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RANDOM MAPPINGS WITH NONUNIQUE RANDOM FIXED POINTS IN POLISH SPACES

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Abstract. We present some nonunique random fixed point theorems for random mappings in a separable complete metric space. Our study includes the special cases of orbitally complete metric space, ordered metric space, metric space with two metrics and the metric space satisfying the minimal class condition.

Keywords: Orbitally complete metric space; Random mapping; Fixed point theorem; Ordered metric space; PPF dependence.

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1. Introduction

Let (X, d) be a metric space and let $T : X \to X$ be a mapping. Given an element $x \in X$, we define an orbit $\mathcal{O}(x;T)$ of T at x by

$$\mathcal{O}(x;T) = \big\{x, Tx, T^2x, ..., T^nx, \dots\big\}. \tag{1.1}$$

Then T is called T-orbitally continuous on X if for any sequence $\{x_n\} \subseteq \mathcal{O}(x;T)$, we have that $x_n \to x^*$ implies $Tx_n \to Tx^*$ for each $x \in X$. The metric space X is called T-orbitally complete if every Cauchy sequence $\{x_n\} \subseteq \mathcal{O}(x;T)$ converges to a point x^* in X. Notice that continuity implies that T-orbitally continuity and completeness implies

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T-orbitally completeness of a metric space X, but the converse may not be true. Ćirić [4] proved the following nonunique fixed point theorem for T-orbitally continuous mappings in T-orbitally complete metric spaces.

Theorem 1.1 (Ćirić [4]) *Let $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} &\min\big\{d(Tx,Ty), d(x,Tx), d(y,Ty)\big\} \\ &- \min\big\{d(x,Ty), d(y,Tx)\big\} \le qd(x,y) \end{aligned} \tag{1.2}$$

for all $x, y \in X$, where $0 \le q < 1$. Further, if T is T-orbitally continuous and X is T-orbitally complete, then T has a fixed point.

The purpose of the present paper is to extend the nonunique fixed point theorems of above type to random mappings in a polish space in different direction. We state and prove our main results in the following section.

2. Random Mappings with a Nonunique Random Fixed Point

Throughout the rest of the paper, let X denote a polish space, i.e., a complete, separable metric space with a metric d. Let $(\Omega, \mathcal{A})$ denote a measurable space with σ-algebra $\mathcal{A}$. A function $x : \Omega \to X$ is said to be a random variable if it is measurable. A mapping $T : \Omega \times X \to X$ is called random mapping if $T(., x)$ is measurable for each $x \in X$. A random mapping on a metric space X is denoted by $T(\omega, x)$ or simply $T(\omega)x$ for $\omega \in \Omega$ and $x \in X$. A random mapping $T(\omega)$ is said to be continuous on X into itself if the mapping $T(\omega, \cdot)$ is continuous on X for each $\omega \in \Omega$. A measurable function $x : \Omega \to X$ is called a random fixed point of the random mapping $T(\omega)$ if $T(\omega)x(\omega) = x(\omega)$ for all $\omega \in \Omega$. Given a random variable $x : \Omega \to X$, by a $T(\omega)$-orbit of $T(\omega)$ at x, we mean a set

$$\mathcal{O}(x; T(\omega)) = \big\{x(\omega), T(\omega)x(\omega), T^2(\omega)x, \dots\big\}, \tag{2.1}$$

for $\omega \in \Omega$. A random mapping $T : \Omega \times X \to X$ is called $T(\omega)$-orbitally continuous, if a sequence $\{x_n\}$ of measurable functions in $\mathcal{O}(x; T(\omega))$ converses to x implies that $T(\omega)x_n \to T(\omega)x$ for each $\omega \in \Omega$. The metric space X is called $T(\omega)$-orbitally complete if every Cauchy sequence of measurable functions $\{x_n\}$ in $\mathcal{O}(x; T(\omega))$ converges to a measurable function x on Ω into X.

The following theorem is essential and frequently used in the theory of random equations and random fixed point theory for random operators in Polish spaces.

Theorem 2.1. *Let X be a Polish space, that is, a complete and separable metric space. Then, the following statements hold in X.*

(a) *If $\{x_n(\omega)\}$ is a sequence of random variables converging to $x(\omega)$ for all $\omega \in \Omega$, then $x(\omega)$ is also a random variable.*

(b) *If $T(\omega, \cdot)$ is continuous for each $\omega \in \Omega$ and $x : \Omega \to X$ is a random variable, then $T(\omega)x$ is also a random variable.*

Our first nonunique random fixed point theorem is as follows.

Theorem 2.2. *Let $T(\omega)$ be a $T(\omega)$-orbitally continuous random mapping on a $T(\omega)$-orbitally complete and separable metric space X into itself satisfying for each $\omega \in \Omega$,*

$$\begin{aligned} &\min\left\{d(T(\omega)x, T(\omega)y), d(x, T(\omega)x), d(y, T(\omega)y)\right\} \\ &- \min\left\{d(x, T(\omega)y), d(y, T(\omega)x)\right\} \leq q(\omega)\, d(x, y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$, where $q : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \leq q(\omega) < 1$. Then $T(\omega)$ has a random fixed point.

Proof. Let $x : \Omega \to X$ be an arbitrary measurable function and consider the sequence $\{x_n\}$ of successive iterates of $T(\omega)$ at x defined by

$$x = x_0,\ x_1 = T(\omega)x, ..., x_n = T(\omega)x_{n-1} \tag{2.3}$$

for each $n \in \mathbb{N}$. Clearly, $\{x_n\}$ is a sequence of measurable functions on Ω into X. We shall show that $\{x_n\}$ is Cauchy sequence in X. Taking $x = x_0$ and $y = x_1$ in (2.2), we obtain

$$\begin{aligned} &\min\left\{d(T(\omega)x_0, T(\omega)x_1), d(x_0, T(\omega)x_0), d(x_1, T(\omega)x_1)\right\} \\ &- \min\left\{d(x_0, T(\omega)x_1), (d(x_1, T(\omega)x_0))\right\} \\ &\leq q(\omega)\, d(x_0, x_1) \end{aligned}$$

which further gives

$$\min\left\{d(x_1,x_2),d(x_0,x_1),d(x_1,x_2)\right\}$$
$$-\min\left\{d(x_0,x_2),d(x_1,x_1)\right\}\le qd(x_0,x_1)$$

or

$$\min\left\{d(x_1,x_2),d(x_0,x_1)\right\}\le qd(x_0,x_1).$$

Since $d(x_0,x_1)\le qd(x_0,x_1)$ is not possible view of $q<1$, one has

$$d(x_1,x_2)\le qd(x_0,x_1).$$

Proceeding in this way, by induction, it follows that

$$d(x_n,x_{n+1})\le qd(x_{n-1},x_n) \tag{2.4}$$

for each $n\in\mathbb{N}$.

From (2.4) it follows that

$$\begin{aligned} d(x_n,x_{n+1}) &\le qd(x_{n-1},x_n)\\ &\le q^2d(x_{n-2},x_{n-1})\\ &\vdots\\ &\le q^nd(x_0,x_1). \end{aligned} \tag{2.5}$$

Now for any positive integer p, we obtain by triangle inequality,

$$\begin{aligned} d(x_n,x_{n+p}) &\le d(x_n,x_{n+1})+...+d(x_{n+p-1},x_{n+p})\\ &\le q^nd(x_0,x_1)+...+q^{n+p-1}d(x_0,x_1)\\ &\le \left[q^n+q^{n+1}+...+q^{n+p-1}\right]d(x_0,x_1)\\ &\le \frac{q^n(1-q^{p-1})}{1-q}\\ &\le \frac{q^n}{1-q}\\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{2.6}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. The metric space X being $T(\omega)$-orbitally complete, there is a measurable function $x^* : \Omega \to X$ in X such that $\lim_{n\to\infty} x_n = x^*$. Again as $T(\omega)$ is $T(\omega)$-orbitally continuous, we have

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for each $\omega \in \Omega$. Thus x^* is a random fixed point of the random mapping $T(\omega)$ on X into itself. This completes the proof. □

Corollary 2.1. *Let $T(\omega)$ be a $T(\omega)$-orbitally continuous random mapping on a $T(\omega)$-orbitally complete and separable metric space X into itself satisfying for each $\omega \in \Omega$,*

$$d(T(\omega)x, T(\omega)y) \le q(\omega)\, d(x, y) \tag{2.2}$$

for all $x, y \in X$, where $q : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \le q(\omega) < 1$. Then $T(\omega)$ has a random fixed point.

When $T(\omega)x = Tx$ for all $\omega \in \Omega$ in Theorem 2.2, we obtain Theorem 1.1 as a corollary which again includes the famous Banach fixed point theorem for contraction mappings on a metric space X into X..

Theorem 2.3. *Let $T(\omega)$ be a $T(\omega)$-orbitally continuous random selfmapping of a $T(\omega)$-orbitally complete and separable metric space X satisfying for each $\omega \in \Omega$,*

$$\begin{aligned}
&\min\left\{[d(T(\omega)x, T(\omega)y)]^2, d(T(\omega)x, T(\omega)y)d(x, y), d(x, T(\omega)x)d(y, T(\omega)y)\right\}\\
&-\min\left\{d(x, T(\omega)x)d(y, T(\omega)y), d(x, T(\omega)y)d(y, T(\omega)x)\right\} \qquad (2.7)\\
&\le q(\omega)d(x, T(\omega)x)d(y, T(\omega)y)
\end{aligned}$$

for all $x, y \in X$, where $q : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \le q(\omega) < 1$. Then $T(\omega)$ has a random fixed point.

Proof. Let $x : \Omega \to X$ be an arbitrary measurable function and consider the sequence $\{x_n\}$ of successive iterates of $T(\omega)$ at x defined by

$$x_0 = x,\ x_{n+1} = T(\omega)x_n,\ n = 0, 1, 2, ...$$

Clearly, $\{x_n\}$ is a sequence of measurable functions on Ω into X. We show that $\{x_n\}$ is a Cauchy sequence in X. Let $x = x_0$ and $y = x_1$ in (2.7) we obtain

$$\begin{aligned} &\min\big\{[d(T(\omega)x_0, T(\omega)x_1))]^2, d(x_0, T(\omega)x_0))d(x_1, T(\omega)x_1)), \\ &\qquad d(T(\omega)x_0, T(\omega)x_1))d(x_0, x_1)\big\} \\ &\quad \le q\, d(x_0, T(\omega)x_0))d(x_1, T(\omega)x_1) \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad &\min\big\{[d(x_1, x_2)]^2, d(x_0, x_1)d(x_1, x_2)d(x_1, x_2)d(x_0, x_1) \\ &\quad - \min\big\{d(x_0, x_1)d(x_1, x_2), d(x_0, x_2)d(x_1, x_1)\big\} \\ &\quad \le qd(x_0, x_1)d(x_1, x_2) \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad &\min\big\{[d(x_1, x2)]^2, d(x_0, x_1)d(x_1, x_2)\big\} \\ &\quad \le qd(x_0, x_1)d(x_1, x_2). \end{aligned} \tag{2.8}$$

Since

$$d(x_0, x_1)d(x_1, x_2) \le qd(x_0, x_1)d(x_1, x_2)$$

is not possible in view of $q < 1$, one has

$$[d(x_1, x_2)]^2 \le qd(x_0, x_1)d(x_1, x_2)$$

$$i.e. \qquad d(x_1, x_2) \le qd(x_0, x_1).$$

Proceeding in this way, by induction,

$$d(x_n, x_{n+1}) \le qd(x_{n-1}, x_n)$$

for each $n = 1, 2, 3, ...$ The rest of the proof is similar to Theorem 2.2 and hence, we omit the details. □

As a consequence of Theorem 2.2 we obtain the following corollary.

Corollary 2.2. *Let T be a T-orbitally continuous selfmapping of a T-orbitally complete metric space X satisfying*

$$\begin{aligned}
&\min\left\{[d(Tx,Ty)]^2, d(Tx,Ty)d(x,y), d(x,Tx)d(y,Ty)\right\} \\
&-\min\left\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\right\} \qquad (2.9)\\
&\leq qd(x,Tx)d(y,Ty)
\end{aligned}$$

for all $x, y \in X$, where $0 \leq q < 1$. Then T has a fixed point.

Sometimes it possible that a metric space may be complete w.r.t. a metric but may not be complete w.r.t. another metric defined on it. Therefore, it is interesting to obtain the fixed point theorems in such situation. Next we prove a couple of nonunique random fixed point theorem in a metric space with two metrics defined on it.

Theorem 2.4. *Let X be a metric space with two metrics d_1 and d_2. Let $(\Omega, \mathcal{A})$ be a measurable space and let $T : \Omega \times X \to X$ be a random mapping satisfying the condition (2.2) w.r.t. d_2 for each $\omega \in \Omega$. Further suppose that*

(i) *$d_1(x, y) \leq d_2(x, y)$ for all $x, y \in X$*
(ii) *$T(\omega)$ is a $T(\omega)$-orbitally continuous w.r.to d_1.*
(iii) *X is $T(\omega)$-orbitally complete w.r.t. d_1, and*
(iv) *X is separable metric space.*

Then $T(\omega)$ has a random fixed point.

Proof. Let $x : \Omega \to X$ be an arbitrary measurable function and consider the sequence $\{x_n\}$ of successive iterations of $T(\omega)$ defined by (2.3). Then, $\{x_n\}$ is a sequence of measurable functions from Ω into X. Now proceeding as in the proof of Theorem 2.2, we obtain,

$$d_2(x_n, x_{n+p}) \leq \frac{q^n}{(1-q)}$$

for some positive integer p. By hypothesis (i), we have

$$\begin{aligned} d_1(x_n, x_{n+p}) &\leq d_2(x_n, x_{n+p}) \\ &\leq \frac{q^n}{(1-q)} d_2(x_0, x_1) \qquad (2.10) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X w.r.t. the metric d_1. The metric space (X, d_1) being $T(\omega)$-orbitally complete, there is a measurable function $x^* : \Omega \to X$ such that

$$\lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for each $\omega \in \Omega$. From the above limit, it follows that

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for each $\omega \in \Omega$. Thus $T(\omega)$ has a random fixed point and the proof of Theorem 2.4 is complete.

Theorem 2.5. *Let X be a metric space with two metrics d_1 and d_2. Let $(\Omega, \mathcal{A})$ be a measurable space and let $T : \Omega \times X \to X$ be a random mapping satisfying the condition (2.7) w. r. t. the metric d_2 for each $\omega \in \Omega$. Suppose that the following conditions hold in X.*

(i) *$d_1(x, y) \leq d_2(x, y)$ for all $x, y \in X$.*

(ii) *$T(\omega)$ is $T(\omega)$-orbitally continuous w.r.t. d_1*

(iii) *X is $T(\omega)$-orbitally complete and separable w.r.t. d_1.*

Then $T(\omega)$ has a random fixed point.

Proof. The proof is similar to Theorem 2.3 and therefore, we omit the details. □

3. Generalization of Ćirić Type Random Mappings with Nonunique Random Fixed Points

In this section ,we generalize the class of Ćirić [4] type random mappings and prove some nonunique random fixed point theorems in a separable and complete metric space.

Theorem 3.1. *Let $T(\omega)$ be a $T(\omega)$-orbitally continuous random selfmapping of a $T(\omega)$ -orbitally complete and separable metric space X satisfying for each $\omega \in \Omega$,*

$$\begin{aligned} &\min\big\{d(T(\omega)x, T(\omega)y), d(x, T(\omega)x), d(y, T(\omega)y)\big\} \\ &- \min\big\{d(x, T(\omega)y), d(y, T(\omega))\big\} \\ &\leq p(\omega)\ \min\{d(x, T(\omega)x), d(y, T(\omega)y)\} + q(\omega)\, d(x, y) \end{aligned} \tag{3.1}$$

for all $x, y \in X$, where $p, q : \Omega \to \mathbb{R}_+$ are measurable functions such that

$$p(\omega) + q(\omega) < 1 \tag{3.2}$$

for all $\omega \in \Omega$. Then $T(\omega)$ has a random fixed point.

Proof. Let $x : \Omega \to X$ be an arbitrary measurable function and consider the sequence $\{x_n\}$ of measurable functions from Ω into X defined by (2.3). Then taking $x = x_0$ and $y = x_1$ in (3.1), we obtain

$$\begin{aligned} &\min\big\{d(T(\omega)x_0, T(\omega)x_1), d(x_0, T(\omega)x_0), d(x_1, T(\omega)x_1)\big\} \\ &\qquad - \min\big\{d(x_0, T(\omega)x_1), d(x_1, T(\omega)x_0)\big\} \\ &\leq p(\omega) \min\big\{d(x_0, T(\omega)x_0), d(x_1, T(\omega)x_1)\big\} + q(\omega)d(x_0, x_1) \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad &\min\big\{d(x_1, x_1), d(x_0, x_1)\big\} \\ &\qquad - \min\big\{d(x_0, x_2), 0\big\} \\ &\leq p(\omega) \min\big\{d(x_0, x_1), d(x_1, x_2)\big\} + q(\omega)d(x_0, x_1) \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad &\min\big\{d(x_1, x_2), d(x_0, x_1)\big\} \\ &\leq p(\omega) \min\big\{d(x_0, x_1), d(x_1, x_2)\big\} \\ &\qquad + q(\omega)d(x_0, x_1) \end{aligned} \tag{3.3}$$

As $\min\big\{d(x_1, x_2), d(x_0, x_1)\big\} = d(x_0, x_1)$ is not possible, we have that

$$\min\big\{d(x_1, x_2), d(x_0, x_1)\big\} = d(x_1, x_2).$$

Hence, from the above inequality (3.3), we obtain

$$d(x_1, x_2) \leq \alpha(\omega) d(x_0, x_1),$$

where $\alpha(\omega) = p(\omega) + q(\omega) < 1$ for all $\omega \in \Omega$. Now, proceeding as in the proof of Theorem 2.2, it is proved that $T(\omega)$ has a random fixed point.

Corollary 3.1. *Let T be a T-orbitally continuous selfmapping of a T-orbitally complete metric space X satisfying*

$$\begin{aligned} &\min\left\{d(Tx,Ty), d(x,Tx), d(y,Ty)\right\} \\ &-\min\left\{d(x,Ty), d(y,Tx)\right\} \leq p\,\min\{d(x,Tx), d(y,Ty)\} + q\,d(x,y) \end{aligned} \tag{3.5}$$

for all $x, y \in X$, where p and q are nonnegative real numbers such that $p + q < 1$. Then T has a fixed point.

Theorem 3.2. *Let X be a metric space with two metrics d_1 and d_2. Let $(\Omega, \mathcal{A})$ be a measurable space and let $T : \Omega \times X \to X$ be a random mapping satisfying the inequality (3.1) w. r. to the metric d_2 for each $\omega \in \Omega$. Suppose that the following conditions hold in X.*

(i) *$d_1(x,y) \leq d_2(x,y)$ for all $x, y \in X$.*
(ii) *$T(\omega)$ is $T(\omega)$ -orbitally continuous w.r.t. d_1.*
(iii) *X is $T(\omega)$-orbitally complete ad separable w.r.t. d_1.*

Then the random mapping $T(\omega)$ has a random fixed point.

Proof. The proof is similar to Theorem 2.5 and hence we omit the details. □

4. Random Fixed Points Mappings in Ordered Metric Spaces

We define an order relation $\leq$ in X which is a reflexive, antisymmetric and transitive relation in X. The metric space X together with the order relation $\leq$ becomes a partially ordered metric space. A random mapping $T : \Omega \times X \to X$ is called nondecreasing if for any $x, y \in X$ with $x \leq y$ we have that $T(\omega)x \leq T(\omega)y$ for all $\omega \in \Omega$. Similarly random mapping $T : \Omega \times X \to X$ is called nonincreasing if for any $x, y \in X$, $x \leq y$ implies $T(\omega)x \geq T(\omega)y$ for all $\omega \in \Omega$. A monotone random mapping which is either nondecreasing or nonincreasing on X.

The investigation of the existence of fixed points in a partially ordered metric space was first considered in Ram and Reuriungs [10]. This study was continued in Nieto and Rodriguer-Lopez [12] by assuming the existence of only lower solution instead of usual approach where both the lower and upper solutions are assumed to exist for the nonlinear equation. These fixed point theorems are then applied to obtain existence and uniqueness results for nonlinear ordinary differential equations in the same paper. A further extension of this idea was considered in Bhaskar and Lakshmikanthan [3] for the coupled fixed point theorems in partially ordered metric spaces. Below we prove some nonunique random fixed point theorems for monotone random mappings in separable and complete metric spaces.

Theorem 4.1. *Let $(\Omega, \mathcal{A})$ be a measurable space and let X be a separable and complete partially ordered metric space. Let $T : \Omega \times X \to X$ be a monotone nondecreasing random mapping satisfying the contraction condition (2.2) for all comparable elements x and y in X. Further if $T(\omega)$ is continuous and if there exists an element $x_0 \in X$ such that $x_0 \leq T(\omega)x_0$ for all $\omega \in \Omega$, then the random mapping $T(\omega)$ has a random fixed point. Further, if every pair of elements $x, y \in X$ has a lower bound and an upper bound, then $T(\omega)$ has a unique random fixed point.*

Proof. Let $x : \Omega \to X$ be an arbitrary measurable function and define a sequence $\{x_n\}$ of successive approximations of $T(\omega)$ by

$$x_{n+1} = T(\omega)x_n, n = 0, 1, 2, \ldots.$$

Clearly $\{x_n\}$ is a sequence of measurable functions from Ω into X such that

$$x_0 \leq x_1 \leq ... \leq x_n \leq \ldots. \tag{4.1}$$

We show that $\{x_n\}$ is a Cauchy sequence in X. Taking $x = x_0$ and $y = x_1$ in (2.2) we obtain

$$d(x_1, x_2) \leq qd(x_0, x_1).$$

Processing in this way, by induction ,

$$d(x_n, x_{n+1}) \leq qd(x_{n-1}), x_n)$$

for each $n = 1, 2, ...$ Then, by repeated application of the above inequality, we obtain

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1).$$

Now for any positive integer $m > n$, by triangle inequality, we get

$$\begin{aligned} d(x_m, x_n) &= d(x_n, x_m) \\ &\leq d(x_n, x_{n+1}) + .. + d(x_{m-1}, x_m) \\ &\leq (q^n + q^{n+1} ... + q^{m-n}) d(x_0, x_1) \\ &\leq \frac{q^n(1 - q^{m-n})}{1 - q} d(x_0, x_1) \\ &\leq \frac{q^n}{1 - q} d(x_0, x_1) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.2}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. The ordered metric space X being complete, there is a measurable function $x^* : \Omega \to X$ such that $\lim_{n\to\infty} x_n = x^*$. From the continuity of the random mapping $T(\omega)$ it follows that

$$x^*(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = T(\omega) \lim_{n\to\infty} x_n(\omega) = T(\omega)x^*(\omega) \tag{4.3}$$

for all $\omega \in \Omega$. Thus x^* is a random fixed point of the random mapping $T(\omega)$ on X.

If every pair of elements $x, y \in X$ has a lower bound and an upper bound, then it can be shown as in Ran and Reurings [10] that $\lim\limits_{n\to\infty} T^n(\omega)x = x^*(\omega)$ for all measurable functions $x : \Omega \to X$, where $x^* = \lim\limits_{n\to\infty} T^n(\omega)x_0$. Thus $T(\omega)$ has a unique random fixed point and the proof of the theorem is complete. □

Corollary 4.1. (Dhage [7]) *Let $(\Omega, \mathcal{A})$ be a measurable space and let X be a separable and complete partially ordered metric space. Let $T : \Omega \times X \to X$ be a monotone nondecreasing random mapping satisfying*

$$d(T(\omega)x, T(\omega)y) \leq q(\omega)d(x, y)$$

for all comparable elements $x, y \in X$, where $q : \Omega \to \mathbb{R}_+$ is a measurable function such that $q(\omega) < 1$ for all $\omega \in \Omega$. Further, if $T(\omega)$ is continuous and if there exists an element $x_0 \in X$ such that $x_0 \leq T(\omega)x_0$ for all $\omega \in \Omega$, then the random mapping $T(\omega)$ has a *random fixed point. Further, if every pair of elements $x, y \in X$ has a lower bound and an upper bound, then $T(\omega)$ has a unique random fixed point.*

Corollary 4.2. *Let X be a partially ordered complete metric space and let $T : X \to X$ be a monotone nondecreasing mapping satisfying the contraction condition (2.7) for all comparable elements $x, y \in X$. Further if T is continuous and if there exists an element $x_0 \in X$ such that $x_0 \leq Tx_0$, then the mapping T has a fixed point.*

Corollary 4.2. (Nieto and Rodriguez-Lopez [12]) *Let X be a complete metric space and let $T : X \to X$ be a monotone nondecreasing mapping satisfying*

$$d(Tx, Ty) \leq qd(x, y) \tag{4.4}$$

for comparable elements $x, y \in X$, where $0 \leq q < 1$. Further if T is continuous and if there exists an element $x_0 \in X$ such that $x_0 \leq Tx_0$, then the mapping T has a fixed point.

Theorem 4.2. *Let $(\Omega, \mathcal{A})$ be a measurable space and let (X, d) be a partially ordered complete separable metric space. Let $T : \Omega \times X \to X$ be a random mapping satisfying for each $\omega \in \Omega$,*

$$\begin{aligned}
&\min\left\{[d(T(\omega)x, T(\omega)y)]^2, d(x, T(\omega)x)d(y, T(\omega)y), d(T(\omega)x, T(\omega)y)d(x, y)\right\} \\
&- \min\left\{d(x, T(\omega)y)d(y, T(\omega)y), d(x, T(\omega)x)d(y, T(\omega)y)\right\} \\
&\leq q(\omega)d(x, T(\omega))d(y, T(\omega)y)
\end{aligned} \tag{4.3}$$

for all comparable elements $x, y \in X$, where $q : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \leq q(\omega) < 1$ for all $\omega \in \Omega$. Further if there exists an element $x_0 \in X$ such that $X_0 \leq T(\omega)x_0$, then $T(\omega)$ has a fixed point.

Proof. The proof is similar to Theorem 4.1 and therefore, we omit the details. □

Theorem 4.3. *Let $(\Omega, \mathcal{A})$ be a measurable space and let (X, d) be a partially ordered separable metric space. Let $T : \Omega \times X \to X$ be a continuous random mapping satisfying* *for each* $\omega \in \Omega$*,*

$$\begin{aligned} &\min\left\{d(T(\omega)x, T(\omega)y), d(x, T(\omega)x), d(y, T(\omega)y)\right\} \\ &- \min\left\{d(x, T(\omega)y), d(y, T(\omega)x)\right\} \qquad (4.4) \\ &\le p(\omega)\min\left\{d(x, T(\omega)x), d(y, T(\omega)y)\right\} + q(\omega)d(x, y) \end{aligned}$$

for all comparable elements $x, y \in X$*, where* $p, q : \Omega \to \mathbb{R}_+$ *are measurable functions such that* $0 \le p(\omega) + q(\omega) < 1$ *for all* $\omega \in \Omega$*. If there exists an element* $x_0 \in X$ *such that* $x_0 \le T(\omega)x_0$ *for each* $\omega \in \Omega$*, then* $T(\omega)$ *has a random fixed point.*

Proof. The proof is simple and can be obtained by closely observing the proof of Theorem 4.1. Hence we omit the details. □

Next, we deal with the case of metric space X with two metrics d_1 and d_2 defined on it and prove some nonunique random fixed point theorems on separable partially ordered metric spaces. □

Theorem 4.4. *Let* (Ω, A) *be a measurable space and let* X *be an partially ordered metric space with two metrics* d_1 *and* d_2*. Let* $T : \Omega \times X \to X$ *be a nondecreasing random mapping satisfying the contractive condition on (2.2) w.r.t.* d_2 *for all comparable elements* $x, y \in X$*. Suppose that the following conditions hold in* X*.*

(i) $d_1(x, y) \le d_2(x, y)$ *for all* $x, y \in X$*.*

(ii) $T(\omega)$ *is continuous w.r.t.* d_1*.*

(iii) X *is Polish space w.r.t.* d_1*.*

Furthermore, if there exists an element $x_0 \in X$ *such that* $x_0 \le T(\omega)x_0$ *for all* $\omega \in \Omega$*, then* $T(\omega)$ *has a random fixed point.*

Proof. Consider the sequence $\{x_n\}$ of successive iterations of $T(\omega)$ at x_0 defined by

$$x_{n+1} = T(\omega)x_n,\ n = 0, 1, 2, \ldots.$$

Clearly, $\{x_n\}$ is a sequence of measurable functions from Ω into X w.r.t. the metric d_1 such that

$$x_0 \le x_1 \le \ldots \le x_n \le \ldots.$$

Then, it can be shown as in the proof of Theorem 4.1 that $\{x_n\}$ is a Cauchy sequence in X w.r.t. the metric d_2, that is, for any positive integer $m > n$,

$$d_2(x_m, x_n) \leq \frac{q^n}{1-q} d_2(x_0, x_1).$$

From the hypothesis (i), it follows that

$$d_1(x_m, x_n) \leq \frac{q^n}{1-q} d_2(x_0, x_1) \to 0 \quad \text{as } n \to \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence w.r.t. the metric d_1. The metric space (X, d_1) being complete and separable, there exists a measurable function $x^*; \Omega \to X$ such that $\lim_{n\to\infty} x_n(\omega) = x^*(\omega)$ for each $\omega \in \Omega$. From the continuity of $T(\omega)$ w.r.t. d_1, it follows that

$$T(\omega)x^*(\omega) = \lim_{n\to\infty} T(\omega)x_n(\omega) = \lim_{n\to\infty} x_{n+1}(\omega) = x^*(\omega)$$

for all $\omega \in \Omega$. This proves that $T(\omega)$ has a random fixed point in X. This completes the proof. □

Remark 4.1. The conclusion of Theorem 4.4 also remains true if we replace the condition (2.2) with those of (4.3) and(4.4).

5. Nonunique PPF Dependant Random Fixed Point Theory

The fixed point theory of nonlinear operators with PPF dependence which is depending upon past, present and future data was developed in Bernfield et.al.[1]. The domain space of the nonlinear operator was taken as $C(I, E)$, $I = [a, b] \subset \mathbb{R}$ and the range space as E, a Banach space. An important example of such a nonlinear operator is a delay differential equation. The PPF dependent fixed point theorems are applied to ordinary nonlinear functional differential equations for proving the existence of solutions. Random fixed point theory for random operators in separable Banach spaces is initiated by Hans [8] and Spacek [13] and further developed by several authors in the literature. A brief survey of such random fixed point theorems appears in Joshi and Bose [9].

In the present section we obtain a successful fusion of above two ideas and prove some PPF dependent random fixed point theorems for random mappings in a separable metric space. In the PPF dependent classical fixed point theory, the Razumikkin or minimal class of functions plays a significant role both in proving existence as well as uniqueness of PPF dependent fixed points. Let E be a metric space and let I be a given closed and bounced interval in $\mathbb{R}$, the set of real numbers. Let $E_0 = C(I, E)$ denote the class of continuous mappings from I to E. We equip the class $C(J, E)$ with metric d_0 defined by

$$d_0(x, y) = \sup_{t \in J} d(x(t), y(t)).$$

Lemma 5.1. If (E, d) is complete then the metric space (E_0, d_0) is also complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence. Then for $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$d_0(\phi_m, \phi_n) < \epsilon$$

for all $m > n \geq n_0$. Since

$$d(\phi_m(t), \phi_n(t) < \epsilon$$

for all $m, n \geq n_0$. Thus $\phi_n(t)$ is a Cauchy sequence in E. So there exists a function $\phi^* \in E_0$ such that

$$\lim_{n \to \infty} \phi_n(t) = \phi^*(t)$$

for all $t \in J$ or

$$\lim_{n \to \infty} d(\phi_n(t), \phi^*(t)) = 0$$

for all $t \in J$. Now

$$\lim_{n \to \infty} d(\phi_n, \phi) = \lim_{n \to \infty} \sup_{t \in J} d(\phi_n(t), \phi^*(t) = 0.$$

Hence $\phi_n \to \phi$ in E_0 and the proof the lemma is complete. □

When E is a Banach space and let $E_0 = C(J, E)$ be a space of continuous E -valued functions defined on J Then the minimal class of functions related to a fixed $c \in J$ is defined as

$$\mathcal{M}_c = \{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\}.$$

Now we are in a position to state and prove our random fixed point results concerning the existence of random fixed points with PPF dependence.

Theorem 5.1. *Let $(\Omega, \mathcal{A})$ be a measurable space and E, a separable complete metric space. Let $T : \Omega \times E_0 \to E$ be a continuous random mapping satisfying for each $\omega \in \Omega$,*

$$\begin{aligned}
&\min\big\{d(T(\omega)\phi, T(\omega)\psi), d(\phi(c,\omega), T(\omega)\phi), d(\psi(c,\omega), T(\omega)\psi)\big\} \\
&- \min\big\{d(\phi(c,\omega), T(\omega)\psi), d(\psi(c,\omega), T(\omega)\phi)\big\} \\
&\leq q(\omega)\, d(\phi, \psi)
\end{aligned} \tag{5.1}$$

for all $\phi, \psi \in E_0$, where $q : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \leq q(\omega) < 1$ for all $\omega \in \Omega$ and $c \in I$ is a fixed point. Then $T(\omega)$ has a random fixed point with PPF dependence.

Proof. Let $\phi_0 : \Omega \to E_0$ be an arbitrary measurable function and define a sequence $\big\{x_n\big\}$ in E_0 as follows. Suppose that $T(\omega)\phi_0 = x_1$ for some $x_1 \in E$. Then choose $\phi_1 \in E_0$ such that $\phi_1(c, \omega) = x_1$ for some fixed $c \in I$ and

$$d_0(\phi_0, \phi_1) = d(\phi_0(c,\omega), \phi_1(c,\omega))$$

for all $\omega \in \Omega$. Again let $T(\omega)\phi_1 = x_2$ for some $x_2 \in E$. Then choose $\phi_2(c,\omega) = x_2$ for the fixed $c \in I$ and

$$d_0(\phi_1, \phi_2) = d(\phi_1(c,\omega), \phi_2(c,\omega))$$

for all $\omega \in \Omega$. Proceeding in this way, we obtain a sequence $\{\phi_n\}$ of points in E_0 of iterations ofn $T(\omega)$ at ϕ_0 as

$$T(\omega)\phi_{n-1} = x_n = \phi_n(c,\omega) \tag{5.2}$$

with

$$d_0(\phi_{n-1}, \phi_n) = d(\phi_{n-1}(c,\omega), \phi_n(c,\omega) \tag{5.3}$$

for all $\omega \in \Omega$. Clearly, $\big\{\phi_n\big\}$ is a sequence of measurable functions from Ω into E_0. Consequently $\big\{\phi_n(c)\big\}$ is a sequence of measurable functions from Ω into E. We show that $\{\phi_n(c,\omega)\}$ is a Cauchy sequence in E. Taking $\phi = \phi_0$ and $\phi = \phi_1$ in the inequality (5.1), we obtain

$$\begin{aligned}
&\min\big\{d(T(\omega)\phi_0, T(\omega)\phi_1), d(\phi_0(c,\omega), T(\omega)\phi_0), d(\phi_1(c,\omega), T(\omega)\phi_1)\big\}\\
&-\min\big\{d(\phi_0(c,\omega), T(\omega)\phi_1), d(\phi_1, T(\omega)\phi_0)\big\} \qquad (5.4)\\
&\le q(\omega)\, d_0(\phi_0,\phi_1)
\end{aligned}$$

which further gives

$$\begin{aligned}
\min&\big\{d_0(\phi_1,\phi_2), d_0(\phi_0,\phi_1)\big\}\\
&= \min\big\{d(\phi_1(c,\omega),\phi_2(c,\omega)), d(\phi_0(c,\omega),\phi_1(c,\omega))\big\}\\
&= \min\big\{d(\phi_1(c,\omega),\phi_2(c,\omega)), d(\phi_0(c,\omega),\phi_1(c,\omega)), d(\phi_1(c,\omega),\phi_2(c,\omega))\big\} \qquad (5.5)\\
&\le q d_0(\phi_0,\phi_1).
\end{aligned}$$

Since $d_0(\phi_0,\phi_1) \le q d_0(\phi_0,\phi_1)$, $q < 1$, is not possible, one has

$$d_0(\phi_1,\phi_2) \le d_0(\phi_0,\phi_1).$$

Proceeding in this way by induction,

$$d_0(\phi_n,\phi_{n+1}) \le q d_0(\phi_{n-1},\phi_n) \qquad (5.6)$$

for all $n, n = 1, 2, 3, \ldots$. By a repeated application of the inequality (3)

$$\begin{aligned}
d_0(\phi_n,\phi_{n+1}) &\le q d_0(\phi_{n-1},\phi_n)\\
&\ \vdots \qquad (5.7)\\
&\le q^n d_0(\phi_0,\phi_1)
\end{aligned}$$

Now for any positive integer p, by triangle inequality,

$$\begin{aligned}
d_0(\phi_n,\phi_{n+p}) &\le d_0(\phi_n,\phi_{n+1}) + \cdots + d_0(\phi_{n+p-1},\phi_{n+p})\\
&\le q^n\big(1 + q + \cdots + q^{p-1}\big) d_0(\phi_0,\phi_1)\\
&\le \frac{q^n}{(1-q)} d_0(\phi_0,\phi_1) \qquad (5.8)\\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Since

$$d(\phi_n(c,\omega),\phi_{n+p}(c,\omega)) = d_0(\phi_n(\omega),\phi_{n+1}(\omega))$$

for all $\omega \in \Omega$, we have that $\{T(\omega)\phi_n\}$ is also Cauchy sequence in E. As E is a complete metric space, there exists a measurable function $\phi^* : \Omega \to E_0$ such that $\phi_n \to \phi^*$ and

$$T(\omega)\phi_n = \phi_{n+1}(c, \omega) \to \phi^*(c, \omega)$$

as $n \to \infty$. To prove that ϕ^* is a PPF dependent random fixed point of $T(\omega)$, we first observe that since $T(\omega)$ is continuous on E_0, $T(\omega)$ is a continuous at ϕ^*. Hence for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d_0(\phi_{n+1}, \phi^*) < \delta \implies d(T\phi_{n+1}, T\phi^*) < \frac{\epsilon}{2}.$$

Also since $T(\omega)\phi_n \to \phi^*(c, \omega)$, for $\gamma = \min\left\{\frac{\epsilon}{2}, \delta\right\}$ there exists $n_0 \in \mathbb{N}$ such that

$$d(T(\omega)\phi_n, \phi^*(c, \omega)) < \gamma$$

for $n \geq n_0$. Thus,

$$\begin{aligned} d(T(\omega)\phi^*, &\phi^*(c, \omega)) \\ &\leq d(T(\omega)\phi^*, T(\omega)\phi_n) + d(T(\omega)\phi_n, \phi^*(c, \omega)) \\ &< \frac{\epsilon}{2} + \gamma < \epsilon. \end{aligned} \tag{5.6}$$

Since ϵ is arbitrary, we have

$$T(\omega)\phi^*(\omega) = \phi^*(c, \omega)$$

for all $\omega \in \Omega$. This completes the proof. □

As a consequence of Theorem 5.1 we obtain the following PPF dependent random fixed result of Dhage [6] as a special case.

Corollary 5.1. (Dhage [6])*Let* $(\Omega, \mathcal{A})$ *be a measurable space and* E*, a separable complete metric space. Let* $T : \Omega \times E_0 \to E$ *be a continuous random mapping satisfying for each* $\omega \in \Omega$*,*

$$d(T(\omega)\phi, T(\omega)\psi) \leq q(\omega)\, d_0(\phi, \psi)$$

for all $\phi, \psi \in E_0$*, where* $q : \Omega \to \mathbb{R}_+$ *is a measurable function satisfying* $0 \leq q(\omega) < 1$ *for all* $\omega \in \Omega$ *and* $c \in I$ *is a fixed point. Then* $T(\omega)$ *has a random fixed point with PPF dependence.*

Again, As a consequence of Theorem 5.1 we obtain the following corollary which is also new to the literature.

Corollary 5.2. *Let E a complete metric space. Let $T : \Omega \times E_0 \to E$ be a continuous mapping satisfying*

$$\begin{aligned}&\min\left\{d(T\phi, T\psi), d(\phi(c), T\phi), d(\psi(c), T\psi)\right\}\\&-\min\left\{d(\phi(c), T\psi), d(\psi(c), T\phi)\right\}\\&\leq q\, d_0(\phi, \psi)\end{aligned} \tag{5.7}$$

for all $\phi, \psi \in E_0$, where $0 \leq q < 1$ and $c \in I$ is a fixed point. Then T has a fixed point with PPF dependence.

From Corollary 5.2 we obtain

Corollary 5.3. (Bernfeld et. al. [1]) *Let E a complete metric space. Let $T : E_0 \to E$ be a continuous mapping satisfying*

$$d(T\phi, T\psi) \leq q\, d_0(\phi, \psi)$$

for all $\phi, \psi \in E_0$, where $0 \leq q(\omega) < 1$ and $c \in I$ is a fixed point. Then T has a fixed point with PPF dependence.

Theorem 5.2. *Let $(\Omega, \mathcal{A})$ be a measurable space and E a complete separable metric space. Let $T : \Omega \times E_0 \to E$ be a continuous random mapping satisfying for each $\omega \in \Omega$,*

$$\begin{aligned}&\min\left\{d(T(\omega)\phi, T(\omega)\psi), d(\phi(c,\omega), T(\omega)\phi), d(\psi(c,\omega), T(\omega)\psi)\right\}\\&-\min\left\{d(\phi(c,\omega), T(\omega)\psi), d(\psi(c,\omega), T(\omega)\phi)\right\}\\&\leq p(\omega)\min\left\{d(\phi(c,\omega), T(\omega)\phi), d(\psi(c,\omega), T(\omega)\psi)\right\} + q(\omega)d_0(\phi, \psi)\end{aligned} \tag{5.8}$$

for all $\phi, \psi \in E_0$, where $c \in I$ is a fixed point and $p, q : \Omega \to \mathbb{R}_+$ are measurable functions satisfying $0 \leq p(\omega) + q(\omega) < 1$ for all $\omega \in \Omega$. Then $T(\omega)$ has a PPF dependent random fixed point.

Proof. The proof is similar to Theorem 5.1 with appropriate modifications. We omit the details. □

Corollary 5.2. *Let E be a complete metric space and let $T : E_0 \to E$ be a continuous mapping satisfying*

$$\begin{aligned} &\min\big\{d(T\phi,T\psi),d(\phi(c),T\phi),d(\psi(c),T\psi)\big\}\\ &-\min\big\{d(\phi(c),T\psi),d(\psi(c),T\phi)\big\}\\ &\leq p\min\big\{d(\phi(c),T\phi),d(\psi(c),T\psi)\big\}+qd_0(\phi,\psi)\end{aligned}\tag{5.9}$$

for all $\phi,\psi \in E_0$, where $c \in I$ is a fixed point and p,q are nonnegative real numbers such that $p+q<1$. Then T has a PPF dependent fixed point.

Corollary 5.2 is again new to the subject of PPF dependent classical fixed point theory initiated by Bernfeld *et. al* [1] and includes some well-known classical PPF dependent fixed point theorems in Banach spaces. □

Theorem 5.2 also remains true if we replace the contractive condition (5.1) by

$$\begin{aligned} &\min\big\{[d(T(\omega)\phi,T(\omega)\psi]^2,d(\phi(c,\omega),T(\omega)\phi)d(\psi(c,\omega),T(\omega)\psi),\\ &\qquad d(T(\omega)\phi,T(\omega)\psi),d(\psi(c,\omega),\phi(c,\omega))\big\}\\ &-\min\big\{d(\phi(c,\omega),T(\omega)\phi)d(\psi(c,\omega),T(\omega)\psi),d(\phi(c,\omega),T(\omega)\psi)d(\psi(c,\omega),T(\omega)\phi)\big\}\\ &\leq q(\omega)d(\phi(c,\omega),T(\omega)\phi)d(\psi(c,\omega),T(\omega)\psi)\end{aligned}\tag{5.10}$$

for all $\omega \in \Omega$ and for all $\phi,\psi \in E_0$, where $c \in I$ is fixed and $q : \Omega \to \mathbb{R}_+$ is a measurable function satisfying $0 \leq q(\omega) < 1$ for all $\omega \in \Omega$.

Finally, while concluding this paper we mention that the random fixed point results of this paper may be extended to two, three and four mappings to prove the random common fixed point theorems in Polish spaces along the similar lines with appropriate modifications. Some of the results along this line will be reported elsewhere.

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