u,u,\ldots u))\in P$ for all $m\geq 1$. Since $\frac{c}{m}\to\theta$ as $m\to\infty$ and P is closed, $-H(T(x_n,x_{n+1},\ldots x_{n+k-1}),T(u,u,\ldots u))\in P$, but $P\bigcap(-P)=\{\theta\}$. Therefore, $H(T(x_n,x_{n+1},\ldots x_{n+k-1}),T(u,u,\ldots u))=\theta$ for all $n\geq N_2$ and so the sequence $\{T(x_n,x_{n+1},\ldots x_{n+k-1})\}$ converges to $T(u,u,\ldots u)$ with respect to the cone metric H. Since $y_{k+n}=f(x_{k+n})\in T(x_n,x_{n+1},\ldots,x_{n+k-1}),n=1,2,\ldots$ we have $Lim_{n\to\infty}y_{k+n}\in Lim_{n\to\infty}T(x_n,x_{n+1},\ldots,x_{n+k-1})$, i.e. $Lim_{n\to\infty}y_n=v=fu\in T(u,u,...u)$. Thus $C(f,T)\neq\emptyset$ and $Lim_{n\to\infty}y_n=v$ is a point of coincidence.

Theorem 3.2. *Let (X,d) be a cone metric space with solid cone P contained in a real Banach space E. For any positive integer k, let $T:X^k\to X$ and $f:X\to X$ be mappings satisfying (3.1), (3.2) with $\lambda\in(0,\frac{1}{k})$,(3.3), (3.4) and*

$$u\in C(T,f)\Rightarrow T(u,u,...u)=\{fu\}. \tag{3.7}$$

Then T and$f$ have a unique point of coincidence. Further if f and T are weakly compatible, then f and T have a unique common fixed point. Moreover if $x_1,x_2,\ldots,x_k$ are arbitrary points in X and for $n\in N, y_{n+k}=f(x_{n+k})=T(x_n,x_{n+1},\ldots x_{n+k-1}),n=1,2,\ldots,$ then the sequence $<y_n>$ is convergent and $lim\ y_n=f(lim\ y_n)=T(lim\ y_n,lim\ y_n,\ldots,lim\ y_n)$.

Proof: By Theorem 3.1, there exists $v,u\in X$ such that $Lim_{n\to\infty}y_n=v=f(u)\in T(u,\ldots u)$. We will prove that v is the unique point of coincidence. Suppose there exists another point of coincidence $v'\in X$ such that $v'=fu'\in T(u',\ldots u')$ for some $u'\in C(T,f)$. Then by (3.6) $\{v\}=\{fu\}=T(u,\ldots u)$ and $\{v'\}=\{fu'\}=T(u',\ldots u')$. By (3.2) we have,

$d(v',v)=H(\{v'\},\{v\})=H(T(u',u',\ldots u'),T(u,u,\ldots u))$

$\leq H(T(fu',u',\ldots u'),T(u',u',\ldots u',u))+H(T(u',u',\ldots fu,u),$

$T(u',u',\ldots,u,u))+\cdots+H(T(u',u,\ldots u,u),T(u,u,\ldots u))$

$\leq\lambda\phi(d(fu',fu'),\ldots d(fu',fu'),d(fu',fu))+\lambda\phi(d(fu',fu'),\ldots d(fu',fu),$

$d(fu, fu)) + \cdots \lambda\phi(d(fu', fu), \ldots d(fu, fu), d(fu, fu))$
$= \lambda\phi(\theta, \theta, \theta, \ldots d(fu', fu)) + \lambda\phi(\theta, \theta \ldots \theta, d(fu', fu), \theta) + \cdots .\lambda\phi(d(fu', fu), \theta, \theta \ldots \theta) = k\lambda d(fu', fu) = k\lambda d(v', v)$.

Repeating this process n times we get, $d(v', v) \leq k^n\lambda^n d(v', v)$. So $k^n\lambda^n d(v', v) - d(v', v) \in P$ for all $n \geq 1$. Since $k^n\lambda^n \to 0$ as $n \to \infty$ and P is closed, $-d(v', v) \in P$, but $P \bigcap(-P) = \{\theta\}$. Therefore, $d(v', v) = \theta$ and so $v' = v$ i.e. $Lim_{n\to\infty} y_n = v$ is the unique point of coincidence . Also since f and T are weakly compatible $f(T(u, u, \ldots u) \in T(fu, fu, fu \ldots fu)$ i.e.$fv \in T(v, v, \ldots v)$. But since $Lim_{n\to\infty} y_n = v$ is the unique point of coincidence, we have,$Lim_{n\to\infty} y_n = v = fv \in T(v, v, \ldots v)$. Thus $Lim_{n\to\infty} y_n = v$ is the unique common fixed point of f and T.

Remark 3.3. *For $k = 1$ and $f = Id$(identity mapping), Theorem 3.2 becomes set valued contraction of Nadler type in cone metric space as introduced by Wardowski[16]. However we dont require normality condition for the cone.*

Example 3.4. *Let $E = R^2, P = \{(x, y) \in E\backslash x, y \geq 0\}, X = [0, 2]$ and $d : X \times X \to E$ such that $d(x, y) = (\mid x - y \mid, \mid x - y \mid)$. Then, d is a cone metric on X. Let $\mathcal{A}$ be the collection of all non empty subsets of X of the form $\mathcal{A} = \{[0, x] : x \in X\}$*
We define H-cone metric $H : \mathcal{A} \times \mathcal{A} \to E$ with respect to d as follows :

$H(A, B) = (\mid x - y \mid, \mid x - y \mid)$ *for* $A = [0, x]$ *and* $B = [0, y]$.

Let $T : X^2 \to X$ and $f : X \to X$ be defined as follows:

$T(x, y) = [0, \frac{(x^2+y^2)}{4} + \frac{1}{2}]$ *if* $(x, y) \in [0, 1) \times [0, 1)$

$T(x, y) = [0, \frac{(x+y)}{4} + \frac{1}{2}]$ *if* $(x, y) \in [1, 2] \times [1, 2]$

$T(x, y) = [0, \frac{(x^2+y)}{4} + \frac{1}{2}]$ *if* $(x, y) \in [0, 1) \times [1, 2]$

$T(x, y) = [0, \frac{(x+y^2)}{4} + \frac{1}{2}]$ *if* $(x, y) \in [1, 2] \times [0, 1)$

$f(x) = x^2$ *if* $x \in [0, 1)$

$f(x) = x$ *if* $x \in [1, 2]$

T and f satisfies condition (3.2) as follows:

Case 1. $x, y, z \in [0, 1)$

$d(T(x,y), T(y,z)) = (\mid T(x,y) - T(y,z) \mid, \mid T(x,y) - T(y,z) \mid)$
$= (\mid \frac{x^2-z^2}{4} \mid, \mid \frac{x^2-z^2}{4} \mid)$
$\leq (\mid \frac{x^2-y^2}{4} \mid + \mid \frac{y^2-z^2}{4} \mid, \mid \frac{x^2-y^2}{4} \mid + \mid \frac{y^2-z^2}{4} \mid)$
$\leq \frac{1}{2}.max\{d(fx, fy), d(fy, fz)\}$

Case 2. $x, y \in [0, 1)$ *and* $z \in [1, 2]$

$d(T(x,y), T(y,z)) = (\mid \frac{x^2+y^2}{4} - \frac{y^2+z}{4} \mid, \mid \frac{x^2+y^2}{4} - \frac{y^2+z}{4} \mid)$
$\leq (\mid \frac{x^2-y^2}{4} \mid + \mid \frac{y^2-z}{4} \mid, \mid \frac{x^2-y^2}{4} \mid + \mid \frac{y^2-z}{4} \mid)$
$\leq \frac{1}{2}.max\{(fx, fy), d(fy, fz)\}$

Case 3. $x \in [0, 1)$ *and* $y, z \in [1, 2]$

$d(T(x,y), T(y,z)) = (\mid \frac{x^2+y}{4} - \frac{y+z}{4} \mid, \mid \frac{x^2+y}{4} - \frac{y+z}{4} \mid)$
$=(\mid \frac{x^2-z}{4} \mid, \mid \frac{x^2-z}{4} \mid)$
$\leq (\mid \frac{x^2-y}{4} \mid + \mid \frac{y-z}{4} \mid, \mid \frac{x^2-y}{4} \mid + \mid \frac{y-z}{4} \mid)$
$\leq \frac{1}{./2}.max\{d(fx, fy), d(fy, fz)\}$

Case 4. $x, y, z \in [1, 2]$

$d(T(x,y), T(y,z)) = (\mid \frac{x+y}{4} - \frac{y+z}{4} \mid, \mid \frac{x+y}{4} - \frac{y+z}{4} \mid)$
$\leq (\mid \frac{x-y}{4} \mid + \mid \frac{y-z}{4} \mid, \mid \frac{x-y}{4} \mid + \mid \frac{y-z}{4} \mid)$
$\leq \frac{1}{2}.max\{(fx, fy), d(fy, fz)\}.$

Similarly in all other cases $d(T(x,y), T(y,z)) \leq \frac{1}{2}.max\{(fx, fy), d(fy, fz)\}$*. Thus, f and T satisfy condition (3.2) with* $\phi(x_1, x_2) = max\{x_1, x_2\}$*. We see that* $C(f, T) = \{0, 1\}$*, f and T commute at 0 and 1 so weakly compatible. Finally,* $Fix(f, T) = \{0, 1\}$*. However f and T do not satisfy condition (3.6) and so the common fixed point of f and T is not unique.*

Example 3.5. *Let* $E = R^2, P = \{(x, y) \in E \backslash x, y \geq 0\}, X = [0, 2]$ *and* $d : X \times X \to E$ *such that* $d(x, y) = (\mid x - y \mid, \mid x - y \mid)$. *Then,* d *is a cone metric on* X. *Let* $\mathcal{A}$ *be the collection of all non empty subsets of* X *of the form* $\mathcal{A} = \{[0, x] : x \in X\} \bigcup \{\{x\} : x \in X\}$ *We define* H*-cone metric* $H : \mathcal{A} \times \mathcal{A} \to E$ *with respect to* d *as follows :*

(3.8)

$$H(A, B) = \begin{cases} (\mid x - y \mid, \mid x - y \mid) \ for \ A = [0, x] \ and \ B = [0, y] \\ (\mid x - y \mid, \mid x - y \mid) \ for \ A = \{x\} \ and \ B = \{y\} \\ (max\{y, \mid x - y \mid\}, max\{y, \mid x - y \mid)\} \ for \ A = [0, x] \ and \ B = \{y\} \\ (max\{x, \mid x - y \mid\}, max\{x, \mid x - y \mid\}) \ for \ A = \{x\} \ and \ B = [0, y] \end{cases}$$

Let $T : X^2 \to X$ *and* $f : X \to X$ *be defined as follows:*

$$(3.9) \qquad T(x, y) = \begin{cases} [0, \frac{x^2+y^2}{8}] \ if \ (x, y) \in [0, 1) \times [0, 1) \\ [0, \frac{x+y}{8}] \ if \ (x, y) \in [1, 2] \times [1, 2] \\ [0, \frac{x^2+y}{8}] \ if \ (x, y) \in [0, 1) \times [1, 2] \\ [0, \frac{x+y^2}{8}] \ if \ (x, y) \in [1, 2] \times [0, 1) \end{cases} \qquad f(x) = \begin{cases} \frac{x^2}{2} \ if \ x \in [0, 1) \\ x \ if \ x \in [1, 2] \end{cases}$$

As in the previous example we can show that f *and* T *satisfy condition (3.1) with* $\phi(x_1, x_2) = max\{x_1, x_2\}$ *and* $\lambda = \frac{1}{4}$. *Clearly* $C(f, T) = \{0\}$ *and* $T(0, 0) = \{0\}$. *Thus all conditions of Theorem 3.2 are satisfied and* $Fix(f, T) = \{0\}$.

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