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NON-UNIQUE FIXED POINT THEOREM ON CONE METRIC SPACE

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Abstract: In the present paper, some results of Ćirić [6] on a non-unique fixed point theorem on the class of metric spaces are extended to the class of cone metric space. Namely, non-unique fixed point theorem is proved in orbitally T complete cone metric spaces under the assumption that the cone is strongly minihedral.

Keywords: Cone metric space, strongly minihedral cone, orbitally continuous mapping.

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1. Introduction.

In 1980, by generalizing the fixed point theorems of Maia type [16], Rzepecki [19] introduced a generalized metric d_E on a set X in a way that $d_E : X\times X \to S$ where E is a Banach space and S is a normal cone in E with partial order $\leq$. In 1987, Lin [13] considered the notion of K-metric spaces by replacing real numbers with cone K in the metric function, that is, $d : X\times X \to K$. In that manuscript, some results of Khan and Imdad [12] on fixed point theorems were considered for K-metric spaces. Without mentioning the papers of Lin [13] and Rzepecki [19], in 2007, Huang and Zhang [9] announced the notion of cone metric spaces (CMSs) by replacing real numbers with an ordering Banach space. In that paper, they also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces.

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Recently, many results on fixed point theory have been extended to cone metric spaces (see, e.g., [1, 2, 3, 9, 10, 11, 18, 20, 21]). Ćirić type non-unique fixed point theorems were considered by many authors (see, e.g.,[4, 6, 8, 14, 15, 17, 22]). In this paper, we extend the results of Ćirić [6] to cone metric spaces.

2. Preliminaries.

Throughout this paper $E := (E, \|.\|)$ stands for a real Banach space.

Definition 2.1. Let $P := P_E$ always be a closed non-empty subset of E. P is called *cone* if $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b where $P \cap (-P) = \{0\}$ and $P \neq \{0\}$.

Definition 2.2. For a given cone P, one can define a partial ordering (denoted by $\leq$ or $\leq_P$) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$ while $x \ll y$ will show $y - x \in \operatorname{int} P$, where int P denotes the interior of P. From now on, it is assumed that $\operatorname{int} P \neq \phi$.

Definition 2.3. The cone P is called *normal* if there is a number $K \geq 1$ for which $0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|$ holds for all $x, y \in E$. The least positive integer K, satisfying this equation, is called the normal constant of P.

Definition 2.4. The cone P is said to be *regular* if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \| x_n - x \| = 0$.

Lemma 2.1.

(i) Every regular cone is normal.

(ii) For each $k > 1$, there is a normal cone with normal constant $K > k$.

(iii) The cone P is regular if every decreasing sequence which is bounded from below is convergent.

Proof of (i) and (ii) are given in [6] and the last one follows from definition.

Definition 2.5. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(M_1) $0 \leq d(x, y)$ for all $x, y \in X$,

(M_2) $d(x, y) = 0$ if and only if $x = y$,

(M_3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y \in X$,

(M_4) $d(x, y) = d(y, x)$ for all $x, y \in X$,

then d is called cone metric on X, and the pair (X, d) is called a cone metric space (CMS).

Example 2.1. Let $E = R^3$, $P = \{(x, y, z) \in E : x, y, z \geq 0\}$, and X = R. Define $d : X \times X \to E$ by $d(x, \tilde{x}) = (\alpha|x - \tilde{x}|, \beta|x - \tilde{x}|, \gamma|x - \tilde{x}|$, where α, β and γ are positive constants. Then (X, d) is a CMS. Note that the cone P is normal with the normal constant K = 1.

Definition 2.6. Let (X, d) be a CMS, $x \in X$, and $\{x_n\}_{n\geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $d(x_n, x) \ll c$ for all $n \geq N$. It is denoted by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 2.2. [9] Let (X, d) be a CMS, P a normal cone with normal constant K, and $\{x_n\}$ a sequence in X. Then,

(i) the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ (or equivalently $\| d(x_n, x) \| \to 0$),

(ii) the sequence $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ (or equivalently $\| d(x_n, x_m) \| \to 0$ as $m, n \to \infty$),

(iii) the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y, then $d(x_n, y_n) \to d(x, y)$.

Lemma 2.3. [20] Let (X, d) be a CMS over a cone P in E. Then

(1) $\text{int}(P) + \text{int}(P) \subseteq \text{int}(P)$ and $\lambda\text{int}(P) \subseteq \text{int}(P)$, $\lambda > 0$.

(2) If $c \gg 0$, then there exists $\delta > 0$ such that $\| b \| < \delta$ implies that $b \ll c$.

(3) For any given $c \gg 0$ and $c_0 \gg 0$ there exists $n_0 \in N$ such that $c_0/n_0 \ll c$.

(4) If a_n, b_n are sequences in E such that $a_n \to a$, $b_n \to b$, and $a_n \le b_n$, for all n, then $a \le b$.

Definition 2.7. [7] P is called *minihedral cone* if sup{x, y} exists for all $x, y \in E$ and strongly minihedral if every subset of E which is bounded from above has a supremum (equivalently, if every subset of E which is bounded from below has an infimum).

Lemma 2.4.

(i) Every strongly minihedral normal (not necessarily closed) cone is regular.

(ii) Every strongly minihedral (closed) cone is normal.

The proof of (i) is straightforward, and for (ii) see, for example, [22].

Example 2.2. Let E = C[0,1] with the supremum norm and $P = \{f \in E : f \ge 0\}$. Then P is a cone with normal constant M = 1 which is not regular. This is clear, since the sequence x^n is monotonically decreasing but not uniformly convergent to 0. This cone, by Lemma 2.4, is not strongly minihedral. However, it is easy to see that the cone mentioned in Example 2.1 is strongly minihedral.

Definition 2.8. A mapping T on CMS (X, d) is said to be *orbitally continuous* if $\lim_{i\to\infty} T^{n_i}(x) = z$ implies that $\lim_{i\to\infty} T\left(T^{n_i}(x)\right) = Tz.$ A CMS (X, d) is called *T orbitally complete* if every Cauchy sequence of the form $\left\{T^{n_i}(x)\right\}_{i=1}^{\infty}$, $x \in X$, converges in (X, d).

Remark 2.1. It is clear that orbital continuity of T implies orbital continuity of T^m for any $m \in N$.

Definition 2.9. A point z is said to be a *periodic point* of function T of period m if $T^m(z) = z,$ where $T^0(x) = x$ and $T^m(x)$ is defined recursively by $T^m(x) = T\left(T^{m-1}(x)\right).$

3. Main Result.

Theorem 3.1. Let $T : X \to X$ be an orbitally continuous mapping on CMS (X, d) over strongly minihedral normal cone P. Suppose that CMS (X, d) is T orbitally complete and that T satisfies the condition

$$\text{(3.1)} \quad \min\{[d(Tx, Ty)]^2, d(x, y)\, d(Tx, Ty), [d(y, Ty)]^2\} - \min\{d(x, Tx)\, d(y, Ty), d(x, Ty)\, d(y, Tx)\} \le q d(x, Tx)\, d(y, Ty)$$

for all $x, y \in X$ and $q \in (0, 1)$, then for each $x \in M$, the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges to a fixed point of T.

Proof. Fix $x_0 \in X$.

For $n \ge 1$, set $x_1 = T(x_0)$ and recursively $x_{n+1} = T(x_n) = T^{n+1}(x_0)$.

It is clear that the sequence $\{x_n\}$ is Cauchy when the equation $x_{n+1} = x_n$ holds for some $n \in N$. Consider the case $x_{n+1} \ne x_n$ for all $n \in N$.

By replacing x and y with x_{n-1} and x_n, respectively, in (3.1), we get

$$\min\{[d(Tx_{n-1}, Tx_n)]^2, d(x_{n-1}, x_n)\, d(Tx_{n-1}, Tx_n), [d(x_n, Tx_n)]^2\} - \min\{d(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_n), d(x_{n-1}, Tx_n)\, d(x_n, Tx_{n-1})\} \le q d(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_n)$$

or
$$\min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), [d(x_n, x_{n+1})]^2\} - \min\{d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), 0\} \le q d(x_{n-1}, x_n)\, d(x_n, x_{n+1})$$

i.e.
$$\min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)\, d(x_n, x_{n+1})\} \le q d(x_{n-1}, x_n)\, d(x_n, x_{n+1}).$$

Since $d(x_{n-1}, x_n)\, d(x_n, x_{n+1}) \le q d(x_{n-1}, x_n)\, d(x_n, x_{n+1})$ is impossible (as $q < 1$), we have

$$d(x_n, x_{n+1}) \le q d(x_{n-1}, x_n).$$

Recursively, we get

$$d(x_n, x_{n+1}) \le q d(x_{n-1}, x_n) \le q^2 d(x_{n-2}, x_{n-1}) \le \ldots \le q^n d(x_0, Tx_0).$$

Hence, by using the triangle inequality, for any $p \in I^+$ one has

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, T(x_{n+p}))$$

$$\leq \left(q^n + q^{n+1} + \ldots + q^{n+p-1}\right) d(x_0, T(x_0))$$

$$= q^n \left(1 + q + \ldots + q^{p-1}\right) d(x_0, T(x_0))$$

$$\leq \frac{q^n}{1-q} d(x_0, T(x_0)).$$

Let $c \in \text{int}(P)$. Choose a natural number M_0 such that

$$\left(\frac{q^n}{1-q}\right) d(x_0, T(x_0)) \ll c \text{ for all } n > M_0.$$

Thus, for any $p \in N$,

$$d(x_{n+p}, x_n) \ll c \text{ for all } n > M_0.$$

So $\{x_n\}$ is a Cauchy sequence in (X, d).

Since (X, d) is T orbitally complete, there is some $z \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = z.$$

Regarding the orbital continuity of T,

$$T(z) = \lim_{n\to\infty} T\left(T^n(x_0)\right) = \lim_{n\to\infty} T^{n+1}(x_0) = z,$$

that is, z is a fixed point of T.

Theorem 3.2. Let $B = B(x_0, r)\{x \in M \mid d(x_0, x) \leq r\}$ where (X, d) is a orbitally complete cone metric space. Let T be an orbitally continuous mapping of B into M and satisfies (3.1) for $x, y \in B$ and

(3.2) $$d(x_0, Tx_0) \leq (1 - q)r.$$

Then T has a fixed point.

Proof. By (3.2), we have

$$x_1 = Tx_0 \in B(x_0, r)$$

and by (3.1) for $x = x_0$ and $y = x_1$, we have

$$\min\{[d(Tx_0, Tx_1)]^2, d(x_0, x_1)\, d(Tx_0, Tx_1), [d(x_1, Tx_1)]^2\}$$

$$- \min\{d(x_0, Tx_0)\, d(x_1, Tx_1), d(x_0, Tx_1)\, d(x_1, Tx_0)\}$$

$$\leq qd(x_0, Tx_0)\, d(x_1, Tx_1)$$

or $\min\{[d(x_1, x_2)]^2, d(x_0, x_1)\, d(x_1, x_2), [d(x_1, x_2)]^2\}$

$$- \min\{d(x_0, x_1)\, d(x_1, x_2), d(x_0, x_2)\, d(x_1, x_2)\}$$

$$\leq qd(x_0, x_1)\, d(x_1, x_2)$$

which implies

$$d(x_1, x_2) \leq qd(x_0, x_1) \leq q(1 - q)r.$$

Hence

$$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$$

$$\leq (1 - q)r + q(1 - q)r = (1 + q)(1 - q)r.$$

Suppose that

$$d(x_0, x_n) \leq (1 + q + \ldots + q^{n-1})(1 - q)r$$

and $d(x_{n-1}, x_n) \leq q^{n-1}(1 - q)r.$

Then by (3.1) for $x = x_{n-1}$ and $y = x_n$, we have

$$\min\{[d(Tx_{n-1}, Tx_n)]^2, d(x_{n-1}, x_n)\, d(Tx_{n-1}, Tx_n), [d(x_n, Tx_n)]^2\}$$

$$- \min\{d(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_1), d(x_{n-1}, Tx_n)\, d(x_n, Tx_{n-1})\}$$

$$\leq qd(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_n)$$

or $\min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), [d(x_n, x_{n+1})]^2\}$

$$- \min\{d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\, d(x_n, x_n)\}$$

$$\leq q d(x_{n-1}, x_n)\, d(x_n, x_{n+1})$$

which implies

$$d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n) \leq q^n(1 - q)r.$$

Therefore,

$$\begin{aligned} d(x_0, x_{n+1}) &\leq d(x_0, x_n) + d(x_n, x_{n+1}) \\ &\leq (1 + q + \ldots + q^{n-1})(1 - q)r + q^n(1 - q)r \\ &= (1 + q + \ldots + q^{n-1})(1 - q)r \leq r. \end{aligned}$$

Thus, the sequence x_0, $x_{n+1} = Tx_n$, $n \geq 0$ is contained in B.

Also

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \ldots + d(x_{m-1}, x_m) \\ &\leq (q^n + \ldots + q^{m-1})(1 - q)r \leq q^n r \to 0. \end{aligned}$$

Since B is also orbitally complete, so $u = \lim_{n\to\infty} T^n(x)$ for some $u \in B$. By orbital continuity of T, we have

$$Tu = \lim_{n\to\infty} T\left(T^n(x)\right) = u.$$

Thus, u is a fixed point of T.

This completes the proof of the theorem.

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