Available online at http://scik.org
Advances in Fixed Point Theory, 3 (2013), No. 2, 248-262
ISSN: 1927-6303

ON A NEW GENERALIZED WEAKLY CONTRACTION MAPPING IN $G-$ METRIC SPACES

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Abstract. In this paper, we establish some common fixed point results for two self-mappings f and g on a generalized metric space (X, G). Our result generalizes and improves some known results in the literature.

Keywords: Fixed point, coincidence point, $G-$ metric spaces, contraction.

2000 AMS Subject Classification: 47H10; 54H25

1. Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity, see [14, 15, 16, 17, 18, 21, 22, 24, 25, 26, 27, 40]. The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhage [1, 2]. Recently, Mustafa and Sims [30, 31, 32] have shown that most of the results concerning Dhage's D-metric spaces are invalid. In [30, 33, 34, 35], they introduced a improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces and fixed point results, one can refer to the papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 23, 28, 36, 37, 38, 39], some of them have

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Received January 8, 2013

given some applications to matrix equations, ordinary differential equations, and integral equations.

2. Preliminaries

Definition 2.1. ([29]). Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties

: (1) $G(x, y, z) = 0$ if $x = y = z$.
  (2) $0 < G(x, x, y)$ for all x, $y \in X$ with $x \neq y$
  (3) $G(x, x, y) \leq G(x, y, z)$ forall $x, y, z \in X$ with $y \neq z$.
  (4) $G(x, y, z) = G(x, z, y) = G(y, z, x) =$ (symmetry in all three variables).
  (5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X and the pair (X, G) is called a $G-$metric space.

Definition 2.2. ([29]). Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is $G-$ convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence x_n and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.1. ([29]) Let (X, G) be a G-metric space. The following are equivalent :

(1) (x_n) is G-convergent to x
(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$
(4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$

Definition 2.3. ([29]) Let (X, G)be a G-metric space. A sequence (x_n) is called a $G-$ Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.2. ([29]) Let (X, G) be a G-metric space. Then the following are equivalent :

(1) The sequence (x_n) is G-Cauchy

(2) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 2.3. ([31] Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$, it follows that:

(1) $G(x, y, z) \leq \frac{2}{3}\left[G(x, y, a) + G(x, a, z) + G(a, y, z)\right]$
(2) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$

Proposition 2.4. ([29]) Let (X, G)be a G-metric space. A mapping $f : X \to X$ is $G-$continuous at $x \in X$ if and only if it is $G-$ sequentially continuous at x, that is , whenever (x_n) is G-convergent to x, $f(x_n)$ is G-convergent to $f(x)$.

Proposition 2.5. ([29]) Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous all three of its variables.

Definition 2.4. ([29]) A G-metric space (X, G) is called $G-$ complete if every $G-$ Cauchy sequence is $G-$ convergent in (X, G).

Definition 2.5. (weakly compatible mappings ([29])). Two mappings $f, g : X \to X$ are weakly compatible if they commute at their coincidence points, that is $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

Proposition 2.6. ([29]) Let X be a non-empty set and S, T self-mappings of X. A point $x \in X$ is called a coincidence point of S and T if $Sx = Tx$. A point $w \in X$ is said to be a point of coincidence of S and T if there exists $x \in X$ so that $w = Sx = Tx$.

Definition 2.6. ($g-$ Non decreasing Mapping ([29]). Suppose $(X, \preceq)$ is a partially ordered set and $f, g : X \to X$ are mappings. f is said to be $g-$ Non decreasing if for $x, y \in X$, $gx \preceq gy$ implies $fx \preceq fy$. Khan et all ([40])introduced the concept of altering distance function that is a control function employed to alter the metric distance between two points enabling one to deal with relatively new classes of fixed point problems.

Let us denote by Ψ the class of the set of altering distance functions $\psi : [0, +\infty[\to [0, +\infty[$ which satisfies the following conditions:

(1) ψ is nondecreasing
(2) ψ is continuous
(3) $\psi(t) = 0 \iff t = 0$

and by Φ the class of the set of continuous functions $\varphi : [0,+\infty[\to [0,+\infty[$, nondecreasing.

3. Main results

We start with the following remark

Remark 3.1.

If $\psi \in \Psi$ and if $\varphi \in \Phi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, then $\varphi(0) = 0$.

Proof. Since $\varphi(t) < \psi(t)$ for all $t > 0$, then we have

$$0 \leq \varphi(0) \leq \liminf_{t\to 0} \varphi(t) \leq \lim_{t\to 0} \psi(t) = \psi(0) = 0$$

This finishes the proof.

Theorem 3.1. Let (X, G) be a G-metric space and $f, g : X \to X$ be two mappings. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, such that

$$\psi\left(G(fx, fy, fz)\right) \leq \varphi\left(\frac{1}{3}\left(G(gx, fy, fy) + G(gy, fz, fz) + G(gz, fx, fx)\right)\right) \tag{1}$$

If $f(X) \subset g(X)$, $g(X)$ is a complete subset of (X, G) and the pair $\{f, g\}$ is weakly compatible. Then f and g have a unique common fixed point.

Proof. By the fact that $f(X) \subset g(X)$, we can construct a sequence $\{x_n\}$ in X such that

$$fx_{n+1} = gx_n \text{ for any } n \in \mathbb{N}$$

If for some n, $gx_{n+1} = fx_n$, then $fx_{n,} = gx_n$ that is f and g have a common fixed point. Thus we may assume that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, then bf (1) and by the rectangle inequality, we get

$$(2)\quad \begin{aligned} \psi\left(G(gx_n\ ,gx_{n+1}\ ,gx_{n+1})\right) &= \psi\left(G(fx_{n-1}\ ,fx_{n,}\ ,fx_{n,}\right) \\ &\leq \varphi\begin{pmatrix} \frac{1}{3}(G(gx_{n-1},fx_{n,}\ ,fx_{n,} \\ +G(gx_n,fx_{n,}\ ,fx_{n,})+G(gx_n,fx_{n-1},fx_{n-1})) \end{pmatrix} \\ &= \varphi\begin{pmatrix} \frac{1}{3}(G(gx_{n-1},gx_{n+1,}\ ,gx) \\ +G(gx_n,gx_{n+1},gx_{n+1})+G(gx_n,gx_n,gx_n)) \end{pmatrix} \\ &\leq \varphi\begin{pmatrix} \frac{1}{3}(G(gx_{n-1},gx_{n+1,}\ ,gx) \\ +G(gx_n,gx_{n+1},gx_{n+1})) \end{pmatrix} \\ &\leq \varphi\begin{pmatrix} \frac{1}{3}(G(gx_{n-1},gx_{n,}\ ,gx) \\ +\frac{2}{3}G(gx_n,gx_{n+1},gx_{n+1}) \end{pmatrix} \end{aligned}$$

By the condition of the theorem, we have

$$(3)\quad \begin{aligned} G(gx_n\ ,gx_{n+1}\ ,gx_{n+1}) &\leq \frac{1}{3}((G(gx_{n-1},gx_{n+1,}\ ,gx_{n+1,})+G(gx_n,gx_{n+1},gx_{n+1}) \\ &\leq \frac{1}{3}\left(G(gx_{n-1},gx_{n,}\ ,gx_{n,})+\frac{2}{3}G(gx_n,gx_{n+1},gx_{n+1}\right) \end{aligned}$$

Then, it follows easily that

$$(4)\qquad G(gx_n\ ,gx_{n+1}\ ,gx_{n+1}) \leq G(gx_{n-1},gx_{n,}\ ,gx_{n,}) \text{ for any } n \in \mathbb{N}$$

Therefore $\{G(gx_n\ ,gx_{n+1}\ ,gx_{n+1})\}$ is a non-increasing sequence. Hence there exists $r \geq 0$ such that

$$(5)\qquad \lim_{n\to\infty} G(gx_n\ ,gx_{n+1}\ ,gx_{n+1}) = r$$

letting $n \to \infty$ in (3), we get

$$r \leq \frac{1}{3}\lim_{n\to\infty} G(gx_{n-1},gx_{n+1},gx_{n+1}) + \frac{r}{3} \leq \frac{1}{3}r + \frac{2}{3}r = r$$

which implies

$$(6)\qquad \lim_{n\to\infty} G(gx_{n-1},gx_{n+1},gx_{n+1}) = 2r$$

Again from (2) we have

$$\psi\left(G(gx_n,gx_{n+1},gx_{n+1})\right)\leq\varphi\left(\frac{1}{3}\left(\begin{array}{c}G\left(gx_{n-1},gx_{n+1},gx_{n+1}\right)\\+G(gx_n,gx_{n+1},gx_{n+1}\end{array}\right)\right)$$

Letting $n\to\infty$ and using (5), (6) and the continuities of ψ and φ, we get

$$\psi\left(r\right)\leq\varphi\left(r\right)$$

using the condition of the theorem, we get $r=0$, this means that

$$\lim_{n\to\infty}G(gx_n,gx_{n+1},gx_{n+1})=0 \tag{7}$$

Next, we show that $\{gx_n\}$ is a $G-$ Cauchy sequence. Suppose, on the contrary, that $\{gx_n\}$ is not a $G-$ Cauchy sequence, that is

$$\lim_{n,m\to\infty}G(gx_m,gx_n,gx_n)\neq 0$$

then, there exists $\varepsilon>0$ for which we can find two sequences $\{gx_{m(i)}\}$ and $\{gx_{n(i)}\}$ of $\{gx_n\}$ such that $n(i)$ is the smallest index for which

$$n(i)>m(i)>i,\quad G(gx_{m(i)},gx_{n(i)},gx_{n(i)})\geq\varepsilon \tag{8}$$

This means that

$$G(gx_{m(i)},gx_{n(i)-1},gx_{n(i)-1})<\varepsilon \tag{9}$$

Now, from (8), (9), rectangular inequality and proposition 6, we have that

$$\begin{aligned}
\varepsilon &\leq G(gx_{m(i)}, gx_{n(i)}, gx_{n(i)}) \\
&\leq G(gx_{m(i)}, gx_{m(i)+1}, gx_{m(i)+1}) \\
&\quad +G(gx_{m(i)+1}, gx_{n(i)}, gx_{n(i)}) \\
&\leq G(gx_{m(i)}, gx_{m(i)+1}, gx_{m(i)+1}) \\
&\quad +G(gx_{m(i)+1}, gx_{n(i)-1}, gx_{n(i)-1}) \\
&\quad +G(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)}) \\
&\leq 3G(gx_{m(i)}, gx_{m(i)+1}, gx_{m(i)+1}) \\
&\quad +G(gx_{m(i)}, gx_{n(i)-1}, gx_{n(i)-1}) \\
&\quad +G(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)}) \\
&< 3G(gx_{m(i)}, gx_{m(i)+1}, gx_{m(i)+1}) \\
&\quad +\varepsilon + G(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)})
\end{aligned}$$

letting $i \to \infty$ in the above inequalities and using (7), we get that

$$\begin{aligned}
\lim_{i\to\infty} G(gx_{m(i)}, gx_{n(i)}, gx_{n(i)}) &= \lim_{i\to\infty} G(gx_{m(i)+1}, gx_{n(i)}, gx_{n(i)}) \\
&= \lim_{i\to\infty} G(gx_{m(i)}, gx_{n(i)-1}, gx_{n(i)-1}) \\
&= \varepsilon
\end{aligned} \tag{10}$$

By (1), we have

$$\begin{aligned}
&\psi\left(G(gx_{m(i)+1}, gx_{n(i)}, gx_{n(i)})\right) \\
&= \psi\left(G(fx_{m(i)}, fx_{n(i)-1}, fx_{n(i)-1})\right) \\
&\leq \varphi\left(\frac{1}{3}\left(\begin{array}{c} G(gx_{m(i)}, fx_{n(i)-1}, fx_{n(i)-1}) + G(gx_{n(i)-1}, fx_{n(i)-1}, fx_{n(i)-1}) \\ +G(gx_{n(i)-1}, fx_{m(i)}, fx_{m(i)}) \end{array}\right)\right) \\
&\leq \varphi\left(\frac{1}{3}\left(\begin{array}{c} G(gx_{m(i)}, gx_{n(i)}, gx_{n(i)}) + G(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)}) \\ +G(gx_{n(i)-1}, gx_{m(i)+1}, gx_{m(i)+1}) \end{array}\right)\right)
\end{aligned} \tag{11}$$

Once again, from the condition of the theorem, we get

$$G(gx_{m(i)+1}, gx_{n(i)}, gx_{n(i)}) \leq \frac{1}{3}\begin{pmatrix} G(gx_{m(i)}, gx_{n(i)}, gx_{n(i)}) \\ +G(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)}) \\ +G(gx_{n(i)-1}, gx_{m(i)+1}, gx_{m(i)+1}) \end{pmatrix}$$

Then, by the rectangular inequality and proposition 6, we have

$$\begin{aligned} G(gx_{m(i)+1}, gx_{n(i)}, gx_{n(i)}) \;\; &\leq \;\; \frac{1}{3}\begin{pmatrix} G(gx_{m(i)}, gx_{n(i)}, gx_{n(i)}) \\ +G(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)}) \\ +G(gx_{n(i)-1}, gx_{m(i)+1}, gx_{m(i)+1}) \end{pmatrix} \\ & \frac{1}{3}\begin{pmatrix} G(gx_{m(i)}, gx_{n(i)}, gx_{n(i)}) \\ +G(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)}) \\ +2G(gx_{n(i)-1}, gx_{n(i)-1}, gx_{m(i)+1}) \end{pmatrix} \\ & \frac{1}{3}\begin{pmatrix} G(gx_{m(i)}, gx_{n(i)}, gx_{n(i)}) \\ +G(gx_{n(i)-1}, gx_{n(i)}, gx_{n(i)}) \\ +2G(gx_{n(i)-1}, gx_{n(i)-1}, gx_{m(i)}) \\ +2G(gx_{m(i)}, gx_{m(i)}, gx_{m(i)+1}) \end{pmatrix} \end{aligned}$$

letting $i \to \infty$ in the above inequalities and using (7) and (10), we get that

$$\lim_{i\to\infty} G(gx_{n(i)-1}, gx_{m(i)+1}, gx_{m(i)+1}) = 2\varepsilon \tag{12}$$

Now letting $i \to \infty$ in(11) and using (7), (10), (12) and the continuities of of ψ and φ, we have

$$\psi(\varepsilon) \leq \varphi(\varepsilon)$$

Therefore by using the condition of the theorem, we get $\varepsilon = 0$, which is a contradiction. Thus $\{gx_n\}$ is a $G-$ Cauchy sequence in $g(X)$. Since $(g(X), G)$ is complete, then there exist t, $u \in X$ such that $\{gx_n\}$ converges to $t = gu$, that is

$$\lim_{n\to\infty} G(gx_n, gx_n, gu) = \lim_{n\to\infty} G(gx_n, gu, gu) = 0 \tag{13}$$

Since G is continuous on its variables, we have

$$\lim_{n\to\infty} G(gx_n, gx_n, fu) = G(gu, gu, fu) \tag{14}$$

and

$$\lim_{n\to\infty} G(gx_n, fu, fu) = G(gu, gu, fu) \tag{15}$$

Let us show that $fu = t$. By (1), we have

$$\begin{aligned}
\psi\left(G(gx_{n+1}, gx_{n+1}, fu)\right) &= \psi\left(G(fx_n, fx_n, fu)\right) \\
&\leq \varphi\left(\frac{1}{3}\left(\begin{array}{c} G(gx_n, fx_n, fx_n) \\ +G(gx_n, fu, fu) + G(gu, fx_n, fx_n) \end{array}\right)\right) \\
&\leq \varphi\left(\frac{1}{3}\left(\begin{array}{c} G(gx_n, gx_{n+1}, gx_{n+1}) \\ +G(gx_n, fu, fu + G(gu, gx_{n+1}, gx_{n+1}) \end{array}\right)\right)
\end{aligned}$$

Letting $n \to \infty$ and using (7), (13), (15) and the continuities of ψ and φ, we get

$$\psi\left(G(gu, gu, fu)\right) \leq \varphi\left(\frac{1}{3} G(gu, fu, fu)\right) \tag{16}$$

By proposition 6, we have $G(gu, fu, fu) \leq 2G(gu, gu, fu)$. Hence using the fact that φ is increasing, (16) becomes

$$\psi\left(G(gu, gu, fu)\right) \leq \varphi\left(\frac{2}{3} G(gu, fu, fu)\right)$$

Therefore from the condition of the theorem, it follows

$$G(gu, gu, fu) \leq \frac{2}{3} G(gu, fu, fu)$$

which implies that $G(gu, gu, fu) = 0$ and hence $fu = gu = t$. Then, u is a coincidence point of f and g, since the pair $\{f, g\}$ is weakly compatible, we have $ft = gt$

Now, we prove that $ft = gt = t$. By (1), we have

$$\begin{aligned}
\psi\left(G(gt, gx_{n+1}, gx_{n+1})\right) &= \psi\left(G(ft, fx_n, fx_n)\right) \\
&\leq \varphi\left(\frac{1}{3}\left(\begin{array}{c} G\left(gt, fx_n, fx_n\right) \\ +\ G\left(gx_n, fx_n, fx_n\right) + G(gx_n, gt, gt) \end{array}\right)\right) \\
&= \varphi\left(\frac{1}{3}\left(\begin{array}{c} G\left(gt, gx_{n+1}, gx_{n+1}\right) \\ +\ G\left(gx_n, gx_{n+1}, gx_{n+1}\right) + \left(G\left(gx_n, gt, gt\right)\right) \end{array}\right)\right)
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} & \psi\left(G(gt,gu,gu)\right) \\ \leq\ & \varphi\left(\frac{1}{3}G\left(gt,gu,gu\right)+0+G(gu,gt,gt\right) \\ \leq\ & \varphi\left(\frac{1}{3}G\left(gt,gu,gu\right)+\frac{2}{3}G\left(gt,gu,gu\right)\right) \\ =\ & \varphi\left(G\left(gt,gu,gu\right)\right) \end{aligned}$$

By the condition of the theorem, we have $G\left(gt,gu,gu\right)=0$ which implies that $gt=gu=t$. We conclude that

$$gt=gu=t.$$

and so t is a common fixed point of f and g. To prove the uniqueness, let s be another common fixed point of f and g. By (1), we have

$$\begin{aligned} \psi\left(G(t,s,s)\right) &= \psi\left(G(ft,fs,fs)\right) \\ &\leq \varphi\left(\frac{1}{3}\left(G\left(t,fs,fs\right)+G\left(s,fs,fs\right)+G(s,ft,ft)\right)\right) \\ &= \varphi\left(\frac{1}{3}\left(G\left(t,s,s\right)+G\left(s,t,t\right)\right)\right) \\ &\leq \varphi\left(\frac{1}{3}\left(2G\left(t,t,s\right)+G\left(s,t,t\right)\right)\right) \\ &= \varphi\left(G\left(t,t,s\right)\right) \end{aligned}$$

By the condition of the theorem, we have $G(t,s,s)=0$, which implies that $s=t$.

Example 3.1. Let $X=[0,2]$, $G(x,y,z)=\max\{|x-y|,\ |y-z|\,,\ |z-x|\}$, $\psi\left(t\right)=t^2$, $\varphi\left(t\right)=\ln\left(1+t^2\right)$, $fx=1$, $gx=2-x$. It is easy to see $\psi\left(t\right)>\varphi\left(t\right)$ for all $t>0$. We have

$$\psi\left(G\left(fx,fy,fz\right)\right)=0$$

and

$$\begin{aligned} & \varphi\left(\frac{1}{3}\left(G(gx,fy,fy)+G(gy,fy.fy)+G(gz,fy,fy)\right)\right) \\ =\ & \ln\left((1+(|x-y|+|y-z|\ +|z-x|)^2\right) \end{aligned}$$

Now, condition (1) is trivially satisfied. Clearly, $f(X) \subset g(X)$, $g(X)$ is a complete subset of (X,G) and the pair $\{f,g\}$ is weakly compatible.Then all the hypothesis of Theorem are satisfied and so f and g have a unique common fixed point, that is $x = 1$.

Let S denotes the class of the continuous functions $\beta : [0,\infty) \to [0,1)$ which is non-increasing and satisfies the condition $\beta(t_n) \to 1$ implies $t_n \to 0$.

Corollary 3.1. Let (X,G) be a G-metric space and $f,g : X \to X$ be two mappings. If there exist $\psi \in \Psi$ and $\beta \in S$ with the condition $\psi(t) > \beta(t)t$ for all $t > 0$, such that

$$\begin{aligned}\psi(G(fx,fy,fz)) \leq\ & \beta\left(\frac{1}{3}(G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx))\right)\\ & \left(\frac{1}{3}(G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx))\right)\end{aligned}$$

If $f(X) \subset g(X)$, $g(X)$ is a complete subset of (X,G) and the pair $\{f,g\}$ is weakly compatible. Then f and g have a unique common fixed point.

Corollary 3.2. Let (X,G) be a G-metric space and $f,g : X \to X$ be two mappings. If there exist $\psi \in \Psi$, φ_1 and $-\psi_1 \in \Phi$ with the condition $\psi(t) > (\varphi_1 - \psi_1)(t)$ for all $t > 0$, such that

$$\begin{aligned}\psi(G(fx,fy,fz)) \leq\ & \varphi_1\left(\frac{1}{3}(G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx))\right)\\ & -\psi_1\left(\frac{1}{3}(G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx))\right)\end{aligned}$$

If $f(X) \subset g(X)$, $g(X)$ is a complete subset of (X,G) and the pair $\{f,g\}$ is weakly compatible. Then f and g have a unique common fixed point.

Proof. It follows by replacing in Theorem14, $\varphi(t)$ by $\phi(t) = \varphi_1(t) - \psi_1(t)$.

From the previous obtained results, we deduce some coincidence point results for mappings satisfying a contraction of an integral type. For this purpose, let

$$Y = \left\{\begin{array}{c}\chi, \chi : \mathbb{R}^+ \to \mathbb{R}^+, \text{ satisfies that } \chi \text{ is Lesbesgue integrable,}\\ \text{summable on each compact of subset of } \mathbb{R}^+\\ \text{and } \int_0^\epsilon \chi(t)\,dt > 0 \text{ for each } \epsilon > 0\end{array}\right\}$$

Theorem 3.2. Let (X,G) be a G-metric space and $f,g:X\to X$ be two mappings. If there exist $\psi\in\Psi$ and $\varphi\in\Phi$ with the condition $\psi(t)>\varphi(t)$ for all $t>0$, such that

$$\int_0^{\psi(G(fx,fy,fz))}\chi(t)\,dt\le\int_0^{\varphi\left(\frac{1}{3}(G(gx,fy.fy)+G(gy,fy.fy)+G(gz,fy.fy))\right)}\chi(t)\,dt$$

If $f(X)\subset g(X)$, $g(X)$ is a complete subset of (X,G) and the pair $\{f,g\}$ is weakly compatible. Then f and g have a unique common fixed point.

Proof. For $\chi\in Y$, consider the function $\Lambda:\mathbb{R}^+\to\mathbb{R}^+$ defined by $\Lambda(x)=\int_0^x\chi(t)\,dt$ we note that $\Lambda\in\Psi$. thus the inequality (**??**) becomes

$$\text{(17)}\quad \Lambda\left(\psi(G(fx,fy,fz))\right)\le\Lambda\left(\varphi\left(\frac{1}{3}\left(G(gx,fy.fy)+G(gy,fy.fy)+G(gz,fy.fy)\right)\right)\right)$$

Setting $\Lambda\circ\psi=\psi_1$, $\psi_1\in\Psi$, and $\Lambda\circ\varphi=\varphi_1$, $\varphi_1\in\Phi$, $\psi_1(t)>\varphi_1(t)$ for all $t>0$. So, we obtain

$$\psi_1(G(fx,fy,fz))\le\varphi_1\left(\frac{1}{3}\left(G(gx,fy.fy)+G(gy,fy.fy)+G(gz,fy.fy)\right)\right)$$

Therefore by Theorem 14 above, f and g have a unique common fixed point.

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