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ON ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN q-HYPERCONVEX T_0-QUASI-METRIC SPACES

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Abstract. In this note a well known result of Khamsi [Proc. Amer. Math. Soc. 132 (2004), 365-373] on approximate fixed points for asymptotically nonexpansive mappings on bounded hyperconvex spaces is generalized to the setting of q-hyperconvex T_0-quasi-metric spaces.

Keywords: Nonexpansive mappings; asymptotically nonexpansive mappings; fixed point; q-hyperconvexity

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1. Introduction

Let (X,d) be a metric space. A mapping $T : X \to X$ is called *nonexpansive* if

$$d(T(x),T(y)) \leq d(x,y)$$

for all $x,y \in X$. $T : X \to X$ is called *asymptotically nonexpansive* (see Goebel and Kirk [3]) if there exists a sequence of positive numbers $(k_n)_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} k_n = 1$, such that

$$d(T(x),T(y)) \leq k_n d(x,y)$$

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for all $x, y \in X$. It is known (see [3]) that the class of *asymptotically nonexpansive mappings* is wider than the class of *nonexpansive mappings*.

A well known result which was proved independently by Sine [9] and Soardi [10] in hyperconvex spaces (see [1], [2]) states that the fixed point property for noexpansive mappings holds in a bounded hyperconvex space. Further, it has been proved by Khamsi [5] that: if $T : H \to H$, where (H, ρ) is a bounded hyperconvex metric space and T is an asymptotically nonexpansive mapping, then T has approximate fixed points, that is, $\inf\{\rho(x, Tx) : x \in\in H\} = 0$. Recently, Künzi and Otafudu [6] have introduced and studied the concept of q-hyperconvexity in T_0-quasi-metric spaces and obtained certain fixed point theorems there in. In this note we continue our studies of this concept by generalizing the above result of Khamsi [5] and show that an asymptotically nonexpansive mapping on a bounded q-hyperconvex T_0-quasi-metric space has approximate fixed points.

2. Preliminaries

For the convenience of the reader and in order to fix our terminology we recall the following concepts.

Definition 2.1. Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then d is called a *quasi-pseudometric* on X if

(a) $d(x, x) = 0$ for all $x \in X$,

(b) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We shall say that d is a T_0-*quasi-metric* provided that d also satisfies the following condition: For each $x, y \in X$,

$d(x, y) = 0 = d(y, x)$ implies that $x = y$.

Remark 2.2. In some cases we need to replace $[0, \infty)$ by $[0, \infty]$ (where for a d attaining the value ∞ the triangle inequality is interpreted in the obvious way). In such a case we shall speak of an *extended quasi-pseudometric*. In the following we sometimes apply concepts from the theory of quasi-pseudometrics to extended quasi-pseudometrics (without changing the usual definitions of these concepts).

Remark 2.3. Let d be a quasi-pseudometric on a set X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate quasi-pseudometric of* d. As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric.* Note that for any T_0-quasi-pseudometric d, $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric).

Let (X, d) be a quasi-pseudometric space. For each $x \in X$ and $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ denotes the *open* ϵ*-ball* at x. The collection of all "open" balls yields a base for a topology $\tau(d)$. It is called the *topology induced by* d on X. Similarly we set for each $x \in X$ and $\epsilon \geq 0$, $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$. Note that this latter set is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general.

3. $q-$Hyper convexity

In this section we recall some results on q-hyperconvexity. Some recent further work about q-hyperconvexity can be found in [4], [6] and [7].

Definition 3.1. [4, Definition 2]. A quasi-pseudometric space (X, d) is called q*-hyperconvex* provided that for each family $(x_i)_{i\in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds: If $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$, then

$$\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset.$$

Remark 3.2. If d and d^{-1} are identical and $r_i = s_i$ for $i \in I$ in Definition 3.1, then $(C_d(x_i, r_i))$ and $(C_{d^{-1}}(x_i, s_i))$ coincide and then we recover the well known definition of hyperconvexity due to Aronszajn and Panitchpakdi [1].

The following examples are basic, but important.

Example 3.3. ([4, Example 1], compare [8, Example 2). Let the set $\mathbb{R}$ of the reals be equipped with the T_0-quasi-metric $u(x, y) = \max\{x - y, 0\}$ whenever $x, y \in \mathbb{R}$. Then $(\mathbb{R}, u)$ is q-hyperconvex.

Corollary 3.4. *([4, Corollary 1]). The quasi-pseudometric subspace* $[0, \infty)$ *of* $(\mathbb{R}, u)$ *is q-hyperconvex.*

Example 3.5. ([4, Example 2]). Let $\mathbb{R}$ be equipped with its standard metric $u^s(x, y) = |x - y|$ whenever $x, y \in \mathbb{R}$. Then $(\mathbb{R}, u^s)$ is not q-hyperconvex.

Proposition 3.6. *([4, Proposition 2]) (a) If* (X, d) *is a(n extended) q-hyperconvex (resp. q-hypercomplete, metrically convex) quasi-pseudometric space, then* (X, d^{-1}) *is q-hyperconvex (resp. q-hypercomplete, metrically convex).*

(b) If (X, d) *is a q-hyperconvex (resp. q-hypercomplete) quasi-pseudometric space, then the metric space* (X, d^s) *is hyperconvex (resp. hypercomplete). However, the corresponding statement for "metrically convex" does not hold.*

The following definition can be found in [6] (compare [5] and [9]).

Definition 3.7. ([6, Definition 8]). Let (X, d) be a T_0-quasi-metric space. We say that a mapping $T : (X, d) \to (X, d)$ has *approximate fixed points* if $\inf_{x \in X} d^s(x, T(x)) = 0$.

4. Main Result

We first recall the following interesting result due to Khamsi [5].

Theorem 4.1. *Let* (H, ρ) *be a bounded hyperconvex metric space and* $T : H \longrightarrow H$ *be asymptotically nonexpansive mapping. Then* T *has approximate fixed points, i.e.* $\inf\{\rho(x, T(x)) : x \in H\} = 0$.

The following result generalizes the above theorem to the setting of q-hyperconvex T_0-quasi-metric spaces.

Theorem 4.2. *Let* (X, d) *be a bounded q-hyperconvex* T_0*-quasi-metric space and* $T : X \to X$ *be asymptotically nonexpansive mapping. Then* T *has approximate fixed points, i.e.* $\inf_{x \in X} d^s(x, T(x)) = 0$.

Proof. Since $T : X \longrightarrow X$ is asymptotically nonexpansive, there exists a sequence of nonnegative real numbers $(k_n)_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} k_n = 1$, such that

$$d(T^n(x), T^n(y)) \leq k_n d(x, y)$$

for all $x, y \in X$.

We shall first show that $T : (X, d^s) \to (X, d^s)$ is asymptotically nonexpansive. Since for any $x, y \in X$, we have

$$d^{-1}(T^n(x), T^n(y)) = d(T^n(y), T^n(x)) \leq k_n d(y, x) = k_n d^{-1}(x, y)$$

with $\lim_{n\to\infty} k_n = 1$, we see that $T : (X, d^{-1}) \to (X, d^{-1})$ is asymptotically nonexpansive. Therefore

$$d(T^n(x), T^n(y)) \leq k_n d(x, y) \leq k_n d^s(x, y)$$

and

$$d^{-1}(T^n(x), T^n(y)) \leq k_n d^{-1}(x, y) \leq k_n d^s(x, y)$$

for all $x, y \in X$. Hence

$$d^s(T^n(x), T^n(y)) \leq k_n d^s(x, y$$

for all $x, y \in X$ with $\lim_{n\to\infty} k_n = 1$ and so, $T : (X, d^s) \to (X, d^s)$ is asymptotically nonexpansive.

By assumption (X, d^s) is bounded and by Proposition 3.1 (b) it is hyperconvex. Therefore by Theorem 4.1 T has approximative fixed points, i.e. $\inf_{x\in X} d^s(x, T(x)) = 0$ and the conclusion holds.□

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