Available online at http://scik.org
Adv. Fixed Point Theory, 2022, 12:12
https://doi.org/10.28919/afpt/7715
ISSN: 1927-6303

DYNAMICAL SAMPLING IN $\ell^2(\mathbb{Z})^{(d)}$ AND VECTOR SHIFT-INVARIANT SPACES

ZHENLI GUO*, YUE SU

Institute of Operations Research and Systems Engineering, College of Science, Tianjin University of Technology, Tianjin 300384, China

Abstract. Dynamical sampling problem in which we seek to reconstruct a sequence or function in evolutionary systems via spatiotemporal trade-off. In this paper, we present the problem of dynamical sampling in $\ell^2(\mathbb{Z})^{(d)}$ and vector shift-invariant spaces. We first give a sufficient and necessary condition under which $c \in \ell^2\left(\mathbb{Z}^d\right)^{(d)}$ can be recovered by its spatial and temporal samples. Then to illustrate our main result, we give two examples to show that the sufficient and necessary condition in the main result is feasible. Finally, we study the dynamical sampling problem in vector shift-invariant spaces.

Keywords: sampling theory; dynamical sampling; vector shift-invariant spaces; $\ell^2(\mathbb{Z})^{(d)}$.

2010 AMS Subject Classification: 42C15, 42C40, 41A58.

1. Introduction

The sampling theorem has attracted wide attention since its publication. It plays an important role in digital signal processing, digital communication, and other fields. There are many types of sampling theorem, such as Shannon sampling theorem [1, 2], sampling theorem in shift invariant spaces [3], average sampling theorem [4, 5, 6], multi-channel sampling theorem [7,

*Corresponding author
E-mail address: guozhenli98@163.com
Received September 05, 2022

8, 9, 10], vector sampling theorem [12], compressed sampling [13, 14, 15, 16] and dynamic sampling [17, 18, 19, 20, 21, 22, 23] so on.

The sampling theory is very important because it is a link between the real digital world and the continuous function analog world. Converting a continuous function into a sequence of real or complex is the beginning of many digital signal processing. However, in many applications, only the function f that is sampled is not enough. To reduce costs, dynamic sampling was proposed by Aldroubi and his collaborators. Dynamic sampling is a new sampling way since it recovers a sequence or function in evolutionary systems via spatiotemporal trade-off. Specifically, dynamic sampling is not only the sequence or function that is sampled but also its various states at different times.

Aldroubi et al mainly studied the uniform dynamical sampling on infinite-dimensional separable Hilbert spaces $\ell^2(\mathbb{Z})$ [19], finite-dimensional spaces[17] and shift-invariant spaces[18, 21]. And Zhang, Li, and Liu discussed periodic nonuniform dynamical sampling in $\ell^2(\mathbb{Z})$ and shift-invariant spaces [22]. For high dimensional cases, Zhang and Li provided a sufficient and necessary condition for dynamical sampling [23].

Vector sampling is motivated by applications in multichannel deconvolution and multiple source separation. It appears in many practical applications. For example, in multiuser or multi-access wireless communications and space-time coding with antenna arrays or telephone digital subscriber loops [24, 25, 26, 27], multitrack magnetic recording [28], multisensor biomedical signals [29, 30], multiple speaker (or other acoustic source) separation with microphone arrays [31, 32], geophysical data processing [33], and multichannel image restoration [34, 35]. To the best of our knowledge, there are no results published about vector dynamical sampling. We want to fill this gap.

In this paper, we mainly present the dynamical sampling in $\ell^2(\mathbb{Z})^{(d)}$ and vector shift-invariant spaces. We give that how to recover $c \in \ell^2(\mathbb{Z})^{(d)}$ or $f \in V(\phi)$ from the measurements. Here $V(\phi)$ is a vector shift-invariant space (see (4.1)). Our results generalize similar ones for the scalar dynamical sampling.

2. Preliminaries

Here and after, $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integer numbers. $\mathbb{R}^d$ is the d-dimensional Euclidean space. We use $\mathscr{F}$ to denote the Fourier transform operator. For any $c \in \ell^2(\mathbb{Z})$, define the discrete Fourier transform

$$\mathscr{F}(c) = \hat{c}(\xi) = \sum_{k \in \mathbb{Z}} c_k e^{-2\pi i k \xi}.$$

Let $\hat{f}(\xi)$ denote the Fourier transform of $f \in L^2(\mathbb{R})$:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i2\pi \xi x} dx.$$

Let super Hilbert space $\ell^2(\mathbb{Z})^{(d)} := \ell^2(\mathbb{Z}) \times \cdots \times \ell^2(\mathbb{Z})$. Let $C = (c_1, c_2, \cdots, c_d)^T \in \ell^2(\mathbb{Z})^{(d)}$. We define $\widehat{C} = (\widehat{c}_1, \widehat{c}_2, \cdots, \widehat{c}_d)^T$.

Let super Hilbert space $L^2(\mathbb{R})^{(d)} := L^2(\mathbb{R}) \times \cdots \times L^2(\mathbb{R})$. Let $F = (f_1, f_2, \cdots, f_d)^T \in L^2(\mathbb{R})^{(d)}$. We define $\widehat{F} = \left(\widehat{f}_1, \widehat{f}_2, \cdots, \widehat{f}_d\right)^T$.

Given $f = (f_1, f_2, \cdots, f_d)^T$, $g = (g_1, g_2, \cdots, g_d)^T \in L^2(\mathbb{R})^{(d)}$, we define the inner product of f and g by

$$\langle f, g \rangle_{L^2(\mathbb{R})^{(d)}} = \int_{\mathbb{R}} \sum_{q=1}^{d} f_q(x) \overline{g_q(x)} dx.$$

The norm of f is defined by $\|f\|_{L^2(\mathbb{R})^{(d)}} = \sqrt{\langle f, f \rangle_{L^2(\mathbb{R})^{(d)}}}$.

A measurable function f belongs to the Wiener amalgam space $W(L^p) := W(L^p(\mathbb{R}))$, $1 \le p < \infty$, if it satisfies

$$\|f\|^p_{W(L^p)} := \sum_{k \in \mathbb{Z}} \operatorname{esssup}\{|f(x+k)|^p; x \in [0,1]\} < \infty.$$

It is easy to see that $W(L^1) \subseteq L^2(\mathbb{R})$. Because ideal sampling makes sense only for continuous functions, we work in the amalgam spaces

$$W_0(L^p(\mathbb{R})) := W(L^p(\mathbb{R})) \cap C(\mathbb{R}).$$

Three kinds of convolution will be used throughout this paper. For any $f, g \in L^2(\mathbb{R})$, define their convolution by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y) dy.$$

For any $c,d\in\ell^{2}(\mathbb{Z})$, define the discrete convolution by

$$(c*_{\mathbb{Z}}d)(l)=\sum_{k\in\mathbb{Z}}c(k)d(l-k).$$

For any $c\in\ell^{2}(\mathbb{Z}), f\in L^{2}(\mathbb{R})$, define the semi-discrete convolution by

$$(c*_{sd}f)(x)=\sum_{k\in\mathbb{Z}}c(k)f(x-k).$$

Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d}\\ a_{21} & a_{22} & \cdots & a_{2d}\\ \vdots & \vdots & \ddots & \vdots\\ a_{d1} & a_{d2} & \cdots & a_{dd}\end{pmatrix},$$

where $a_{ij}\in\ell^{2}(\mathbb{Z}), 1\le i\le d, 1\le j\le d$ or $a_{ij}\in W_{0}(L^{1}), 1\le i\le d, 1\le j\le d$. The Fourier transform of A is

$$\widehat{A}=\begin{pmatrix} \widehat{a_{11}} & \widehat{a_{12}} & \cdots & \widehat{a_{1d}}\\ \widehat{a_{21}} & \widehat{a_{22}} & \cdots & \widehat{a_{2d}}\\ \vdots & \vdots & \ddots & \vdots\\ \widehat{a_{d1}} & \widehat{a_{d2}} & \cdots & \widehat{a_{dd}}\end{pmatrix}.$$

The definition of A^{j} is $A^{j}=\underbrace{A*_{\mathbb{Z}}A*_{\mathbb{Z}}\cdots*_{\mathbb{Z}}A}_{j}$ or $A^{j}=\underbrace{A*A*\cdots*A}_{j}$. Here

$$A*_{\mathbb{Z}}A=\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1d}\\ c_{21} & c_{22} & \cdots & c_{2d}\\ \vdots & \vdots & \ddots & \vdots\\ c_{d1} & c_{d2} & \cdots & c_{dd}\end{pmatrix},$$

where $c_{i,j}=a_{i1}*_{\mathbb{Z}}a_{1j}+a_{i2}*_{\mathbb{Z}}a_{2j}+...+a_{id}*_{\mathbb{Z}}a_{dj}, 1\le i\le d, 1\le j\le d$. And

$$A*A=\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1d}\\ c_{21} & c_{22} & \cdots & c_{2d}\\ \vdots & \vdots & \ddots & \vdots\\ c_{d1} & c_{d2} & \cdots & c_{dd}\end{pmatrix},$$

where $c_{i,j}=a_{i1}*a_{1j}+a_{i2}*a_{2j}+...+a_{id}*a_{dj}, 1\le i\le d, 1\le j\le d$.

Semi-discrete convolution between A and C is

$$\begin{aligned} A *_{sd} C &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dd} \end{pmatrix} *_{sd} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix} \\ &= \begin{pmatrix} a_{11} *_{sd} c_1 + a_{12} *_{sd} c_2 + \ldots + a_{1d} *_{sd} c_d \\ a_{21} *_{sd} c_1 + a_{22} *_{sd} c_2 + \ldots + a_{2d} *_{sd} c_d \\ \vdots \\ a_{d1} *_{sd} c_1 + a_{d2} *_{sd} c_2 + \ldots + a_{dd} *_{sd} c_d \end{pmatrix}. \end{aligned}$$

Take Fourier transform at both sides of above formula, then

$$(\widehat{A *_{sd} C}) = \widehat{A}\widehat{C}. \tag{2.1}$$

The following is the Poisson sum formula [36, Proposition 1.4.2] which is useful in sampling theory. If $\sum_{k\in\mathbb{Z}} \hat{f}(\xi + k) \in L^2(\mathbb{T})$ and $\sum_{k\in\mathbb{Z}} |f(k)|^2 < \infty$, then

$$\sum_{k\in\mathbb{Z}} \hat{f}(\xi + k) = \sum_{k\in\mathbb{Z}} f(k) e^{-2\pi i k \xi}, \quad a.e.\ \xi \in \mathbb{R}.$$

For some fixed factor $m \in \mathbb{N}$, the subsampling operator

$$S_m : \ell^2(\mathbb{Z})^{(d)} \longrightarrow \ell^2(\mathbb{Z})^{(d)}$$

is defined by $(S_m C)(k) = C(mk), k \in \mathbb{Z}^d$.

3. Dynamical Sampling in $\ell^2(\mathbb{Z})^{(d)}$

3.1. Main result. The following is our main result in this section.

Theorem 3.1. *Let E be the unit matrix and*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dd} \end{pmatrix},$$

where $\widehat{a_{ij}} \in L^\infty(\mathbb{T}), i,j = 1,...d$. *Define*

$$\mathscr{A}_m(\xi) = \begin{pmatrix} E & E & \cdots & E \\ \widehat{A}(\frac{\xi}{m}) & \widehat{A}(\frac{\xi+1}{m}) & \cdots & \widehat{A}(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{A}^{m-1}(\frac{\xi}{m}) & \widehat{A}^{m-1}(\frac{\xi+1}{m}) & \cdots & \widehat{A}^{m-1}(\frac{\xi+m-1}{m}) \end{pmatrix} \tag{3.1}$$

$\xi \in \mathbb{T}$. *Then a vector* $x \in \ell^2(\mathbb{Z})^{(d)}$ *can be recovered in a stable way , i.e. the inverse is bounded, from the measurements* $y_n = S_m A^n x, n = 0,1,...,m-1$, *if and only if there exists* $\alpha > 0$ *such that the set* $\{\xi : |det\mathscr{A}_m(\xi)| < \alpha\}$ *has zero measure.*

Proof. Since

$$\begin{aligned} \frac{1}{m}\sum_{l=0}^{m-1}\widehat{C}\left(\frac{\xi+l}{m}\right) &= \frac{1}{m}\sum_{l=0}^{m-1}\sum_{n\in\mathbb{Z}} C(n)e^{\frac{-2\pi in(\xi+l)}{m}} \\ &= \sum_{n\in\mathbb{Z}} C(n)e^{\frac{-2\pi in\xi}{m}}\frac{1}{m}\sum_{l=0}^{m-1}e^{\frac{-2\pi inl}{m}} \\ &= \sum_{n=mk} C(mk)e^{\frac{-2\pi imk\xi}{m}} \\ &= \sum_{k\in\mathbb{Z}} C(mk)e^{-2\pi ik\xi}, \quad \xi\in[0,1], \end{aligned}$$

we have

$$\widehat{(S_mC)}(\xi) = \frac{1}{m}\sum_{l=0}^{m-1}\widehat{C}\left(\frac{\xi+l}{m}\right).$$

Here the third equality is obtained by

$$\sum_{l=0}^{m-1} e^{\frac{-2\pi inl}{m}} = \begin{cases} m, & n = 0 \bmod m, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Writing the above formula in matrix form, we have

$$\widehat{(S_mC)}(\xi) = \frac{1}{m}\sum_{l=0}^{m-1}\widehat{C}(\frac{\xi+l}{m}) = \frac{1}{m}\begin{pmatrix} E & E & \cdots & E \end{pmatrix}\begin{pmatrix} \widehat{C}(\frac{\xi}{m}) \\ \widehat{C}(\frac{\xi+1}{m}) \\ \vdots \\ \widehat{C}(\frac{\xi+m-1}{m}) \end{pmatrix}. \tag{3.3}$$

Replacing the C in above formula with the $A *_{\mathbb{Z}} C$, we have

$$\begin{aligned}
\mathscr{F}[S_m(A *_{\mathbb{Z}} C)](\xi) &= \frac{1}{m}\sum_{l=0}^{m-1}(\widehat{A *_{\mathbb{Z}} C})\left(\frac{\xi+l}{m}\right) \\
&= \frac{1}{m}\sum_{l=0}^{m-1}\widehat{A}\left(\frac{\xi+l}{m}\right)\widehat{C}\left(\frac{\xi+l}{m}\right) \\
&= \frac{1}{m}\begin{pmatrix} \widehat{A}(\frac{\xi}{m}) & \widehat{A}(\frac{\xi+1}{m}) & \cdots & \widehat{A}(\frac{\xi+m-1}{m}) \end{pmatrix}\begin{pmatrix} \widehat{C}(\frac{\xi}{m}) \\ \widehat{C}(\frac{\xi+1}{m}) \\ \vdots \\ \widehat{C}(\frac{\xi+m-1}{m}) \end{pmatrix}.
\end{aligned} \tag{3.4}$$

Similarly, for any $2 \leq j \leq m-1$, we have

$$\begin{aligned}
\mathscr{F}[S_m(A^j *_{\mathbb{Z}} C)](\xi) &= \frac{1}{m}\sum_{l=0}^{m-1}\left(\widehat{A^j *_{\mathbb{Z}} C}\right)\left(\frac{\xi+l}{m}\right) \\
&= \frac{1}{m}\sum_{l=0}^{m-1}\widehat{A}^j\left(\frac{\xi+l}{m}\right)\widehat{C}\left(\frac{\xi+l}{m}\right) \\
&= \frac{1}{m}\begin{pmatrix} \widehat{A}^j(\frac{\xi}{m}) & \widehat{A}^j(\frac{\xi+1}{m}) & \cdots & \widehat{A}^j(\frac{\xi+m-1}{m}) \end{pmatrix}\begin{pmatrix} \widehat{C}(\frac{\xi}{m}) \\ \widehat{C}(\frac{\xi+1}{m}) \\ \vdots \\ \widehat{C}(\frac{\xi+m-1}{m}) \end{pmatrix}.
\end{aligned} \tag{3.5}$$

Writing (3.3), (3.4) and (3.5) in block matrix form, we have

$$\begin{pmatrix} \mathscr{F}(S_mC)(\xi) \\ \mathscr{F}[S_m(A *_{\mathbb{Z}} C)](\xi) \\ \vdots \\ \mathscr{F}[S_m(A^{m-1} *_{\mathbb{Z}} C)](\xi) \end{pmatrix} = \frac{1}{m}\begin{pmatrix} E & E & \cdots & E \\ \widehat{A}(\frac{\xi}{m}) & \widehat{A}(\frac{\xi+1}{m}) & \cdots & \widehat{A}(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{A}^{m-1}(\frac{\xi}{m}) & \widehat{A}^{m-1}(\frac{\xi+1}{m}) & \cdots & \widehat{A}^{m-1}(\frac{\xi+m-1}{m}) \end{pmatrix} \times \begin{pmatrix} \widehat{C}(\frac{\xi}{m}) \\ \widehat{C}(\frac{\xi+1}{m}) \\ \vdots \\ \widehat{C}(\frac{\xi+m-1}{m}) \end{pmatrix}.$$

Let

$$\mathrm{y}(\xi)=\begin{pmatrix} m\mathscr{F}(S_mC)(\xi) \\ m\mathscr{F}[S_m(A*_{\mathbb{Z}}C)](\xi) \\ \vdots \\ m\mathscr{F}[S_m(A^{m-1}*_{\mathbb{Z}}C)](\xi) \end{pmatrix}$$

and

$$\mathscr{C}(\xi)=\begin{pmatrix} \widehat{C}(\frac{\xi}{m}) \\ \widehat{C}(\frac{\xi+1}{m}) \\ \vdots \\ \widehat{C}(\frac{\xi+m-1}{m}) \end{pmatrix}, \quad \widehat{C}(\xi)\in L^2(\mathbb{T})^{(d)}.$$

Then we have

$$\mathrm{y}(\xi)=\mathscr{A}_m(\xi)\mathscr{C}(\xi). \tag{3.6}$$

We can solve the equation (3.6) with respect to $\mathscr{C}(\xi)$ (which we use to produce C) if $\mathscr{A}_m(\xi)$ is invertible. Since there exists $\alpha>0$ such that the set $\{\xi:|det\mathscr{A}_m(\xi)|<\alpha\}$ has zero measure, the $\mathscr{A}_m(\xi)$ has a bounded inverse. Thus a vector $x\in\ell^2(\mathbb{Z})^{(d)}$ can be recovered in a stable way from the measurements $y_n=S_mA^nx, n=0,1,...,m-1$. □

3.2. **Auxiliary lemma.** Now, we introduce a lemma about Vandermonde block matrix.

Lemma 3.2. *Let E be the unit matrix of order k, and $A_i\,(i=1,2,...,n)$ be square matrices of order k. Here k is any natural number. If for any $i,j=1,...,n$ have $A_iA_j=A_jA_i$. Then*

$$\begin{vmatrix} E & E & \cdots & E \\ A_1 & A_2 & \cdots & A_n \\ A_1^2 & A_2^2 & \cdots & A_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ A_1^{n-1} & A_2^{n-1} & \cdots & A_n^{n-1} \end{vmatrix}=\prod_{1\le j<i\le n}|A_i-A_j| \tag{3.7}$$

Proof. Use mathematical method of induction for n. When $n=2$, we have

$$\begin{pmatrix} E & 0 \\ -A_1 & E \end{pmatrix}\begin{pmatrix} E & E \\ A_1 & A_2 \end{pmatrix}=\begin{pmatrix} E & E \\ 0 & A_2-A_1 \end{pmatrix}. \tag{3.8}$$

Take the determinant of both sides

$$\begin{vmatrix} E & E \\ A_1 & A_2 \end{vmatrix} = |A_2 - A_1|.$$

Therefore (3.7) is true for $n = 2$.

Suppose that (3.7) holds for $n-1$. By multiplication of block matrices, we have the following matrix equations

$$\begin{pmatrix} E & E & E & \cdots & E \\ 0 & A_2 - A_1 & A_3 - A_1 & \cdots & A_n - A_1 \\ 0 & (A_2 - A_1)A_2 & (A_3 - A_1)A_3 & \cdots & (A_n - A_1)A_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (A_2 - A_1)A_2^{n-2} & (A_3 - A_1)A_3^{n-2} & \cdots & (A_n - A_1)A_n^{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} E & 0 & 0 & \cdots & 0 \\ -A_1 & E & 0 & \cdots & 0 \\ 0 & 0 & E & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & E \end{pmatrix} \begin{pmatrix} E & 0 & 0 & \cdots & 0 \\ 0 & E & 0 & \cdots & 0 \\ 0 & -A_1 & E & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & E \end{pmatrix} \cdots$$

$$\begin{pmatrix} E & 0 & 0 & \cdots & 0 & 0 \\ 0 & E & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E & 0 \\ 0 & 0 & 0 & \cdots & -A_1 & E \end{pmatrix} \begin{pmatrix} E & E & \cdots & E \\ A_1 & A_2 & \cdots & A_n \\ A_1^2 & A_2^2 & \cdots & A_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ A_1^{n-1} & A_2^{n-1} & \cdots & A_n^{n-1} \end{pmatrix}$$

Take the determinant of both sides, we have

$$\begin{vmatrix} E & E & \cdots & E \\ A_1 & A_2 & \cdots & A_n \\ A_1^2 & A_2^2 & \cdots & A_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ A_1^{n-1} & A_2^{n-1} & \cdots & A_n^{n-1} \end{vmatrix} = \begin{vmatrix} A_2 - A_1 & A_3 - A_1 & \cdots & A_n - A_1 \\ (A_2 - A_1)A_2 & (A_3 - A_1)A_3 & \cdots & (A_n - A_1)A_n \\ (A_2 - A_1)A_2^2 & (A_3 - A_1)A_3^2 & \cdots & (A_n - A_1)A_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ (A_2 - A_1)A_2^{n-2} & (A_3 - A_1)A_3^{n-2} & \cdots & (A_n - A_1)A_n^{n-2} \end{vmatrix}. \tag{3.9}$$

Since $A_1, A_2, A_3, ..., A_n$ are interchangeable, we can see that $A_j - A_1$ and $A_j^i (j = 2, 3, ..., n; i = 1, 2, ..., n-2)$ are also interchangeable. Hence, we have

$$\begin{vmatrix} A_2 - A_1 & \cdots & A_n - A_1 \\ (A_2 - A_1)A_2 & \cdots & (A_n - A_1)A_n \\ (A_2 - A_1)A_2^2 & \cdots & (A_n - A_1)A_n^2 \\ \vdots & \vdots & \vdots \\ (A_2 - A_1)A_2^{n-2} & \cdots & (A_n - A_1)A_n^{n-2} \end{vmatrix} = \begin{vmatrix} E(A_2 - A_1) & \cdots & E(A_n - A_1) \\ A_2(A_2 - A_1) & \cdots & A_n(A_n - A_1) \\ A_2^2(A_2 - A_1) & \cdots & A_n^2(A_n - A_1) \\ \vdots & \vdots & \vdots \\ A_2^{n-2}(A_2 - A_1) & \cdots & A_n^{n-2}(A_n - A_1) \end{vmatrix}$$

$$= \left| \begin{pmatrix} E & \cdots & E \\ A_2 & \cdots & A_n \\ A_2^2 & \cdots & A_n^2 \\ \vdots & \vdots & \vdots \\ A_2^{n-2} & \cdots & A_n^{n-2} \end{pmatrix} \operatorname{diag}(A_2 - A_1, A_3 - A_1, \cdots, A_n - A_1) \right|$$

$$= \begin{vmatrix} E & \cdots & E \\ A_2 & \cdots & A_n \\ A_2^2 & \cdots & A_n^2 \\ \vdots & \vdots & \vdots \\ A_2^{n-2} & \cdots & A_n^{n-2} \end{vmatrix} |A_2 - A_1||A_3 - A_1| \cdots |A_n - A_1|. \tag{3.10}$$

According to the hypothesis of induction, we get

$$\begin{vmatrix} E & \cdots & E \\ A_2 & \cdots & A_n \\ A_2^2 & \cdots & A_n^2 \\ \vdots & \vdots & \vdots \\ A_2^{n-2} & \cdots & A_n^{n-2} \end{vmatrix} = \prod_{2 \le j < i \le n} |A_i - A_j|. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), we get that (3.7) is also true for n. Therefore, (3.7) is true for any natural number n according to the principle of induction. This completes the proof. □

3.3. Examples of dynamical sampling in $\ell^2(\mathbb{Z})^{(d)}$. In this section, we give two examples of sampling sequences to illustrate the main results.

Example 3.3. *For simplicity, let $d=2$, $\widehat{a_{11}}(\xi)=\widehat{a_{22}}(\xi)=\xi-[\xi]$, where $[\xi]$ stands for the largest integer no more than ξ. Suppose that*

$$\widehat{A}\left(\frac{\xi}{2}\right)=\begin{pmatrix}\widehat{a_{11}}\left(\frac{\xi}{2}\right) & 0\\ 0 & \widehat{a_{22}}\left(\frac{\xi}{2}\right)\end{pmatrix}$$

and

$$\widehat{A}\left(\frac{\xi+1}{2}\right)=\begin{pmatrix}\widehat{a_{11}}\left(\frac{\xi+1}{2}\right) & 0\\ 0 & \widehat{a_{22}}\left(\frac{\xi+1}{2}\right)\end{pmatrix}.$$

Obviously, $\hat{A}(\frac{\xi}{2})$ and $\hat{A}(\frac{\xi+1}{2})$ are commutative. By Lemma 3.2, for $0\le\xi<2$, we have

$$\begin{aligned}
|\mathscr{A}_m(\xi)| &= \begin{vmatrix} E & E\\ \widehat{A}\left(\frac{\xi}{2}\right) & \widehat{A}\left(\frac{\xi+1}{2}\right)\end{vmatrix}\\
&= \begin{vmatrix}\widehat{a_{11}}\left(\frac{\xi}{2}\right)-\widehat{a_{11}}\left(\frac{\xi+1}{2}\right) & 0\\ 0 & \widehat{a_{22}}\left(\frac{\xi}{2}\right)-\widehat{a_{22}}\left(\frac{\xi+1}{2}\right)\end{vmatrix}\\
&= \left(\widehat{a_{11}}\left(\frac{\xi}{2}\right)-\widehat{a_{11}}\left(\frac{\xi+1}{2}\right)\right)\left(\widehat{a_{22}}\left(\frac{\xi}{2}\right)-\widehat{a_{22}}\left(\frac{\xi+1}{2}\right)\right)\\
&= \left(\frac{\xi}{2}-\left[\frac{\xi}{2}\right]-\frac{\xi+1}{2}+\left[\frac{\xi+1}{2}\right]\right)^2\\
&= \left(-\frac{1}{2}-\left[\frac{\xi}{2}\right]+\left[\frac{\xi+1}{2}\right]\right)^2\\
&= \frac{1}{4}\neq 0.
\end{aligned}$$

Therefore, $\mathscr{A}_m(\xi)$ is reversible and $\mathscr{A}_m(\xi)$ satisfies the condition of Theorem 3.1.

Example 3.4. *Let $d=2$, $\widehat{a_{11}}(\xi)=\widehat{a_{22}}(\xi)=cos(2\pi\xi)$, $\widehat{a_{21}}(\xi)=-sin(2\pi\xi)$, $\widehat{a_{12}}(\xi)=sin(2\pi\xi)$.*

It is easy to see

$$\widehat{a_{11}}\left(\frac{\xi+1}{2}\right)=cos\left(2\pi\frac{\xi+1}{2}\right)=cos(\pi\xi+\pi)=-cos(\pi\xi)=-\widehat{a_{11}}(\frac{\xi}{2}),$$

$\widehat{a_{21}}(\frac{\xi+1}{2}) = -sin(\pi\xi+\pi) = sin(\pi\xi) = -\widehat{a_{21}}(\frac{\xi}{2})$ and $\widehat{a_{12}}(\frac{\xi+1}{2}) = -sin(\pi\xi) = -\widehat{a_{12}}(\frac{\xi}{2})$. Then

$$\widehat{A}\left(\frac{\xi+1}{2}\right) = \begin{pmatrix} \widehat{a_{11}}\left(\frac{\xi+1}{2}\right) & \widehat{a_{12}}\left(\frac{\xi+1}{2}\right) \\ \widehat{a_{21}}\left(\frac{\xi+1}{2}\right) & \widehat{a_{22}}\left(\frac{\xi+1}{2}\right) \end{pmatrix} = \begin{pmatrix} -\widehat{a_{11}}\left(\frac{\xi}{2}\right) & -\widehat{a_{12}}\left(\frac{\xi}{2}\right) \\ -\widehat{a_{21}}\left(\frac{\xi}{2}\right) & -\widehat{a_{22}}\left(\frac{\xi}{2}\right) \end{pmatrix} = -\widehat{A}\left(\frac{\xi}{2}\right)$$

and $\widehat{A}(\frac{\xi}{2})$ and $\widehat{A}(\frac{\xi+1}{2})$ are commutative. By Lemma 3.2, for $\xi \in \mathbb{R}$. We have

$$\begin{aligned} |\mathscr{A}_m(\xi)| &= \begin{vmatrix} E & E \\ \widehat{A}\left(\frac{\xi}{2}\right) & \widehat{A}\left(\frac{\xi+1}{2}\right) \end{vmatrix} \\ &= \begin{vmatrix} \widehat{a_{11}}\left(\frac{\xi}{2}\right) - \widehat{a_{11}}\left(\frac{\xi+1}{2}\right) & \widehat{a_{12}}\left(\frac{\xi}{2}\right) - \widehat{a_{12}}\left(\frac{\xi+1}{2}\right) \\ \widehat{a_{21}}\left(\frac{\xi}{2}\right) - \widehat{a_{21}}\left(\frac{\xi+1}{2}\right) & \widehat{a_{22}}\left(\frac{\xi}{2}\right) - \widehat{a_{22}}\left(\frac{\xi+1}{2}\right) \end{vmatrix} \\ &= \begin{vmatrix} 2cos(\pi\xi) & 2sin(\pi\xi) \\ -2sin(\pi\xi) & 2cos(\pi\xi) \end{vmatrix} \\ &= 4 \neq 0. \end{aligned}$$

Therefore, $\mathscr{A}_m(\xi)$ is reversible and $\mathscr{A}_m(\xi)$ satisfies the condition of Theorem 3.1.

4. Dynamical Sampling in Vector Shift-Invariant Spaces

4.1. Vector shift-invariant spaces.

Let

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1d} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{d1} & \phi_{d2} & \cdots & \phi_{dd} \end{pmatrix},$$

where $\phi_{ij} \in W_0(L^1), 1 \leq i \leq d, 1 \leq j \leq d$, and ϕ satisfies (4.2). A shift-invariant space (SIS) generated by one generator ϕ has the form

$$V(\phi) = \left\{ \sum_{k\in\mathbb{Z}} \phi(\cdot - k)C(k) : C \in \ell^2(\mathbb{Z})^{(d)} \right\}. \tag{4.1}$$

By following Theorem 4.1, $V(\phi)$ is well-defined. Denote $\sigma(A)$ is the spectrum of matrix A.

Theorem 4.1. *If there exist constants $a,\ b > 0$ such that*

$$a \leq \lambda(\xi) \leq b,\ \forall\ \lambda(\xi) \in \sigma\left(\sum_{k\in\mathbb{Z}} \overline{\hat{\phi}(\xi+k)}^T \hat{\phi}(\xi+k) \right), \quad \xi \in \mathbb{T}, \tag{4.2}$$

then

$$\sqrt{a}\|C\|_{\ell^2(\mathbb{Z})^{(d)}} \le \left\|\sum_{k\in\mathbb{Z}} \phi(\cdot-k)C(k)\right\|_{L^2(\mathbb{R})^{(d)}} \le \sqrt{b}\|C\|_{\ell^2(\mathbb{Z})^{(d)}}.$$

Proof. From (2.1), one has

$$\begin{aligned}
\left\|\sum_{k\in\mathbb{Z}} \phi(\cdot-k)C(k)\right\|^2_{L^2(\mathbb{R})^{(d)}} &= \left\langle \sum_{k\in\mathbb{Z}} \phi(\cdot-k)C(k), \sum_{k\in\mathbb{Z}} \phi(\cdot-k)C(k)\right\rangle_{L^2(\mathbb{R})^{(d)}} \\
&= \left\langle \hat{\phi}(\cdot)\widehat{C}(\cdot), \hat{\phi}(\cdot)\widehat{C}(\cdot)\right\rangle_{L^2(\mathbb{R})^{(d)}} \\
&= \int_{\mathbb{R}} \overline{\widehat{C}}^T(\xi)\overline{\hat{\phi}}^T(\xi)\hat{\phi}(\xi)\widehat{C}(\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}} \int_k^{k+1} \overline{\widehat{C}}^T(\xi)\overline{\hat{\phi}}^T(\xi)\hat{\phi}(\xi)\widehat{C}(\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 \overline{\widehat{C}}^T(\xi+k)\overline{\hat{\phi}}^T(\xi+k)\hat{\phi}(\xi+k)\widehat{C}(\xi+k)d\xi \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 \overline{\widehat{C}}^T(\xi)\overline{\hat{\phi}}^T(\xi+k)\hat{\phi}(\xi+k)\widehat{C}(\xi)d\xi \\
&= \int_0^1 \overline{\widehat{C}}^T(\xi)\sum_{k\in\mathbb{Z}} \overline{\hat{\phi}}^T(\xi+k)\hat{\phi}(\xi+k)\widehat{C}(\xi)d\xi,
\end{aligned}$$

by (4.2) and Rayleigh-Ritz theorem, we have

$$a\int_0^1 \overline{\widehat{C}}^T(\xi)\widehat{C}(\xi)d\xi \le \left\|\sum_{k\in\mathbb{Z}} \phi(\cdot-k)C(k)\right\|^2_{L^2(\mathbb{R})^{(d)}} \le b\int_0^1 \overline{\widehat{C}}^T(\xi)\widehat{C}(\xi)d\xi.$$

Note that

$$\int_0^1 \overline{\widehat{C}}^T(\xi)\widehat{C}(\xi)d\xi = \|C\|^2_{\ell^2(\mathbb{Z})^{(d)}}.$$

This completes the proof. □

Assume that $f\in V(\phi)$. Define $f_j = A^j * f$, $h_j = f_j|_{\mathbb{Z}}$, $y_j = S_m h_j$, $\phi_j = A^j * \phi$, $\widehat{\Phi_j}(\xi) = \sum_k \widehat{\phi_j}(\xi+k)$, $0\le j\le m-1$.

Lemma 4.2. *Let $V(\phi)$ is a SIS and $f\in V(\phi)$. Then*

$$\mathscr{F}(S_m h_j)(\xi) = \frac{1}{m}\sum_{l=0}^{m-1} \widehat{\Phi_j}\left(\frac{\xi+l}{m}\right)\widehat{C}\left(\frac{\xi+l}{m}\right). \tag{4.3}$$

Proof. By the proof of Theorem 3.1, we have

$$\mathscr{F}(S_m h)(\xi) = \frac{1}{m}\sum_{l=0}^{m-1} \hat{h}\left(\frac{\xi+l}{m}\right), \tag{4.4}$$

where $\widehat{h}(\xi) = \sum_k h(k)e^{-2\pi ik\xi}$. Using the Poisson summation formula and the 1-periodicity of $\widehat{C}$, we have for $j = 0,1,...,m-1$

$$\begin{aligned}
\mathscr{F}(S_m h_j)(\xi) &= \frac{1}{m}\sum_{l=0}^{m-1}\sum_{k\in\mathbb{Z}} \widehat{f}_j\left(\frac{\xi+l}{m}+k\right) \\
&= \frac{1}{m}\sum_{l=0}^{m-1}\sum_{k\in\mathbb{Z}} \widehat{\phi}_j\left(\frac{\xi+l}{m}+k\right)\widehat{C}\left(\frac{\xi+l}{m}+k\right) \\
&= \frac{1}{m}\sum_{l=0}^{m-1}\left(\sum_{k\in\mathbb{Z}} \widehat{\phi}_j\left(\frac{\xi+l}{m}+k\right)\right)\widehat{C}\left(\frac{\xi+l}{m}\right) \\
&= \frac{1}{m}\sum_{l=0}^{m-1} \widehat{\Phi_j}\left(\frac{\xi+l}{m}\right)\widehat{C}\left(\frac{\xi+l}{m}\right).
\end{aligned}$$

□

Theorem 4.3. *Let*

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1d} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{d1} & \phi_{d2} & \cdots & \phi_{dd} \end{pmatrix},$$

where $\phi_{ij} \in W_0(L^1), 1 \le i \le d, 1 \le j \le d$. *Define*

$$\mathscr{A}_m(\xi) = \begin{pmatrix} \widehat{\Phi_0}\left(\frac{\xi}{m}\right) & \widehat{\Phi_0}\left(\frac{\xi+1}{m}\right) & \cdots & \widehat{\Phi_0}\left(\frac{\xi+m-1}{m}\right) \\ \widehat{\Phi_1}\left(\frac{\xi}{m}\right) & \widehat{\Phi_1}\left(\frac{\xi+1}{m}\right) & \cdots & \widehat{\Phi_1}\left(\frac{\xi+m-1}{m}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{\Phi_{m-1}}\left(\frac{\xi}{m}\right) & \widehat{\Phi_{m-1}}\left(\frac{\xi+1}{m}\right) & \cdots & \widehat{\Phi_{m-1}}\left(\frac{\xi+m-1}{m}\right) \end{pmatrix}, \tag{4.5}$$

$\xi \in \mathbb{T}$. *Then a vector* $f \in V(\phi)$ *can be recovered in a stable way , i.e. the inverse is bounded, from the measurements* y_j, *if and only if there exists* $\alpha > 0$ *such that the set* $\{\xi : |det\mathscr{A}_m(\xi)| < \alpha\}$ *has zero measure.*

Proof. By Lemma 4.2, for $0 \le j \le m-1$

$$\widehat{y_j}(\xi) = \mathscr{F}(S_m h_j)(\xi) = \frac{1}{m}\sum_{l=0}^{m-1} \widehat{\Phi_j}\left(\frac{\xi+l}{m}\right)\widehat{C}\left(\frac{\xi+l}{m}\right).$$

Let

$$\mathrm{y}(\xi) = m \begin{pmatrix} \widehat{y_0}(\xi) \\ \widehat{y_1}(\xi) \\ \vdots \\ \widehat{y_{m-1}}(\xi) \end{pmatrix}$$

and

$$\mathscr{C}(\xi) = \begin{pmatrix} \widehat{C}(\frac{\xi}{m}) \\ \widehat{C}(\frac{\xi+1}{m}) \\ \vdots \\ \widehat{C}(\frac{\xi+m-1}{m}) \end{pmatrix}, \quad \widehat{C}(\xi) \in L^2(\mathbb{T})^{(d)}.$$

Then we have

$$\mathrm{y}(\xi) = \mathscr{A}_m(\xi)\mathscr{C}(\xi). \tag{4.6}$$

We can solve the equation (4.6) with respect to $\mathscr{C}(\xi)$ (which we use to produce C) if $\mathscr{A}(\xi)$ is invertible. Since there exists $\alpha > 0$ such that the set $\{\xi : |det\mathscr{A}_m(\xi)| < \alpha\}$ has zero measure, the $\mathscr{A}_m(\xi)$ has a bounded inverse. Thus a vector $x \in \ell^2(\mathbb{Z})^{(d)}$ can be recovered in a stable way from the measurements $y_n = S_m A^n x, n = 0, 1, ..., m-1$. □

ACKNOWLEDGMENTS

This work was supported partially by the Natural Science Research Project of Higher Education of Tianjin (2018KJ128).

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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